A Matrix Labelling Problem

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Abstract. The following matrix labelling problem is considered: label the rows and columns of an $m \times n$ matrix with positive integers such that the largest positive integer is as small as possible and the entries of the matrix are all distinct. A solution is obtained when $m = 3$ (the $3 \times n$ case) by considering the independence of number of a difference graph.

I. INTRODUCTION

Several articles have been written involving the strength $s(G)$ of a graph [1,2,3,4]. The strength $s(G)$ is defined as the smallest positive integer N such that each edge of G can be given a positive integer label with value at most N with the resulting weighted (or labelled) graph having all its weighted degrees different. Similarly one can define the dual strength by assigning positive integer labels to vertices of G requiring that the resulting graph has all edges with different weights. Here the weight of an edge is the sum of the labels of its incident vertices.

It is the dual strength of the complete bipartite graph $K_{m,n}$, formulated as a matrix labelling problem, which is addressed in this paper. Within the paper the problem is reformulated in a difference graph setting and in this setting a partial solution is obtained.

The Problem: Label the columns and rows of an $m \times n$ matrix with positive integers, assigning the sum of the labels given to the *ith* row and *jth* column to the *ijth* entry. Find the smallest positive integer $N = N(m,n)$ such that all labels have value at most N and all matrix entries are distinct.

Several observations should be made concerning the problem. First there is no loss of generality in assuming that both the row labels and columns labels include the smallest possible label, the positive integer 1, and both sets of labels are strictly increasing sequences on m and n positive integers respectively. Furthermore, since the mn entries are distinct with smallest entry 2, the largest of these entries must be at least $mn + 1$. Hence $N \geq (mn+1)/2$. Equality need not hold and a value for N will be conjectured later which is always larger than $(mn+1)/2$ for $n, m \geq 3$, but of that order of magnitude.

The problem can also be formulated in an alternate way. Assume the rows are given labels $r_1 < r_2 < \cdots < r_m$ and the columns labels $c_1 < c_2 < \cdots < c_n$. Form the sets of

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differences $\mathcal{R} = \{r_j - r_i | 1 \le i < j \le m\}$ and $\mathcal{C} = \{c_j - c_i | 1 \le i < j \le n\}$. It is easy to see that all the entries of the matrix are distinct under this assignment if and only if $\mathcal{R} \cap \mathcal{C} = \phi$.

It is this later formulation of the problem that suggests introduction of a difference graph. Let N be a fixed positive integer and let $\ell_1, \ell_2, \ldots, \ell_{m-1}$ be positive integers (not necessarily distinct). Define the difference graph $G_N(\ell_1, \ell_2, \ldots, \ell_{m-1})$ as one with vertex set $V = \{1, 2, \ldots, N\}$ and edge set $E = \{xy | y \ge x \text{ and } y - x = \sum_{i=t}^{k} \ell_i \text{ for some } t, k, 1 \le t \le k \le m-1\}$. Solving the problem then becomes equivalent to finding the smallest positive integer N for which there exist positive integers $\ell_1, \ell_2, \ldots, \ell_{m-1}$ $\left(\sum_{i=1}^{m-1} \ell_i \le N - 1\right)$ such that the independence number β of $G_N(\ell_1, \ell_2, \ldots, \ell_{m-1})$ is at least n. To see this is the case assume G_N has $\{v_1, v_2, \ldots, v_n\}$ as an independent set. Under this assumption label the rows $1, \ell_1 + 1, \sum_{i=1}^{2} \ell_i + 1, \ldots, \sum_{i=1}^{m-1} \ell_i + 1$ and the columns $v_1, v_2, \ldots, v_n$. Since each pair of row labels are adjacent as a pair of vertices in G_N and each pair of column labels are nonadjacent, the difference sets $\mathcal{R}$ an $\mathcal{C}$ defined earlier satisfy $\mathcal{R} \cap \mathcal{C} = \phi$.

Although the difference graph approach does not give a complete solution, it does lead to a solution when $m = 3$ and n is arbitrary.

