ON A SPECIAL CASE OF THE WALL PROBLEM

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ABSTRACT.

Assume that $I_1, I_2, I_3, \ldots, I_n$ are closed intervals of the real line. The greedy coloring assigns positive integers sequentially to the intervals with the rule that the number assigned to I_j is the smallest k such that I_j does not intersect any interval already colored with k. A conjecture of Woodall (rediscovered by Chrobak and Slusarek) can be stated as follows: the greedy coloring does not use more than constant times the optimal number of colors. Here we prove a special case of this conjecture.

INTRODUCTION.

A family $\mathcal{J}$ of intervals is a finite collection of closed intervals of the real line. The word collection is used to indicate that $\mathcal{J}$ may contain the same interval with arbitrary(finite) multiplicity. A proper coloring of a family $\mathcal{J}$ of intervals is an assignment of positive integers (colors) to the intervals of $\mathcal{J}$ so that two intervals with the same color do not intersect. The greedy (or First-Fit) coloring is a proper coloring which colors the intervals of $\mathcal{J}$ in some order, assigning the smallest available color to the interval to be colored next. The density $d(\mathcal{J})$ of a family $\mathcal{J}$ is the maximum number of pairwise intersecting intervals in $\mathcal{J}$. It is a well-known result of Gallai that $\mathcal{J}$ has a proper coloring with $d(\mathcal{J})$ colors and this coloring is obviously optimal. The following problem is equivalent to a problem of Woodall ([4]), rediscovered by Chrobak and Slusarek ([1]),[2]). Does there exist an absolute constant c such that the greedy coloring uses at most $c\,d(\mathcal{J})$ colors on any family $\mathcal{J}$ of intervals under any ordering of $\mathcal{J}$. Lower bounds for c are $4/(\sqrt{17}-3)$ (Woodall, personal communication), 4 ([2],[5]), 13/3 ([1] without proof), 22/5 (Slusarek, personal commmunication). The obvious upper bound $d^2(\mathcal{J})$ is improved by Witsenhausen in [5] and the best upper bound is due to W.Just who proved that the greedy coloring uses at most $cd(\mathcal{J})\log d(\mathcal{J})$ colors. With his kind permission, his proof is in [3], where greedy and on-line algorithms are studied for several classes of graphs. In this paper a special case of the conjecture is proved. *

We introduce some notations at this point. Script letters are used for families of intervals. If $\mathcal{J}$ is properly colored then $c(I)$ denotes the color of $I \in \mathcal{J}$.

From now on a coloring always means proper coloring. A t-coloring is a coloring using colors $1,2,\dots,t$. Assume that a t-coloring of $\mathfrak{J}$ is given. Then
$\mathfrak{J}(k) = \{I \in \mathfrak{J} : c(I)=k\}$, $\mathfrak{J}(k,l) = \{I \in \mathfrak{J} : k \leq c(I) \leq l\}$.
For $I \in \mathfrak{J}$ set

$$I^{+} = \{J \in \mathfrak{J} : J \cap I \neq \phi, \; c(J) > c(I)\},$$
$$I^{-} = \{J \in \mathfrak{J} : J \cap I \neq \phi, \; c(J) < c(I)\}.$$

Notice that by definition, a t-coloring of $\mathfrak{J}$ is equivalent to

(1) $\mathfrak{J} = \bigcup_{k=1}^{t} \mathfrak{J}(k)$ where $\mathfrak{J}(k)$ contains pairwise disjoint intervals for $k=1,2,..,t$.

It is easy to see that a greedy t-coloring of $\mathfrak{J}$ is equivalent to (1) and (2), where property (2) is defined as follows.

(2) If $1 \leq m < n \leq t$ and $I \in \mathfrak{J}(n)$ then $I^{-} \cap \mathfrak{J}(m) \neq \phi$.

We may express (2) briefly by saying that in a greedy coloring each interval (except the intervals of $\mathfrak{J}(1)$) has lower support. The reason for this is appearent if a t-coloring of $\mathfrak{J}$ is visualized by placing $\mathfrak{J}(j)$ on the horizontal line $y=j$. Then a greedy t-coloring of may be called a wall of height t. These terms have been introduced by Woodall, Chrobak and Slusarek. Using this representation, $d(\mathfrak{J})$, the density of the wall, is the maximum number of intervals a vertical line can meet.

Let $\nu(\mathfrak{J})$ denote the maximum number of pairwise disjoint intervals in $\mathfrak{J}$. A greedy coloring of $\mathfrak{J}$ is of type i (i=1,2 or 3) if

(4) $$\nu(I^+) \leq i \text{ for all } I \in \mathfrak{J} \quad .$$

In this paper we prove the following result.

THEOREM. A type 1 greedy coloring of $\mathfrak{J}$ uses at most 24 d($\mathfrak{J}$) colors.

