7. Covering Complete Graphs by Monochromatic Paths

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A theorem of R. Radó (see in [2]) says that if the edges of the countably infinite complete graph K_∞ are colored with r colors then the vertices of K_∞ can be covered by at most r (finite or one-way infinite) vertex-disjoint monochromatic paths. Edge-coloring theorems are usually extended from finite to infinite, however, in the case of Radó's theorem it seems natural to look for finite versions.

Conjecture 1. If the edges of a finite complete graph K are r-colored then the vertex-set of K can be covered by at most r vertex-disjoint monochromatic paths.

It is easy to see that Conjecture 1. is true for $r = 2$ (see in [1]) but for $r = 3$ it seems to be difficult. It is worth considering the following weaker versions.

Conjecture 2. If the edges of a finite complete graph K are r-colored then the vertex-set of K can be covered by at most r monochromatic paths.

Conjecture 3. There exists a function $f(r)$ with the following property: if the edges of a finite complete graph K are r-colored then the vertex-set of K can be covered by at most $f(r)$ vertex-disjoint monochromatic paths.

Conjectures 2 and 3 are both open even for $r = 3$. For general r we prove here the following result (which is weaker than Conjecture 2 or Conjecture 3).

Theorem. *There exists a function $f(r)$ with the following property: if the edges of a finite complete graph K are r-colored then the vertex-set of K can be covered by at most $f(r)$ monochromatic paths.*

The proof of this theorem is based on the following

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Lemma. *Let r be a natural number and assume that a bipartite graph $G = (A, B)$ satisfies the following two conditions:*

(1) $rd(x) \geq |B|$ for all $x \in A$ ($d(x)$ denotes the degree of x)

(2) $$r\binom{r+1}{2}|A| \leq |B|$$

Then there exist at most r vertex-disjoint paths in G whose vertices cover A.

Proof. We may assume that $r \geq 2$. The required covering is found by the greedy algorithm. Let P_1 be a longest path of G starting in A and terminating in B. If $P_1, P_2, \dots, P_{i-1}$ are already defined for some $i, 2 \leq i \leq r$, then P_i is defined as a longest path of G such that P_i and P_j are vertex-disjoint for all $j < i$, moreover P_i starts in A and terminates in B. The starting point of P_i is denoted by x_i and the number of vertices of P_i is denoted by $2k_i$. Note that $k_i \leq k_j$ if $i > j$, by definition.

Assume indirectly that $y \in A$ is not covered by any of the paths $P_1, P_2, \dots, P_r$. The vertex x_i is not adjacent to any vertex of P_j for $j > i$, thus by (1) it is adjacent to all vertices of a set C_i, where $C_i \subset B, C_i \cap V(P_j) = \emptyset$ for all $j, 1 \leq j \leq r$, moreover

(3) $$|C_i| \geq r^{-1}|B| - \sum_{j=1}^{i} k_j \geq r^{-1}|B| - ik_1.$$

The sets C_i are pairwise disjoint by the choice of the paths P_i therefore the summing of (3) for $i = 1, 2, \dots, r$ gives

(4) $$\left|\bigcup_{i=1}^{r} C_i\right| \geq |B| - \binom{r+1}{2}k_1.$$

The vertex y is not adjacent to any vertex of $\cup_{i=1}^{r} C_i$ by the definition of the paths P_i thus (1) implies

(5) $$r^{-1}|B| \leq d(y) \leq |B| - \left|\bigcup_{i=1}^{r} C_i\right|.$$

Comparing (4) and (5) we find that $r^{-1}\binom{r+1}{2}^{-1}|B| \leq k_1$ which contradicts (2) since $k_1 < |A|$ for $r \geq 2$. □

Proof of the theorem. Let X and Y be two disjoint subsets of the vertex-set of K such that $|X| = |Y| = \lfloor |K|/2 \rfloor = n$. Let G_i denote the bipartite subgraph of K induced by the edges of color i between X and Y. It is clear that we can define a partition of X into sets $X_1, X_2, \dots, X_r$ such that $d_{G_i}(x) \geq r^{-1}n$ ($d_{G_i}(x)$ denotes the degree of x in G_i) holds for all $x \in X_i$. We can partition all X_i further in such a way that each part contains at most $r^{-1}\binom{r+1}{2}^{-1} n$ elements. If X_i' denotes one such part of X_i then the lemma can be applied to the bipartite graph (X_i', Y) induced by the edges of color i. Therefore X_i' can be covered by the vertices of at most r paths of color i. Applying this argument for each part of X_i and for all $i, 1 \le i \le r$, we get a covering of X by at most t monochromatic paths where

$$t = r\left(\sum_{i=1}^{r} \left\lceil |X_i| r \left(\binom{r+1}{2}\right) n^{-1} \right\rceil\right).$$

Since $\sum_{i=1}^{r} |X_i| = n$, we get $t \le r^2(\binom{r+1}{2} + 1)$. Applying the same argument again with changing the role of X and Y, the theorem follows with

$$f(r) = 2r^2(\binom{r+1}{2} + 1) + 1).$$

□

References

[1] A.Gyárfás, Vertex Coverings by Monochromatic Paths and Cycles, *Journal of Graph Theory* **7** (1983),131-135.

[2] R.Rado, Monochromatic Paths in Graphs, *Annals of Discrete Math.* **3**, 191-194.

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