Odd Cycles in Graphs of Given Minimum Degree

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ABSTRACT

The principal result of the paper is that any nonbipartite 2–connected graph on n vertices of minimum degree $\geq 2n/(k+2)$ (k a fixed odd integer and n large) contains a k–cycle or is isomorphic to the following graph H. The graph H has n vertices (with n divisible by k + 2), and is obtained from the k + 2–cycle by replacing each of its k + 2 vertices by an independent set of order n/(k + 2).

1. Results

Let G be a nonbipartite graph of order n and minimum degree δ. A natural extremal question is the following: What is the smallest value of δ such that G contains a cycle C_k of fixed, odd length k?

To address this question first consider two special n–vertex graphs H and L. Let $G_i = G_i(A_i, B_i)$, $1 \leq i \leq 3$, be three vertex disjoint copies of the complete bipartite graph $K_{\lceil (n-3)/6 \rceil, \lceil (n-3)/6 \rceil}$ where A_i and B_i denote the partite sets of G_i. Take a triangle C_3 with vertices a_1, a_2, a_3 (the vertices of C_3 disjoint from each G_i) and for each i join vertex a_i completely to the set of vertices A_i in G_i. Let L denote the graph that results. To ensure that L is an n–vertex graph one can assume that n – 3 is a multiple of 6. For n divisible by k + 2 let H be the n–vertex graph obtained from a C_{k+2} by replacing each of its k + 2 vertices by an independent set of order n/(k + 2).

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Since L has C_3 as its only odd cycle and has minimum degree $\lceil (n-3)/6 \rceil > 2n/(k+2)$ for $k \geq 11$ and n large, minimum degree $\delta = 2n/(k+2)$ is not sufficient to guarantee the existence of a C_k in G. If one insists that G be 2–connected as well as nonbipartite, then H shows $\delta > 2n/(k+2)$. In fact under these conditions G contains a C_k when $\delta \geq 2n/(k+2)$ unless G is isomorphic to H. This is the content of the principal result in the paper and is stated as the first theorem.

Theorem 1 *Let $k \geq 3$ be a fixed odd positive integer. If G is a 2–connected nonbipartite graph on n vertices of minimum degree $\geq 2n/(k+2)$, then for n large ($n \geq f(k)$) either G contains the k–cycle C_k or is isomorphic to H.*

What happens if G is regular? This has particular meaning when n is odd, since then the graph G must be nonbipartite. Also in this case the 2–connected condition can be dropped as is seen in the next theorem.

Theorem 2 *Let $k \geq 3$ be a fixed odd positive integer. If G is $2n/(k+2)$–regular on n vertices with n odd, then for n large ($n \geq g(k)$)G contains a C_k or is isomorphic to H.*

As noted earlier the 2–connectedness of G assumed in Theorem 1 is essential, at least for $k \geq 11$. What happens if $3 \leq k \leq 9$? Theorem 3 shows for these cases that the 2–connectedness can be dropped.

Theorem 3 *Let G be a nonbipartite graph on n vertices of minimum degree $\geq 2n/(k+2)$, where k is a fixed odd integer. For n large ($n \geq l(k)$)G either is isomorphic to H or contains a C_k for $k \in \{3, 5, 7, 9\}$, but may fail to contain a C_k and also not be isomorphic to H when $k \geq 11$.*

Clearly if the conditions of Theorem 1 hold, then for n large G contains all C_{2t+1} for $k < 2t+1 \leq d$, d any fixed number larger than k. Also if $G' = K_{2n/(k+2),n-(2n/(k+2))}$ and G is obtained from G' by adding an edge to its smaller part, then G contains C_{2t+1} for all $3 \leq 2t+1 \leq 4n/(k+2) - 1$ and no larger odd cycle. Thus it is reasonable to inquire whether the conditions of Theorem 1 guarantee all odd cycles C_{2t+1} for $k < 2t+1 \leq 4n/(k+2) - 1$. This is partially answered in Theorem 4.

Theorem 4 *Let G be a 2–connected nonbipartite graph of order n and minimum degree $\geq cn$, $0 < c < \frac{1}{3}$. For each ε, $1 > \varepsilon > 0$, there exist functions $h_1(c,\varepsilon)$ and $h_2(c,\varepsilon)$ such that for large $n (n \geq h_2(c,\varepsilon))$G contains the cycle C_{2t+1} for $h_1(c,\varepsilon) \leq 2t+1 \leq 4(1-\varepsilon)cn/3$.*

From the discussion given above, letting $c = 2/(k + 2)$, it is clear that the upper bound in the last result can at most be improved to $2(1 - \varepsilon)cn$. Although this is most likely true the present proof does not seem to work beyond the bound given in the theorem.