The motivation for the theorem is that all the known "high" walls are of type 1. Moreover, a generalization for type 3 walls would solve the general conjecture as the following proposition shows.

PROPOSITION. If $\mathfrak{J}$ has a greedy t-coloring then there exists $\mathfrak{J}'$ such that d($\mathfrak{J}'$) $\leq$ d($\mathfrak{J}$) and $\mathfrak{J}'$ has a greedy t-coloring of type 3.

PROOF. Assume that $\mathfrak{J}$ has a greedy t-coloring and $I \in \mathfrak{J}$. Select q pairwise disjoint intervals $B_1, B_2, .., B_q \in I^+$ with q= $\nu(I^+)$. If q $\geq$ 4 then I is "broken up" by replacing I with the following q-2 intervals: $I \cap \text{conv}(B_1, B_2)$, $I \cap B_3$ ...$I \cap B_{q-2}$ and $I \cap \text{conv}(B_{q-1}, B_q)$. (The notation conv stands for convex hull.) Repeatedly applying this refinement to the intervals of $\mathfrak{J}(t)$, $\mathfrak{J}(t-1)$,..., $\mathfrak{J}(1)$ in this order (and within a particular $\mathfrak{J}(k)$ in any order) a family $\mathfrak{J}'$ is obtained. It is easy to see that $\mathfrak{J}'$ has a type 3 greedy t-coloring. Adopting the notion of the wall, one can argue that breaking up a particular I, no interval of I^+ can fall through the pieces replacing I because of the definition of q. On the other hand, all pieces replacing I support an identical copy of the piece or a subinterval of the piece and their lower supports are lower supports for the pieces. Since the density does not increase by this refinement, the proposition is proved.

2. PROOF OF THE THEOREM.

Some further notation is introduced at this point. For a family $\mathfrak{J}$ of intervals $\tau(\mathfrak{J})$ denotes the cardinality of a minimum transversal of $\mathfrak{J}$, i.e. the minimum r such that there exists a set of points P with $|P|=r$ and with $P \cap I \neq \emptyset$ for all $I \in \mathfrak{J}$. Assume that $\mathfrak{J}$ is a t-colored family and $\mathcal{A} \subseteq \mathfrak{J}$. Let $c(\mathcal{A})$ denote the set of colors used on the intervals of $\mathcal{A}$, i.e. $c(\mathcal{A}) = \{c(I): I \in \mathcal{A}\}$. A section $\mathcal{S}$ of $\mathcal{A}$ is defined as a set of $|c(\mathcal{A})|$ intervals of $\mathcal{A}$, such that the intervals of $\mathcal{S}$ have distinct colors. Finally, we define

$$\tau^*(\mathcal{A}) = \min \tau(\mathcal{S}), \qquad \nu^*(\mathcal{A}) = \max \nu(\mathcal{S})$$

where the minimum and maximum is taken over all sections of $\mathcal{A}$. It is easy to see that $\tau^*(\mathcal{A}) \leq \nu^*(\mathcal{A})$ but equality does not hold in general.

THEOREM. A type 1 greedy coloring of $\mathfrak{J}$ uses at most $24\, d(\mathfrak{J})$ colors.

PROOF. The proof goes by induction on $d(\mathfrak{J})$. For $d(\mathfrak{J})=1$ a greedy coloring uses one color and the theorem is trivially true. Assume that the theorem is true for all $\mathfrak{J}$ such that $d(\mathfrak{J}) < s$ for some $s \geq 2$.

Let $\mathfrak{I} = \mathfrak{I}(1) \cup \mathfrak{I}(2) \cup \ldots \cup \mathfrak{I}(t)$ be a greedy t-coloring of $\mathfrak{I}$ and $d(\mathfrak{I})=s$.

Define the integers $t_1, t_2, .., t_k \in \{1,2,..,t\}$ and the intervals $A_1, A_2, .., A_k, M_1, M_2, .., M_k$ recursively as follows. Set $t_1=t$ and let $A_1=M_1 \in \mathfrak{I}(t)$. Assume that t_i, A_i and M_i are defined for some $i \geq 1$. Let t_{i+1} be the largest $m \in \{1,2,..,t_i-1\}$ such that