Before pursuing this approach consider a labelling which may give the correct value of N. Assume for the moment that n is even. Label the rows with the labels $1, \frac{n}{2} + 1, n + 1, \ldots, (m-1)\frac{n}{2} + 1$ and the columns with labels $1, 2, \ldots, \frac{n}{2}, m(\frac{n}{2}) + 1, m(\frac{n}{2}) + 2, \ldots, (m+1)\frac{n}{2}$, so that the largest label used is $(m+1)(\frac{n}{2})$. With this labelling it is easy to check that the entries of the matrix are as small as possible, i.e., they include precisely the numbers $2, 3, \ldots, mn + 1$. Likewise if m is even row labels of $1, 2, \ldots, \frac{m}{2}, n(\frac{m}{2}) + 1, n(\frac{m}{2}) + 2, \ldots, (n+1)(\frac{m}{2})$ and column labels of $1, \frac{m}{2} + 1, m + 1, \ldots, (n-1)\frac{m}{2} + 1$ provide a labelling with largest label $(n+1)(\frac{m}{2})$. Adjusting this idea when both m and n are odd gives two natural labellings both of which gives largest label $\frac{(m+1)n+m-1}{2}$. They are (1) row labels $1, \frac{n+1}{2} + 1, n + 2, \ldots, (m-1)(\frac{n+1}{2}) + 1$ and column labels $1, 2, \ldots, \frac{n+1}{2}, m(\frac{n+1}{2}) + 1, m(\frac{n+1}{2}) + 2, \ldots, \frac{(m+1)n+m-1}{2}$, and (2) row labels $1, 2, \ldots, \frac{m+1}{2}, n(\frac{m+1}{2}) + 1, n(\frac{m+1}{2}) + 2, \ldots \frac{(n+1)m+n-1}{2}$ and columns labels $1, \frac{m+1}{2} + 1, m + 2, \ldots, (n-1)(\frac{m+1}{2}) + 1$. It should be noted that the two labellings just described, when n and m are odd, do not yield as matrix entries all mn positive integers from 2 to $mn + 1$. Since the largest entry is $mn + m$ (or $mn + n$), $m - 1$ (or $n - 1$) numbers are skipped between the smallest and largest values. Although that the given labelling may fail to be optimal when both m and n are odd, there are other indicators which suggest it is a best possible one. Thus these labellings suggest the following conjecture.

Conjecture:

$$N = N(m,n) = \begin{cases} (n+1)m/2 \text{ for } m \text{ even, } m \le n, \\ (m+1)n/2 \text{ for } n \text{ even, } m \text{ odd,} m \le n, \\ [(m+1)n + m - 1]/2 \text{ for } m \text{ and } n \text{ both odd.} \end{cases}$$

II. RESULTS.

At this point the principal results can be given and their relation to the given problem explored. The first of these results appears in the following theorem.

Theorem 1. Let $n \geq 3$ be a positive integer, $m = 3$, and $G_N(\ell_1, \ell_2)$ be as defined with $N \leq 2n$, and $\ell_1 + \ell_2 \leq N - 1$. Then the independence number $\beta(G_N(\ell_1, \ell_2)) \leq n$ with equality if and only if $N = 2n, n$ is even and $\ell_1 = \ell_2 = n/2$. When equality occurs $G_N(n/2, n/2) = n/2(K_4 - e), n/2$ disjoint copies of a K_4 minus one edge e.

This theorem has as an immediate corollary the conjectured value for $N(3, n)$. Theorem 1 gives that there is no labelling with distinct matrix entries unless $N \geq 2n (N \geq 2n + 1)$ for n even (odd). In addition the labelling described earlier show these inequalities are equalities. These facts are summarized in the following corollary.

Corollary 2.

$$N(3,n) = \begin{cases} 2n \text{ for } n \text{ even}, n \geq 3 \\ 2n+1 \text{ for } n \text{ odd}, n \geq 3. \end{cases}$$

When n is odd there are at least three different "good" labellings. Two of these were described earlier, prior to the conjecture. A third simply labels the rows $1, 2, \ldots, n$ and the columns $1, n+1, 2n+1$. This last labelling is natural, but can't be extended in the obvious way when $m > 3$. It should also be observed that each of these three labellings have different largest sum which appears in its $3n$ entries. They are $3n + 3$ and $4n$ for the two labellings given earlier and $3n + 1$ for the one just given. This suggests a refinement of the problem to not only find the smallest value of N which gives a "good" labelling, but find the one which makes the largest of the mn entries in the matrix as small as possible.