It is easy to give examples which show that the degree condition of Theorem 1 cannot be replaced by a reasonable edge condition, even if the graph G contains a very small odd cycle. For example take a cycle C_{k+2}, and assume k is not too small, say k is fixed and odd with $k \geq 15$. Mark four consecutive vertices of the cycle x_1, x_2, x_3, x_4. Join a new vertex x to precisely x_1 and x_2, and replace each of x_3 and x_4 by an independent set of order $(n - k - 1)/2$. The resulting graph has n vertices, approximately $n^2/4$ edges, a C_3, and no odd cycles strictly between 3 and $k + 2$.

Throughout the paper notation will, unless otherwise specified, follow that found in standard texts. Before giving the proofs of the above results two well known extremal theorems are stated. These two theorems are used frequently in the proofs that follow.

Theorem A [1] (Erdös, Gallai) *A graph G on n vertices with at least $[n(k-2)+1]/2$ edges contains a path P_k on k vertices. Furthermore, when $n = (k-1)t$ the graph tK_{k-1} (the union of t vertex disjoint copies of K_{k-1}) contains the maximum number of edges in an n vertex graph with no P_k and is the unique such graph.*

The second extremal result deals with the well known problem of Zarankiewicz. Let Z(n;t) denote the maximal size of a bipartite graph G(n,n) having both parts with n vertices such that G(n,n) contains no $K_{t,t}$. The bound given in the theorem below is an improvement by Znám [3] of the bound proved by Kövári, Sós, and Turán [2].

Theorem B [3] (Problem of Zarankiewicz)

(1) If $2 \leq t < n$, then $Z(n;t) < (t-1)^{1/t}n^{2-1/t} + (t-1)n/2$.

(2) If a graph of order n does not contain a $K_{t,t}$ then its size is at most $((t-1)^{1/t}n^{2-1/t} + (t-1)/2^n)/2$.

2. Proofs

Before proving the main theorem several lemmas and propositions are needed.

Lemma 1 *Let C_t be a cycle of odd length in a graph G. If some vertex x not on the cycle C_t is adjacent to at least five vertices of C_t, then G contains a cycle C_p of odd length for some $p, t/5 \leq p < t$.*

Proof Let $C_t = (x_1, x_2, \ldots, x_t, x_1)$ and assume x is adjacent to $x_{j_i}, 1 \leq i \leq 5$, where $j_{i-1} < j_i$ for $2 \leq i \leq 5$. Set $j_i - j_{i-1} - 1 = a_{i-1}$ for $2 \leq i \leq 5$ and $t + j_1 - j_5 - 1 = a_5$.

Therefore a_{i-1} counts the number of vertices which are strictly between $x_{j_{i-1}}$ and x_{j_i} along the cycle from $x_{j_{i-1}}$ to $x_{j_i} (2 \le i \le 5)$ and a_5 counts the number strictly between x_{j_5} and x_{j_1}. Without loss of generality assume $a_1 = \max_{1 \le i \le 5}\{a_i\}$ so that $a_1 \ge (t-5)/5$.

Consider the following possibilities: (1) a_1 is even, (2) a_1 is odd and a_2 or a_5 is even, (3) a_1, a_2, a_5 are all odd and exactly one of a_3 or a_4 is even. Note that since

$$t = \sum_{i=1}^{5} a_i + 5$$

and t is odd, one of the three possibilities occur. If (1) occurs let $C_p = (x, x_{j_1}, x_{j_{1+1}}, \ldots, x_{j_2}, x)$. If (2) occurs assume without loss of generality that a_2 is even and let $C_p = (x, x_{j_1}, x_{j_1+1}, \ldots, x_{j_3}, x)$. If (3) occurs assume without loss of generality that a_4 is even and let $C_p = (x, x_{j_2}, x_{j_2-1}, \ldots, x_{j_1}, x_{j_1-1}, \ldots, x_{j_5}, x_{j_5-1}, \ldots, x_{j_4}, x)$. It is easy to see that C_p as defined is such that p is odd with $t/5 \le p < t$. ❑

