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INDUCED MATCHINGS IN BIPARTITE GRAPHS

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Dedicated to the memory of our friend Tory Parsons

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1. Introduction

All graphs in this paper are understood to be finite, undirected, without loops or multiple edges. The graph $G' = (V', E')$ is called an *induced* subgraph of $G = (V, E)$ if $V' \subseteq V$ and $uv \in E'$ if and only if $\{u, v\} \subseteq V'$, $uv \in E$.

The following two problems about induced matchings have been formulated by Erdős and Nešetřil at a seminar in Prague at the end of 1985:

1. Determine $f(k, d)$, the maximum number of edges in a graph which has maximum degree d and contains no induced $(k+1)$-matching (an induced matching of $k+1$ edges). For $k = 1$ this was asked earlier by Bermond, Bond and Peyrat (see [1]).

2. Let $q^*(G)$ denote the minimum integer t for which the edge set of G can be partitioned into t induced matchings of G. (We will call $q^*(G)$ the strong chromatic index of G.) As is done in Vizing's theorem, find the best upper bound of $q^*(G)$ when G has maximum degree d.

It was shown in [1] that (for d even) $f(1, d) = \frac{5}{4}d^2$ and the extremal graph is unique (each vertex of a five cycle is multiplied by $d/2$). This result suggests that $f(k, d) = \frac{5}{4}d^2k$. Perhaps a stronger conjecture is also true, namely, that $q^*(G) \leq \frac{5}{4}d^2$ when G has maximum degree d.

In this paper the analogous extremal problem for bipartite graphs is considered. It is shown that bipartite graphs of maximum degree d without an induced $(k+1)$-matching have at most kd^2 edges (Theorem 1). Extremal graphs for $k > 1$ are not unique but can be completely described (Theorem 2). It is also shown (Theorem 3) that when the extremal problem is restricted to connected bipartite graphs, the extremal number drops by at least d (if $k > 2$). We

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conjecture that the connectivity restricts the extremal number to decrease to $kd^2 - ckd$ for some constant $c > 0$, if k and d are large.

It is probably true that $q^*(G) \leq d^2$ for all bipartite graphs of maximum degree d (a conjecture which is obviously stronger than our extremal result). It is clear that there is no loss of generality in considering only regular graphs in this conjecture. However, we are not able to prove the first non-trivial case: The strong chromatic index of any 3-regular bipartite graph is at most 9.

2. Results

Throughout this section $G = (A, B)$ will denote a bipartite graph with vertex classes A and B. The edge set of G will be denoted by $E(G)$. We use the notation $\Gamma(x)$ ($\Gamma(X)$) for the set of vertices adjacent to x (some element of X).

A bipartite graph G of maximum degree d with no isolated vertices and no induced $(k+1)$-matching is called (k, d)-extremal if it has the maximum number of edges with respect to these conditions.

Assume that $G = (A, B)$ is (k, d)-extremal. Choose the smallest p such that $X = \{x_1, x_2, \ldots, x_p\} \subseteq A$ and $\Gamma(X) = B$. The choice of p shows that $\Gamma(X - \{x_i\}) \neq B$, $1 \leq i \leq p$, and so it follows that G has an induced p-matching. Since G has maximum degree d and $p \leq k$, we have

$$|E(g)| \leq |B|\, d = |\Gamma(X)|\, d \leq p \cdot \max_{1 \leq i \leq p} |\Gamma(x_i)| \cdot d \leq kd^2. \tag{1}$$

Observe that $kK_{d,d}$ has no induced $(k+1)$-matching, has maximum degree d and contains no isolated vertices, so (1) gives the following result:

Theorem 1. *A (k, d)-extremal graph has kd^2 edges.*

The next goal is to describe the structure of the (k, d)-extremal graphs. If G is (k, d)-extremal then all inequalities in (1) are in fact equalities. This implies that for all i, j ($i \neq j$, $1 \leq i \leq p$),

$$|\Gamma(x_i)| = \emptyset,\ \Gamma(x_i) \cap \Gamma(x_j) = \emptyset, \quad p = k. \tag{2}$$

Moreover, all vertices of B must be of degree d. Since the role of A and B can be interchanged in this argument, we have

$$a\ (k, d)\text{-extremal graph is } d\text{-regular.} \tag{3}$$

It is appropriate to introduce some additional notation at this point. For $i = 1, 2, \ldots, k$ set

$$A_i = \{x \mid x \in A,\ \Gamma(x) = \Gamma(x_i)\},$$

$$H_i = \{x \mid x \in A - A_i,\ \Gamma(x) \cap \Gamma(x_i) \neq \emptyset\}.$$

Notice that $A_i \neq \emptyset$, since $x_i \in A_i$, but the sets H_i can be empty. Since G has maximum degree d, $A_i \cap A_j = \emptyset$ for $i \neq j$ and $H_i \cap A_j = \emptyset$.