The problem is still unanswered in the general case (when $m > 3$). Although this question is not answered here, one can prove a result which gives credibility to the conjectured values of N. Consider the graph $G_N(\ell_1, \ell_2, \ldots, \ell_{m-1})$ when $\ell_1 = \ell_2 = \cdots = \ell_{m-1}$. For convenience denote this graph $G_N(\ell : m-1)$ where $\ell = \ell_1 = \cdots = \ell_{m-1}$. Further assume $N \leq (m+1)n/2$ ($[(m+1)n + (m-1)]/2$) for n even (odd) with $m, n \geq 4$ and $(m-1)\ell < N$. Under these conditions one can prove the following result.

Theorem 3 The clique covering number $\theta(G_N(\ell : m-1)) \leq n$ with equality if and only if

$$\ell = \begin{cases} n/2 \text{ for n even} \\ (n+1)/2 \text{ for n odd.} \end{cases}$$

When equality occurs

$$G_N(\ell : m-1) = \begin{cases} \ell(K_{m+1} - e) \text{ for n even} \\ (\ell - 1)(K_{m+1} - e) \cup K_m \text{ for n odd,} \end{cases}$$

and $\theta(G_N(\ell : m-1)) = \beta(G_N(\ell : m-1))$.

A consequence of this result is that if either the rows or the columns have labels which form an arithmetic progression, then any good labelling needs its largest label to be as large as the conjectured value for N. The details of this interpretation will not be given, since this would focus too heavily on special labellings. The consequence itself is stated formally in the following corollary.

Corollary 4. Restrict the labellings of the $m \times n$ matrix to ones where either the row labels or the column labels form an arithmetic progression. Then the value of $N = N(m,n)$ is as given in the conjecture.

In the remainder of the paper only the proof of Theorem 1 will be given. There are two primary reasons for not including the proof of Theorem 3 and its corollary. They are that these results deal with special labellings away from the focus of the paper, and detailed proofs would require some tedious calculations.

In order to prove Theorem 1 it is best to investigate some of the structure of $G_N = G_N(\ell_1, \ell_2)$. This will be done in a sequence of four lemmas. Since the N vertices of G_N are the first N positive integers, the vertex set of this graph will be assumed ordered (in its natural way). Also it will be assumed throughout the remainder of the paper that $\ell_1 \le \ell_2 \le \cdots \le \ell_{m-1}$ with $\sum_{i=1}^{m-1} \ell_i \le N-1$; in particular for $m = 3, \ell_1 \le \ell_2$ and $\ell_1 + \ell_2 \le N-1$.

Lemma 1. Let H be a subgraph of G_N spanned by any set of $\ell_1 + \ell_2$ consecutive vertices of G_N. Then $H = (\ell_1 \ell_2) C_{(\ell_1+\ell_2)/(\ell_1,\ell_2)}$, i.e. H consists of (ℓ_1, ℓ_2) disjoint cycles on $(\ell_1 + \ell_2)/(\ell_1, \ell_2)$ vertices where (ℓ_1, ℓ_2) is the GCD of ℓ_1 and ℓ_2.

PROOF: Without loss of generality we can assume that the vertices of H are $1, 2, \ldots, \ell_1+\ell_2$. Each vertex i has precisely two of the four adjacencies, $i+\ell_1, i+\ell_2, i-\ell_1, i-\ell_2$, so that H is 2-regular. Furthermore if $i+\ell_1 > \ell_1+\ell_2$, then $1 \le i-\ell_2 < i$ and $i-\ell_2 \equiv i+\ell_1 \pmod{\ell_1+\ell_2}$. Likewise if $i-\ell_1 < 1$, then $i < i+\ell_2 \le \ell_1+\ell_2$ and $i-\ell_1 \equiv i+\ell_2 \pmod{\ell_1+\ell_2}$. Hence one can describe H in an equivalent way by assuming its vertex set is $Z_{\ell_1+\ell_2}$ (the integers modulo $\ell_1+\ell_2$) with two vertices adjacent when their difference is ℓ_1 (or ℓ_2.) For such a graph H it is well known that it is the union of cycles as described. ■

The result of Lemma 1, that each interval of $\ell_1 + \ell_2$ vertices in G_N is a union of cycles, is not surprising. The reader should not mistakenly assume that this makes G_N easy to visualize. Remember for N large with respect of $\ell_1 + \ell_2$, most vertices are of degree six, and each pair of vertices with difference $\ell_1 + \ell_2$ are adjacent.