In each of the remaining lemmas and propositions that preceeds the proof of the main theorem, similar assumptions are needed. Thus the following conditions are assumed through the proof of Theorem 1. The graph G is of order n and minimal degree $\ge 2n/(k+2)$ where k is a fixed odd integer ≥ 3. Also G contains an odd length cycle $C_l, l > k$, but contains no C_{l-2}. Let $C_l = (x_1, x_2, \ldots, x_l, x_1)$, and for $1 \le i < j \le l$, let $A_{ij} = \{v \in V(G) - V(C_l) \mid v$ is adjacent to both x_i and $x_j\}$, $X_i = \{v \in A_{i-1,i+1} \mid v$ has precisely two adjacencies to $C_l\}$, and $Y_i = \{v \in V(G) - V(C_l) \mid v$ is adjacent to precisely x_i on $C_l\}$. For A and B disjoint subsets of $V(G)$, $[A,B]$ will denote the bipartite subgraph of G with parts A and B that contains all edges of G between A and B. Finally assume $l \le 5k$ for each of the lemmas and propositions in this section (but not in the proof of the theorem).

Lemma 2 *Let $h(n)$ be any unbounded nonnegative function such that $\lim h(n)/n \to 0$. For all $1 \le i, j \le l, i \ne j, |i-j| \ne 2$, and n sufficiently large $|A_{ij}| \le h(n)$ so that $|A_{ij}| = o(n)$.*

Proof By assumption G contains a C_l but no C_{l-2}, l is odd, and $d_{G-C_l}(x) \ge 2n/(k+2) - l$ for all $x \in V(G) - V(C_l)$. Partition the vertices of A_{ij} into sets B_{ij} and C_{ij} such that $d_{A_{ij}}(x) \ge n/(k+2) - l/2$ for $x \in B_{ij}$ and $C_{ij} = A_{ij} - B_{ij}$. Note that $d_{[A_{ij}, V(G)-(A_{ij} \cup V(C_l))]}(x) \ge n/(k+2) - l/2$ for $x \in C_{ij}$. Suppose $|A_{ij}| > h(n)$. It will be shown that this supposition leads to a contradiction. Two cases are considered.

Case 1: $|B_{ij}| > h(n)/2$. Let $m-1$ be the distance from x_i to x_j along the cycle C_l. It will be shown for n large that A_{ij} contains a path on $l-m-2$ vertices. Connecting the end vertices of this path in A_{ij} by disjoint edges to x_i and x_j gives the m vertex path on C_l a cycle C_{l-2}, a contradiction. Thus the proof for this case is completed by showing that A_{ij} contains a path on $5k-m-2 \geq l-m-2$ vertices. Note that $l-m-2>0$, since $l \geq k+2$ implies $l-m-2 \geq l-((l+1)/2)-2 \geq (l-5)/2 \geq (k-3)/2 > 0$. The last inequality $(k-3)/2 > 0$ follows since $k = 3$, $l = 5$, and $m = (l+1)/2 = 3$ means $|i-j| = 2$, contrary to the hypothesis of the lemma. But by definition of B_{ij} each of its vertices are adjacent to at least $n/(k+2) - l/2$ vertices of A_{ij} so that $|E(<A_{ij}>)| > h(n)(n/(k+2) - l/2)/4 \geq n(5k-m-4)/2 \geq |A_{ij}|(5k-m-4)/2$ for n sufficiently large. Hence by Erdös–Gallai (Theorem A), A_{ij} contains a path P_{5k-m-2} on $5k-m-2$ vertices.

Case 2: $|C_{ij}| > h(n)/2$. Observe that the number of vertices on the two paths from x_i to x_j on C_l have opposite parities. Thus choose the one with an even number, say m, vertices. Since $|i-j| \neq 2$, $m \leq l-3$ which implies $l-m-2>0$. This time an even length path on $l-m-2$ vertices is found in $[C_{ij}, V(G) - (A_{ij} \cup V(C_l))]$ with its end vertices in C_{ij}. This path on $l-m-2$ vertices has its end vertices joined by disjoint edges to x_i and x_j so that a C_{l-2} results (using the m vertex path from x_i to x_j on C_l), a contradiction.

