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NOTE

A SIMPLE LOWER BOUND ON EDGE COVERINGS BY CLIQUES

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Assume that $G = G(V, E)$ is an undirected graph with vertex set V and edge set E. A clique of G is a complete subgraph. An edge clique-covering is a family of cliques of G which cover all edges of G. The edge clique-cover number, $\theta_e(G)$, is the minimum number of cliques in an edge clique-cover of G. For results and applications of the edge clique-cover number see [1–4].

Observe that $\theta_e(G)$ does not change if isolated vertices are removed from G. We give another obvious operation on G which does not effect $\theta_e(G)$. Call vertices x, y equivalent if $xy \in E$ and for all vertices z different from x and y, $zx \in E$ if and only if $zy \in E$. If x and y are equivalent vertices of G and xy is not an isolated edge then $\theta_e(G) = \theta_e(G')$, where G' denotes the graph we get from G by identifying x and y. Due to these observations it is enough to determine or estimate $\theta_e(G)$ for graphs without isolated or equivalent vertices.

Theorem. *If a graph G has n vertices and G contains neither isolated vertices nor equivalent vertices then* $\theta_e(G) \geq \log_2(n+1)$.

Proof. Assume that $A_1, A_2, \ldots, A_m$ is an edge clique-cover of G. Let $I(x)$ denote the index-set of the A's covering the edges incident to x. Since G has no isolated vertices, $I(x)$ is non-empty for all $x \in V$. We claim that $I(x) \neq I(y)$ if $x, y \in V$ and $x \neq y$. If $xy \notin E$ then the claim is true since $I(x)$ and $I(y)$ are disjoint sets. If $xy \in E$ then the non-equivalence of x and y implies that there exists a vertex $z \in V$ adjacent to exactly one of the vertices x, y. We may clearly assume that $zx \in E$ and $zy \notin E$. Let A_i be a clique covering xz, then $i \in I(x)$ but $i \notin I(y)$ and the claim is proved. We conclude that the sets $I(x)$ are distinct non-empty subsets of $\{1, 2, \ldots, m\}$. Thus $|V(G)| \leq 2^m - 1$ and the theorem is proved. □

For certain graphs G the lower bound in the theorem assimptotically gives $\theta_e(G)$. For instance, if G is the complement of a factor (cocktail party graph) or if G is the complement of a cycle, $\theta_e(G) = \log_2(n) + o(\log_2(n))$ as shown in [3] and [2]. It is also true that the theorem is sharp for infinitely many n. Assume that $n = 2^k$ and let K be a complete graph on n vertices. It is easy to define subsets $A_1, A_2, \ldots, A_k$ of $V(K)$ separating $V(K)$, i.e. for all vertex pairs x, y of $V(K)$, there exists A_i such that $|A_i \cap \{x, y\}| = 1$. (Let $V(K)$ be the set of 0–1 sequences of length k. For $i = 1, 2, \ldots, k$, let A_i contain the elements of $V(K)$ having 0 in the ith position). Define the graph G by adding new vertices $v_1, v_2, \ldots, v_k$ to $V(K)$ and making v_i to be adjacent to all vertices of A_i for $i = 1, 2, \ldots, k$. The graph G has neither isolated vertices nor equivalent vertices and

$$|V(G)| = 2^k + k, \qquad \theta_e(G) \leqslant k + 1 = \lceil \log_2(2^k + k + 1) \rceil .$$

References

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