The set H_i is said to be *matchable* if the bipartite graph induced by $H_i \cup \Gamma(x_i)$ has an induced 2-matching.

A C_8-like graph is one obtained from the cycle C_8 by expanding each of its eight vertices to independent sets of vertices, making two vertices in different sets adjacent if and only if the corresponding vertices are adjacent in C_8.

The following lemma is needed.

Lemma. *Let $G = (A, B)$ be a bipartite graph without induced $(k+1)$-matching, with maximum degree d and without isolated vertices. Assume moreover that* (2) *is true and H_l is matchable for some l, $1 \leq l \leq k$. If G' is the component of G containing $\Gamma(x_l)$, then it is C_8-like with $V(G') \cap B = \Gamma(x_i) \cup \Gamma(x_t)$ for some $t \neq l$, $1 \leq t \leq k$.*

Proof. Since H_l is matchable, there exist $a, b \in H_l$ and $y, z \in \Gamma(x_l)$ such that $\{ay, bz\}$ is an induced 2-matching in G. If for all $t \in \{1, 2, \ldots, k\} - \{l\}$ there exist $y_t \in \Gamma(x_t)$ such that $y_t \notin \Gamma(a) \cup \Gamma(b)$, then ay, bz and the edges $x_t y_t$ gives an induced $(k+1)$-matching in G, a contradiction. Therefore we can choose $t \neq l$ such that

$$\Gamma(a) \cup \Gamma(b) \supseteq \Gamma(x_t). \tag{4}$$

Set $Y_t = \Gamma(a) \cap \Gamma(x_t)$ and $Z_t = \Gamma(b) \cap \Gamma(x_t)$. Clearly $Y_t \neq \emptyset$, $Z_t \neq \emptyset$ and $a, b \in H_t$. Therefore from (4) and from the definition of H_t, $\{a, b\}$ is matchable to elements of $\Gamma(x_t)$. Applying the same argument with t playing the role of l, there exists an $m \in \{1, 2, \ldots, k\} - \{t\}$ such that

$$\Gamma(a) \cup \Gamma(b) \supseteq \Gamma(x_m). \tag{5}$$

Set $Y_m = \Gamma(a) \cap \Gamma(x_m)$ and $Z_m = \Gamma(b) \cap \Gamma(x_m)$.

Since the maximum degree of G is d, and $\{a, b\}$ is matchable to $\Gamma(x_l)$, (4) and (5) imply that $l = m$, and the sets Y_t, Z_t, Y_l, Z_l are pairwise disjoint.

Next it is shown that for any vertex, $x \in H_l \cup H_t$, either $\Gamma(x) = \Gamma(a)$ or $\Gamma(x) = \Gamma(b)$. One may suppose $x \neq a$, $x \neq b$, and $x \in H_l$, $\Gamma(x) \cap Y_l \neq \emptyset$.

If $xz \notin E(G)$ for some $z \in Z_l$ then $\{x, b\}$ is matchable to $\Gamma(x_l)$ and the previous argument implies that $\Gamma(x) \cup \Gamma(b) = \Gamma(x_l) \cup \Gamma(x_t)$, therefore $\Gamma(x) = \Gamma(a)$ as desired. (By symmetry, $\Gamma(x) \cap Z_l \neq \emptyset$ would lead to $\Gamma(x) = \Gamma(b)$.)

The above observations imply that the component of G containing $\Gamma(x_l)$ is C_8-like. The eight sets of independent vertices which replace the vertices of the C_8 are

$$A_l,\ A_t,\ Y_l,\ Z_l,\ Y_t,\ Z_t$$

and the following two sets:

$$\{x \mid x \in H_l \cup H_t,\ \Gamma(x) = \Gamma(a)\} \qquad \{x \mid x \in H_l \cup H_t,\ \Gamma(x) = \Gamma(b)\}. \quad \square$$

Let C_8^d denote a C_8-like graph which is d-regular. The next theorem describes (k, d)-extremal graphs.