The proof is thus completed by showing $[C_{ij}, V(G) - (A_{ij} \cup V(C_l))]$ contains all odd length paths of length at most $5k-m-2$ with end vertices in $[C_{ij}, V(G) - (A_{ij} \cup V(C_l))]$, i.e. contains a path of length $5k-m-1$. But $|E([C_{ij}, V(G) - (A_{ij} \cup V(C_l))])| > h(n)(n/(k+2) - l/2)/2 \geq n(5k-m-3)/2$ for n sufficiently large. Hence by Erdös-Gallai $[C_{ij}, V(G) - (A_{ij} \cup V(C_l))]$ contains the desired length path, so G contains a C_{l-2}. ❑

Lemma 3 *For all $1 \leq i \leq l$ both $n/(k+2) - o(n) \leq |X_i| \leq n/(k+2) + o(n)$ and $|Y_i| \leq o(n)$.*

Proof By Lemma 2 $|A_{ij}| = o(n)$ for all $i \neq j$, $|i-j| \neq 2$ so that each vertex x_i on C_l is adjacent to at least $2n/(k+2) - o(n)$ vertices of $X_{i-1} \cup X_{i+1} \cup Y_i$, i.e. for each $1 \leq i \leq l$

(1) $|X_{i-1}| + |X_{i+1}| + |Y_i| \geq 2n/(k+2) - o(n)$. This gives

$$2\left|\bigcup_{i=1}^{l} X_i\right| + \left|\bigcup_{i=1}^{l} Y_i\right| \geq l(2n/(k+2)) - o(n) \geq 2n - o(n), \text{ since } l \geq k+2.$$

Then $\left|\bigcup_{i=1}^{l} X_i\right| + \frac{1}{2}\left|\bigcup_{i=1}^{l} Y_i\right| \geq n - o(n)$, while $\left|\bigcup_{i=1}^{l} X_i\right| + \left|\bigcup_{i=1}^{l} Y_i\right| \leq n$.

Hence

(2) $|Y_i| \leq \left|\bigcup_{i=1}^{l} Y_i\right| = o(n)$ and $\left|\bigcup_{i=1}^{l} X_i\right| = n - o(n)$.

Suppose, for some fixed $\varepsilon > 0$ and n large, that there exists an i such that $|X_i| \geq n/(k+2) + \varepsilon n$. Then $|X_{i+2}| \leq n/(k+2) - \varepsilon' n (0 < \varepsilon' < \varepsilon)$, otherwise $|X_i| + |X_{i+2}| \geq 2n/(k+2) + (\varepsilon - \varepsilon')n$, contrary to (1) and (2). Likewise $|X_{i+2}| \leq n/(k+2) - \varepsilon' n$ implies $|X_{i+4}| \geq n/(k+2) + \varepsilon'' n$ for some $0 < \varepsilon'' < \varepsilon'$. Hence if for n large $|X_i| \geq n/(k+2) + \varepsilon n$, then there exists a δ, $0 < \delta < \varepsilon$, such that $|X_{i+2j}| \geq n/(k+2) + \delta n$ for $j = 0,2,4,\ldots,2l-2$ and $|X_{i+2j}| \leq n/(k+2) - \delta n$ for $j = 1,3,5,\ldots,2l-1$, where all indices $i + 2j$ are taken modulo l. Since these inequalities are incompatible, it follows that $|X_i| \leq n/(k+2) + o(n)$ for all $1 \leq i \leq l$. Applying (1) gives $|X_{i+2}| \geq n/(k+2) - o(n)$ for all $1 \leq i \leq n$. ❑

Lemma 4 *For n sufficiently large ($n \geq f_2(k)$) each of the bipartite graphs $[X_i, X_{i+1}]$, $1 \leq i \leq l$, contain a path P_l on l vertices.*

Proof First observe, using Lemma 3, that $d_{X_i}(x) \leq n/(k+2) + o(n)$ for each $x \in X_i$. Therefore since $l \leq 5k$ and

$$\left| \bigcup_{j=1}^{l} X_i \right| = n - o(n), \text{ the degree } d_{\bigcup\limits_{j \neq i-1} X_j}(x) \geq n/(k+2) - o(n) \text{ for all } x \in X_i.$$

Partition X_i into two parts, Z_i and $X_i - Z_i$, where Z_i are those vertices of X_i adjacent to at least $n/(2(k+2))$ vertices of X_{i+1}.