There are two additional subgraphs in G_N which are used in the proof of Theorem 1. For convenience these subgraphs will be given special names. The first of these, <u>a sunflower graph</u>, will consist of a cycle $(v_1, v_2, \ldots, v_r)$ together with a set of independent vertices $W = \{w_1, w_2, \ldots, w_t\}$ such that

(1) the adjacencies of each w_i in W consist of three consecutive vertices $v_{k_i}, v_{k_i+1}, v_{k_i+2}$ of the cycle, and

(2) if $v_{k_i}, v_{k_i+1} v_{k_i+2}$ and $v_{k_j}, v_{k_j+1}, v_{k_j+2}$ are the respective adjacencies of $w_i \neq w_j, w_i, w_j \epsilon W$, then $v_{k_i+1} \notin \{v_{k_j}, v_{k_j+1}, v_{k_j+2}\}$ (and thus $v_{k_{j+1}} \notin \{v_{k_i}, v_{k_{i+1}}, v_{k_i+2}\}$)

The second subgraph, a fan graph, consists of a cycle together with two vertices x and y off the cycle such that (1) x is adjacent to y, and (2) there exist three consecutive vertices v_i, v_{i+1}, v_{i+2} on the cycle such that the remaining adjacencies of x are v_i and v_{i+1} and the remaining adjacencies of y are v_{i+1} and v_{i+2}.

Lemma 2. Let $1 \leq t \leq \ell_1 \leq \ell_2$. Consider a set $L = \{v_1, v_2, \ldots, v_{\ell_1+\ell_2}, w_1, w_2, \ldots, w_t\}$ of ordered vertices in G_N where the first $\ell_1 + \ell_2$ are consecutive and where each w_k satisfies $1 \leq w_k - v_{\ell_1+\ell_2} \leq \ell_1$. Then the subgraph spanned by L is a disjoint union of cycles and a nonempty collection of sunflower graphs.

PROOF: The first $\ell_1+\ell_2$ vertices of L form a disjoint union of cycles by Lemma 1. Consider the set $\{w_1, w_2, \ldots, w_t\}$ of remaining vertices of L. Clearly no w_i is adjacent to w_j since $|w_i - w_j| < \ell_1$. Also the adjacencies of w_i in the set of given vertices are $w_i - \ell_1, w_i - \ell_1 - \ell_2, w_i - \ell_2$ and these form a two edge path on one of the cycles of the first $\ell_1 + \ell_2$ vertices. Thus condition (1) of the definition of a sunflower graphs holds. Suppose condition (2) of this definition fails to hold. Then there exists a $w_i \neq w_j$ such that $w_i - \ell_1 - \ell_2$ is equal to one of $w_j - \ell_1, w_j - \ell_1 - \ell_2$ or $w_j - \ell_2$. This gives $|w_i - w_j| = \ell_1$ *or* ℓ_2, contrary to $|w_k - v_{\ell_1+\ell_2}| \leq \ell_1$ for all $1 \leq k \leq t$. Hence the vertices of L span a disjoint union of sunflower subgraphs of G_N and cycles.

Lemma 3. Consider a set $L = \{v_1, v_2, \ldots, v_{\ell_1+\ell_2}, v_{\ell_1+\ell_2+1}, v_{\ell_1+\ell_2+2}\}$ of ordered vertices in G_N where the first $\ell_1 + \ell_2$ are consecutive, $v_{\ell_1+\ell_2+2} - v_{\ell_1+\ell_2+1} = \ell_1$, and $v_{\ell_1+\ell_2+2} - v_{\ell_1+\ell_2} \leq \ell_2$. Then the subgraph spanned by L is a disjoint union of cycles and a fan graph.

PROOF: The first $\ell_1 + \ell_2$ vertices of L span a disjoint union of cycles. Vertices $v_{\ell_1+\ell_2+2}$ and $v_{\ell_1+\ell_2+1}$ are adjacent to each other and commonly adjacent to $v_{\ell_1+\ell_2+2} - \ell_1 - \ell_2$. Also $v_{\ell_1+\ell_2+2}$ is adjacent to $v_{\ell_1+\ell_2+2} - \ell_2$ and $v_{\ell_1+\ell_2+1}$ is adjacent to $v_{\ell_1+\ell_2+1} - \ell_1 - \ell_2$. Since $v_{\ell_1+\ell_2+1} - \ell_1 - \ell_2, v_{\ell_1+\ell_2+2} - \ell_1 - \ell_2, v_{\ell_1+\ell_2+2} - \ell_2$ is a two edge path on one of the cycles, the subgraph spanned by L is as described.