Theorem 2. *A bipartite graph G is (k, d)-extremal if and only if $G = mC_8^d \cup nK_{d,d}$ with $2m + n = k$.*

Corollary. *Each (k, d)-extremal graph is disconnected for $k \geqslant 3$. For $k = 1$ the only extremal graph is $K_{d,d}$. For $k = 2$ the only disconnected extremal graph is $2K_{d,d}$ and each connected extremal graph is a C_8^d.*

Proof of Theorem 2. If $G = mC_8^d \cup nK_{d,d}$ with $2m + n = k$ then clearly G is (k, d)-extremal. Let C be the vertex-set of a component of a (k, d)-extremal graph G. There is an i $(1 \leqslant i \leqslant k)$ such that $\Gamma(x_i) \cap C \neq \emptyset$. If $H_i = \emptyset$ then C induces a complete bipartite graph in G and (3) implies that C is isomorphic to $K_{d,d}$.

If $H_i \neq \emptyset$ then we claim that H_i is matchable. If this is not the case, choose $x \in H_i$ such that $t = |\Gamma(x) \cap \Gamma(x_i)|$ is as large as possible. The definition of H_i implies that $t < d = |\Gamma(x_i)|$.

Choose a $y' \in \Gamma(x_i)$ such that $xy' \notin E(G)$. Since y' has degree d, there is an $x' \in \Gamma(y') \cap H_i$. The choice of x implies the existence of a $y \in \Gamma(x)$ such that xy, $x'y'$ is an induced 2-matching, this proves the claim. The lemma implies that C induces a C_8-like graph, and (3) implies that this subgraph is isomorphic to C_8^d. □

Theorem 3. *If G is a connected (k, d)-extremal graph with $k \geqslant 3$, then $|E(G)| \leqslant kd^2 - d$.*

Proof. If $|\bigcup_{i=1}^{p} \Gamma(x_i)| < kd$ then $|E(G)| \leqslant (kd - 1)d$ and the theorem is proved.

Therefore it is assumed that (2) is satisfied. Moreover the connectivity of G implies that the hypergraph H with edge set $\{H_1, H_2, \ldots, H_k\}$ is connected. If there exists a matchable H_i for some i $(1 \leqslant i \leqslant k)$ then, the lemma implies G has a component which is a C_8-like graph. Since $k \geqslant 3$, that component is not G, which contradicts the connectivity of G. Therefore, no H_i is matchable. Also, since G is connected, no H_i is empty.

Since the hypergraph H is connected there exist $i, j \in \{1, 2, \ldots, k\}$ such that $i \neq j$ and $H_i \cap H_j \neq \emptyset$. Neither H_i nor H_j are matchable, so that $A_i = \{\Gamma(x) \cap \Gamma(x_i) \mid x \in H_i\}$ and $A_j = \{\Gamma(x) \cap \Gamma(x_j) \mid x \in H_j\}$ are both nested non-empty sets. Select $a \in H_i$ and $b \in H_j$ such that $\Gamma(a) \cap \Gamma(x_i)$ and $\Gamma(b) \cap \Gamma(x_j)$ are minimal elements of A_i and A_j respectively. From the choice of a and b, any $c \in H_i \cap H_j$ is adjacent to all vertices of $T = (\Gamma(a) \cap \Gamma(x_i)) \cup (\Gamma(b) \cap \Gamma(x_j))$. Since the degree of c is at most d, $|T| \leqslant d$.

It is next shown that each vertex $y \in (\Gamma(x_i) \cup \Gamma(x_j)) - T$ has degree less than d in G. By symmetry, assume that $y \in \Gamma(x_i) - (\Gamma(x_i) \cap \Gamma(a))$. Let x be an element of $\Gamma(y)$ and $y_0 \in \Gamma(x_i) \cap \Gamma(a)$. If $x \in A_i$, then from the definition of A_i $xy_0 \in E(G)$, and if $x \in H_i$ then from the choice of a, $xy_0 \in E(G)$. Therefore clearly $|\Gamma(y_0)| \geqslant |\Gamma(y)|$ but the inequality is in fact strict, since $y_0 a \in E(G)$ and $ya \notin E(G)$, implying y has degree less than d in G. Since $|(\Gamma(x_i) \cup \Gamma(x_j)) - T| \geqslant d$, there are at least d vertices in $\Gamma(x_i) \cup \Gamma(x_j)$ of degree less than d. Therefore $|E(G) \leqslant |\bigcup_{m=1}^{k} \Gamma(x_m)| \cdot d - d = kd^2 - d$. □

Observe that Theorem 3 is sharp for some small values of k and d, for example when $d = 2$ and $k = 3$ or 4. However, it is probably true, for k and d sufficiently large, that a connected (k, d)-extremal graph has at most $kd^2 - ckd$ edges where c is a positive constant.

References

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