Let G' denote the graph $[X_i, X_j]$ when $j \neq i, i-1, i+1$, and let it denote the graph induced by X_i when $j = i$. Suppose $|X_i - Z_i| \geq |X_i|/2 \geq n/(2(k+2)) - o(n)$. By the pigeonhole principle at least $|X_i|/(2(l-2)) \geq n/(2(l-2)(k+2)) - o(n)$ vertices in $X_i - Z_i$ are adjacent (for some $j \neq i-1, i+1$) to at least $n/(2(l-2)(k+2)) - o(n)$ vertices of X_j. Therefore G' contains at least $n^2/(8(l-2)^2(k+2)^2) - o(n)$ edges. By Erdös-Gallai G' contains, for n large, a path of any fixed length. This means when $i = j$ that vertex x_{i+1} (or x_{i-1}) and a path on $l - 3 \leq 5k - 3$ vertices in G' gives a C_{l-2}, a contradiction. Also if $i \neq j$, then let the even length path from x_i to x_j on C_l contain m vertices. Since $j \neq i-1, i+1, m < l$, the m vertex path from x_i to x_j on C_l can be joined to a path in G' on $l - 2 - m$ vertices with a pair of disjoint edges, one from x_i and another from x_j. But this again gives a C_{l-2}, a contradiction. Hence the supposition that $|X_i - Z_i| \geq |X_i|/2$ is false and $|Z_i| \geq |X_i|/2$.

Since $|Z_i| \geq |X_i|/2$ and each vertex of Z_i is adjacent to at least $n/(2(k+2))$ vertices of X_{i+1}, the graph $[X_i, X_{i+1}]$ contains at least $(n/(2(k+2)) - o(n))(n/(2(k+2)))$ edges. Thus Erdös-Gallai again applies and $[X_i, X_{i+1}]$ contains a P_l for n sufficiently large.

It is easy to check that in all of the usages above (and also those of Lemmas 2 and 3) $o(n)$ depends only on n and k. Thus n sufficiently large, used throughout this proof, means there is an $f_2(k)$ such that $n \geq f_2(k)$. ❑

Proposition 1 *For n sufficiently large ($n \geq f_2(k)$) the cycle C_l has no diagonals.*

Proof If C_l has a diagonal, then G contain an odd length cycle $C_t, t < l$, such that all but one of the edges of C_t are also edges of C_l. Choose any edge $x_i x_{i+1}$ common to C_t and C_l. By Lemma 4 $[X_i, X_{i+1}]$ contains an even length path on $l - t - 2$ vertices. Join the vertices x_i and x_{i+1} to the appropriate end vertices of this $l - t - 2$ vertex path. But then the C_t cycle can be expanded to a C_{l-2} by replacing edge $x_i x_{i+1}$ by the $l - t - 2$ vertex path, a contradiction. ❑

Proposition 2 *For n sufficiently large ($n \geq f_2(k)$) each vertex x of G not on the cycle C_l has at most two adjacencies to vertices of the cycle.*

Proof Suppose there exists an $x \in V(G) - V(C_l)$ which is adjacent to at least three vertices of C_l. Then it is clear that G contains an odd length cycle $C_t, t < l$, such that C_t has at least $t - 2 \geq 1$ edges in common with C_l. Let $x_i x_{i+1}$ be any common edge. In the same way as was done in the last proof, Lemma 4 implies the existence of an even length path on $l - t - 2$ vertices in $[X_i, X_{i+1}]$ which can replace edge $x_i x_{i+1}$. This gives a C_{l-2}, a contradiction. ❑

Proof of Theorem 1

Since G is nonbipartite let C_l be an odd length cycle in the graph. It will be shown that $l \geq k$. Suppose $l < k$ and consider the graph G_A induced by A, where $A = V(G) - V(C_l)$. This graph G_A has at least $(n - l)(\delta(G) - l)/2 \geq (n - l)(n/(k + 2) - l/2)$ edges. By Theorem B there exists a $f_1(k)$ such that for $n \geq f_1(k)$ G_A contains the complete bipartite graph $K_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$ Since G is 2-connected, by Menger's Theorem there exist two vertex disjoint paths P_x and P_y connecting C_l to the graph $K_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$. Let $x(y)$ be the vertex common to the path P_x and C_l (P_y and C_l). Note that x and y are joined by two paths on C_l, one with an even number of vertices and the other with an odd number of vertices. Thus using one of these two paths it is easy to see that G contains an odd length cycle C_t using all vertices of $P_x \cup P_y$ and all but at most one of the vertices of $K_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$. Then $t \geq k + 1$ so G contains a C_l for some odd integer $l, l \geq k$.