(5) $$\nu^*(\mathfrak{I}(m, t_i-1) \cap A_i^-) = 3.$$

If there is no m satisfying (5) then set k=i and the definition is finished. Set

$$p_1=1, \text{ and } p_j=t_{j-1}-t_j \quad \text{for } j=2,3,..,k.$$

Otherwise, t_{i+1} is defined and we proceed to define M_{i+1} and A_{i+1}. As a first step, set

$$\mathcal{A}_{i+1} = \mathfrak{I}(t_{i+1}, t_i-1) \cap A_i^- .$$

The definition of t_{i+1} implies that we can select three pairwise disjoint intervals, $B_1^{i+1}, B_2^{i+1}, B_3^{i+1}$ from $\mathcal{A}_{i+1}$ indexed from left to right. Set

$$M_{i+1} = B_2^{i+1} .$$

Since $\nu^*(\mathcal{A}_{i+1})=3$ implies $\tau^*(\mathcal{A}_{i+1}) \leq 3$, there exist a set of at most three points which meets at least one interval at each level of $\mathcal{A}_{i+1}$. Consequently, one of these points, say x, is contained in at least $\lceil p_{i+1}/3 \rceil = n$ intervals of $\mathcal{A}_{i+1}$. Let $\mathcal{B}$ denote this family of intervals. Let $a_1 \leq a_2 \leq \ldots \leq a_n$ and $b_1 \geq b_2 \geq \ldots \geq b_n$ denote the left and right endpoints of the intervals in $\mathcal{B}$. Mark the intervals of $\mathcal{B}$ one by one in the order $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ of the endpoints until only one or two are unmarked. Define A_{i+1} as an unmarked interval of $\mathcal{B}$. It is easy to check that the density of each point of A_{i+1} in $\mathcal{B}$ is at least $\lceil n/2 \rceil = \lceil \lceil p_{i+1}/3 \rceil /2 \rceil \geq \lceil p_{i+1}/6 \rceil$. This concludes the recursive definition.

Consider a fixed i, $2 \leq i \leq k$. Let $\mathfrak{F}_i$ denote the family of those intervals of $\mathfrak{J}(1, t_i-1)$ which lie inside (a,b), where a is the right endpoint of B_1^i and b is the left endpoint of B_3^i. For each $m \in \{1,2,..,t_i-1\}$, there exists $I_m \in \mathfrak{J}(m) \cap M_i^-$ (property (2) is applied to $\mathfrak{J}$). Since $\mathfrak{J}$ is of type 1, $I_m \subset (a,b)$ and $I_m \in \mathfrak{F}_i$ follows. Therefore $\mathfrak{F}_i$ has at least one interval at levels 1,2,..,t-1. Moreover, if $I \in \mathfrak{F}_i \cap \mathfrak{J}(m)$ for some m such that $1 < m \leq t_i - 1$ and we have an n satisfying $1 \leq n < m$, then I is supported by an interval $J \in \mathfrak{J}(n)$. Since $\mathfrak{J}$ is of type 1, $J \subset (a,b)$ and $J \in \mathfrak{F}_i$ follows. This argument shows that $\mathfrak{F}_i' = \mathfrak{F}_i \cup \{M\ \}$ has a greedy t_i-coloring. Moreover, since $(a,b) \subset A_{i-1}$ and the density of A_{i-1} at each point is at least $\lceil p_{i-1}/6 \rceil$, $d(\mathfrak{F}_i') \leq s - \lceil p_{i-1}/6 \rceil < s$ follows. Therefore the inductive hypothesis can be applied to $\mathfrak{F}_i'$:

$$t_i \leq 24\, d(\mathfrak{F}_i') \leq 24(s - \lceil p_{i-1}/6 \rceil) \leq 24s - 4p_{i-1}.$$

Since $t = t_i - 1 + \sum_{j=1}^{i} p_j$, it follows that

$$t < 24s - 4p_{i-1} + \sum_{j=1}^{i} p_j ,$$

which concludes the inductive proof if $24s - 4p_{i-1} + \sum_{j=1}^{i} p_j \leq 24s$. This condition is equivalent to $p_i \leq 4p_{i-1} - \sum_{j=1}^{i-1} p_j$. We may assume that the induction fails for all i such that $2 \leq i \leq k$, which means that

(6) $$p_i > 4p_{i-1} - \sum_{j=1}^{i-1} p_j \quad \text{for all } i,\ 2 \leq i \leq k.$$

It is easy to see (by induction, for example) that (6) implies

(7) $$\sum_{j=1}^{i} p_j \leq 2p_i \quad \text{for all } i,\ 1 \leq i \leq k.$$

The definition of t_k implies that $\nu^*(\mathcal{J}(1,t_k-1) \cap A_k^-) \leq 2$, therefore $\tau^*(\mathcal{J}(1,t_k-1) \cap A_k^-) \leq 2$, implying $t_k - 1 \leq 2s$. Similarly, $p_k \leq 3s$ follows from $\nu^*(\mathcal{J}(t_k, t_{k-1}-1) \cap A_{k-1}^-) = 3$. Using these inequalities and (7) for i=k, we get

$$t = t_k - 1 + \sum_{j=1}^{k} p_j \leq 2s + 2p_k \leq 2s+6s=8s$$

and the theorem is proved.

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* After the completion of this paper the authors have learned that H.A.Kierstead proved Woodalls conjecture. His proof uses some ideas from preliminary versions of this paper.