For the remainder of this proof let l be the smallest odd positive integer $l \geq k$, such that G contains a C_l. Since it is to be shown that $l = k$ or $G \cong H$, assume throughout that $l \geq k + 2$.

Suppose $l > 5k$. Since l is the length of the smallest odd cycle $\geq k$, each vertex x on C_l is adjacent to at most $2(k-2)$ vertices of the cycle, the $2(k-2)$ vertices closest to x. Thus each vertex of the cycle is adjacent to at least $2n/(k+2) - (2k-4)$ vertices of $G - C_l$. By Lemma 1 no vertex of $G - C_l$ has as many as five adjacencies to C_l. Hence there are at least $l/(2n/(k+2) - 2k - 4)$ edges from C_l to $G - C_l$ and there are at most $4(n - l)$ from $G - C_l$. It follows that $4n \geq l(2n/(k+2) - 2k + 8) \geq 5k(2n/(k+2) - 2k + 8)$, which leads to a contradiction for n large ($n \geq f_3(k)$). Thus we may assume $l \leq 5k$, giving $k + 2 \leq l \leq 5k$.

The reader can check that G now satisfies all of the conditions assumed uniformly throughout the proofs of Lemmas 2, 3, 4 and Propositions 1 and 2. First apply Proposition 1 to C_l. For n sufficiently large each vertex of C_l has at least $2n/(k+2) - 2$ adjacencies to vertices of $G - C_l$. Also by Proposition 2 for n sufficiently large each vertex of $G - C_l$ has at most two adjacencies to C_l. Therefore $l(2n/(k+2) - 2) \leq 2(n - l)$. Since $l \geq k + 2$, a contradiction occurs unless $l = k + 2$ and $k + 2$ divides n. Assume $l = k + 2$ so that each vertex of C_l has precisely $2n/(k+2) - 2$ adjacencies to vertices of $G - C_l$, and each vertex of $G - C_l$ has precisely two adjacencies to C_l. It is shown under these conditions that $G \cong H$.

Consider $G - C_l$. Note that $x \in V(G) - V(C_l)$ implies

$$x \in \bigcup_{j=1}^{l} X_j.$$

If this were not the case, then x has adjacencies x_m and x_j on C_l, where $|m - j| \neq 2$. This gives a C_t, t odd ($t < l$), using edges xx_m, xx_j and the appropriate $t - 2$ edges of the C_l. But as was done in the proofs of Propositions 1 and 2, some edge x_ix_{i+1} common to C_t and C_l can be replaced by a path with an appropriate number of vertices from $[X_i, X_{i+1}]$ together with two edges joining its ends to x_i and x_{i+1}, respectively, to give a C_{l-2}, a contradiction. Thus

$$V(G) - V(C_l) = \bigcup_{j=1}^{l} X_j.$$

Next for each i consider $d_{G-C_l}(x_i) = 2n/(k+2) - 2 = |X_{i-1}| + |X_{i+3}|$. Thus $|X_{i-1}| = |X_{i+3}|$ for each $i, 1 \leq i \leq l$ (addition modulo l). But l is odd, so that this implies $|X_i| = |X_{i+1}|$ for each i and thus $|X_i| = n/(k+2) - 1$.

Finally observe that no vertex of X_i is adjacent to a vertex of X_j for $j \neq i-1, i+1$. If this were not the case, an odd cycle $C_t, t < l$, is again obtained which can be lengthened to a C_{l-2} by Lemma 4, a contradiction. Thus all adjacencies of X_i are to vertices of $X_{i-1} \cup X_{i+1} \cup \{x_{i-1}, x_{i+1}\}$. Since $|X_{i-1} \cup X_{i+1} \cup \{x_{i-1}, x_{i+1}\}| =$

$2n/(k+2)$ and $d(x) \geq 2n/(k+2)$ for $x \in X_i$, it follows that each $x \in X_i$ is adjacent to precisely $X_{i-1} \cup X_{i+1} \cup \{x_{i-1}, x_{i+1}\}$ for all $1 \leq i \leq l$. This gives $G \cong H$.

Throughout the proof there are places where n must be larger than each of $f_1(k), f_2(k)$, and $f_3(k)$. Thus setting $f(k) = \max_{i=1,3}\{f_i(k)\}$ the theorem holds for all $n \geq f(k)$.

Proof of Theorem 2

Since n is odd and G is regular, G is nonbipartite. The reader will recall that in the proof of Theorem 1 the 2-connectedness is only used to show that G contains an odd cycle C_l for some $l \geq k$. Thus this proof becomes a corollary to the proof of Theorem 1 once it is established that G contains a C_l for some odd integer $l, l \geq k$.

Without loss of generality assume G is connected, otherwise one can simply restrict attention to a component of G. Further if G is not 2-connected, consider an end block B of G.

Since B is an end block, it has at most one cut vertex x. Consider an internal vertex y of B (a non-cut-vertex). Since $d(y) \geq 2n/(k+2)$, $d_B(y) \geq 2n/(k+2)$ so that the block B has at least $2n/(k+2)+1$ vertices.

Suppose B fails to contain an odd cycle. Let B be bipartite with partite sets R and S and with cut vertex x in S. Let $m = d_{G-B}(x)$ and note $1 \leq m \leq 2n/(k+2) - 2$. Set $r = |R|$ and $s = |S|$. Counting edges from R to S and then from S to R, it follows that $|E(B)| = r(2n/(k+2)) = s(2n/(k+2)) - m$. Hence $m = (s-r)(2n/(k+2))$ so that $2n/(k+2)$ divides m, a contradiction. Therefore B contains some odd length cycle C.

Assume $|C| < k$. Now $d_B(y) \geq 2n/(k+2)$ for all y in $B - \{x\}$ so that there exists $\delta > 0$ such that for n large enough B has at least $\delta |B|^2$ edges. Hence for n large it follows from Theorem B that $B - C$ contains a $K_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$. But then since B is 2-connected, by Menger's theorem there exist two vertex disjoint paths P_x and P_y connecting C to the $K_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$. In the same fashion as argued in the proof of Theorem 1, the graph $C \cup P_x \cup P_y \cup K_{\lceil \frac{k}{2} \rceil, \lceil \frac{k}{2} \rceil}$ contains an odd cycle C_l with $l \geq k+1$. Hence G contains the desired odd cycle, completing the proof. ❑

Proof of Theorem 3

The graph L given in the first section (prior to the statement of the theorem) shows that G can be different from H and does not need to contain a C_k, k odd, when $k \geq 11$. Thus in what follows k always has one of the values 3, 5, 7, or 9.

As in the proof of Theorem 2 it is assumed that G is connected. If any block B of G contains a non-cut-vertex, then $|B| \geq 2n/(k+2)+1$ and the same type of argument as given in the proof of Theorem 2 applies. This argument implies, when B contains an odd length cycle and n is large, that it contains an odd length cycle C_l for some $l \geq k$. If B (and hence G) contains any such odd length cycle, then the proof of Theorem 1 shows for n large that G contains the desired cycle C_k or is isomorphic to H. Hence assume for the remainder of this proof that each block B of G which contains an odd cycle has no non-cut-vertices. But G is nonbipartite so some block B contains an odd cycle with each of its vertices cut-vertices. This means B is in fact an odd cycle. It will be shown that this assumption leads to a contradiction.

Thus assume G has block $B = C_l = (x_1, x_2, \ldots, x_l, x_1)$, l odd, $l < k$, in which each $x_i (1 \leq i \leq l)$ is a cut-vertex common to both B and some connected subgraph G_i of G. Note that G_i is only assumed connected and is not in general itself a block. Since $k \in \{3, 5, 7, 9\}$ there are several possible values for l all of which are handled similarly. Hence in what follows only the case when $k = 9$ and $l = 3$ will be considered. The reader can check that the remaining possible values for k and l are handled in the same way as this special case.

As described above $B = C_3 = (x_1, x_2, x_3, x_1)$ and each $G_i (1 \leq i \leq 3)$ is a connected graph with x_i a cut-vertex and the only vertex in common to both C_3 and G_i. There are two ways in which a lower bound on $|G_i|$ is determined. First assume G_i is bipartite with partite sets R_i and S_i and with $x_i \in S_i$. Since $d(y) \geq 2n/11$ for all y in $G, d_{G_i}(y) \geq 2n/11$ for y in $G_i - \{x_i\}$ and $d_{G_i}(x_i) \geq 2n/11 - 2$. Thus $|S_i| \geq 2n/11$, $|R_i| \geq 2n/11 - 2$ so that $|G_i| \geq 4n/11 - 2$ for all i where G_i is bipartite. Secondly if G_i contains an odd cycle, let $C = (y_1, y_2, \ldots, y_l, y_1)$ be one of longest odd length. Note that $d_{G_i}(y_i) \geq 2n/11$ for $y_i \neq x_i$ and $d_{G_i}(x_i) \geq 2n/11 - 2$. It is easy to see (since G_i contains no $l+2$ cycle) that $|G_i| \geq 2n/11 + (2n/11 - 2 - l)$ for each G_i that contains an odd cycle. Hence in all cases $|G| \geq |G_1| + |G_2| + |G_3| > n$ for n large, a contradiction.

It has been shown that no block can be an odd cycle. Since G is nonbipartite, the block that contains an odd cycle has a non-cut-vertex and thus an odd cycle C_l for some $l \geq k$ or G is isomorphic to H. This completes the proof of the theorem. ❑

The proof of the final result (Theorem 4) will not be given in great detail. The reasons are that a detailed proof would be very lengthy and that the result itself can most likely be improved. Thus simply a sketch of the ideas of the proof will be given.

Proof of Theorem 4 (A Sketch)

Since the graph G is nonbipartite let C_l be a shortest odd cycle in G and consider $G - C_l$. It is possible to show that $G - C_l$ contains a subgraph H_1 of order at least $2(1 - \varepsilon)cn$ or a pair of vertex disjoint subgraphs H_1 and H_2, each of order less than $2(1 - \varepsilon)cn$, such that the following conditions hold. Each H_i is δn-connected ($\delta = \delta(c,\varepsilon)$) and each H_i has minimum degree $\geq (1 - \varepsilon)cn$.

If $G - C_l$ contains the pair of vertex disjoint subgraphs H_1 and H_2, then connect H_1 and H_2 by a pair of vertex disjoint paths with end vertices x_1, x_2 and y_1, y_2, with $x_1, x_2 \in V(H_1), y_1, y_2 \in V(H_2)$. These vertex disjoint paths exist, since G is 2-connected, and each can be assumed to each have length independent of n, since the minimum degree in G is $\geq cn$. But $d_{H_i}(x) \geq (1 - \varepsilon)cn$ and $|H_i| < 2(1 - \varepsilon)cn$ so that both H_i's are panconnected. Hence both x_1,x_2 in H_1 and y_1,y_2 in H_2 are connected by paths of all possible lengths greater than are equal to the distance between them in H_i. This shows that G contains all cycles of length at least $h_1(c,\varepsilon)$ and at most $2(1 - \varepsilon)cn$.

Thus the only case that remains is when $G - C_l$ contains the subgraph H_1 of order at least $2(1 - \varepsilon)cn$ described above. It can be shown by proper application of Theorem B that H_1 contains a subgraph H or order at least $4(1 - \varepsilon)cn/3$ which is the disjoint union of copies of complete bipartite graphs. Each of these bipartite graphs is a $K_{t,t}$ for some $t \geq c' l nn$ where $c' = c'(\varepsilon,c)$. The idea is to order the $K'_{t,t}$s of H such that consecutive pairs can be joined by two vertex disjoint paths. Also C_l is joined by two vertex disjoint paths to the first $K_{t,t}$ in H under the given order. Further all the paths linking the C_l to a $K_{t,t}$ and consecutive pairs of $K'_{t,t}$s are to be such that they are vertex disjoint. This linking is possible by using the δn-connectivity of H and the fact that the linking paths are each of bounded length independent of n.

Finally the linking is done so that at least some $K_{\alpha\sqrt{lnn},\alpha\sqrt{lnn}}$ in each $K_{t,t}$, α fixed and small, has none of its vertices in a linking path (they are protected from the linking paths). Observe that there are at most $(4(1 - \varepsilon)cn)/(6c' l nn)$ complete bipartite graphs $K_{t,t}$ in H, so that there are at most $2(4(1 - \varepsilon)cn)/(6c' l nn)$ vertices in the linking paths and there are at most $(4(1 - \varepsilon)cn/(6c' l nn))(2\sqrt{lnn})$ protected vertices in the $K'_{t,t}$s. But for n large $2(4(1 - \varepsilon)cn)/(6c' l nn) + (4(1 - \varepsilon)cn)(2\sqrt{lnn})/(6c' l nn) < \delta n$, the connectivity of H, so that the linking described is possible. It is now a matter of checking that the linkage described from the C_l through all the $K'_{t,t}$s in H give odd cycles of all lengths from $h_1(c,\varepsilon)$ to $4(1 - \varepsilon)n/3$. ❑

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