Lemma 4. Let H be either a sunflower subgraph or a fan subgraph of G_N. Then the independence number $\beta(H) < |H|/2$.

PROOF: Note that both graphs are Hamiltonian so the result follows if $|H|$ is odd. Therefore assume $|H|$ is even. In case H is a sunflower subgraph let z be any of the t vertices off the cycle with v_1, v_2, v_3 its adjacencies on the cycle. In case H is a fan subgraph let z and v_3 be its adjacent pair off the cycle, v_2 their common adjacency on the cycle and v_1 the remaining adjacency of z on the cycle. For both cases consider the graph $H - \{v_2, z\}$. This graph is a Hamiltonian path from v_1 to v_3 on an even number of vertices. Thus if A is a largest independent set in H then $|A \cap (V(H - \{v_2, z\}))| \leq (|H| - 2)/2$, and A does not contain both v_1 and v_3. Further if it contains either v_1 or v_3, then A does not contain either z or v_2. Hence $|A| \leq (|H|-2)/2 < |H|/2$.

Proof of Theorem 1. The proof is divided into two cases.

Case I. $\ell_1 = \ell_2$

For this case partition the vertex set into ℓ_1 classes $A_1, A_2, \ldots, A_{\ell_1}$ as follows: for each $i, 1 \leq i \leq \ell_1$, let $A_i = \{x | x$ is a vertex (positive integer) such that ℓ_1 divides $x_i\}$ Thus if $N = q\ell_1 + r$ with $0 \leq r < \ell, A_i$ contains the q vertices $i, i+\ell_1, \ldots, i+(q-1)\ell_1$ and (when $i \leq r$) the additional vertex $i + q\ell_1$.

Assume the elements (vertices) in each A_i are in their natural order. Then from the definition of $G_N = G_N(\ell_1, \ell_2)$ any three consecutive elements in A_i form a K_3 (a triangle) and any four consecutive ones form a $K_4 - e$. Since $N \geq 2\ell_1 + 1$ the vertices of G_N are covered by a disjoint union of edges and traingles. Furthermore this covering can be assumed to include at least one triangle, unless $N = 4\ell_1$, in which case G_N is covered by ℓ_1 disjoint copies of $K_4 - e$. But $N \leq 2n$ so this covering implies $\beta(G_N) \leq n$ with equality occurring precisely when n is even, $N = 2n$, and $\ell_1 = n/2$.

Case II. $\ell_1 \neq \ell_2$.

First set $N = q(\ell_1+\ell_2)+r$ where $1 \leq r \leq \ell_1+\ell_2$ and partition the N vertices of $G_N = G_N(\ell_1, \ell_2)$ into $q+1$ classes $B_1, B_2, \ldots, B_{q+1}$ as follows: let $B_i = \{x | (i-1)(\ell_1+\ell_2)+1 \leq x \leq i(\ell_1+\ell_2)\}$ for $i = 1, 2 \ldots, q$ and let $B_{q+1} = \{q(\ell_1+\ell_2)+1, q(\ell_1+\ell_2)+2 \ldots, q(\ell_1+\ell_2)+r\}$. Since $\ell_1 + \ell_2 \leq N - 1$, it follows that $q \geq 1$ so that there are at least two classes in this partition.

Set $r = s\ell_1 + u$ where $0 \leq u < \ell_1$. Next partition the r elements of class B_{q+1} differently depending on whether $u \neq 0$ with s even, $u \neq 0$ with s odd, $u = 0$ with s odd ,or $u = 0$ with s even. Each of these possibilities is considered as one of four separate subcases. Similarities between the individual subcases will be used to shorten the argument. Each B_i is always assumed ordered in the natural way.

Subcase 1. $u \neq 0$ and s is even.

Split B_{q+1} into two sets B'_u and B''_u as follows: place the first u elements in B'_u and the remainder in B''_u. Since $|B''_u| = s\ell_1$ with s even, the elements of B''_u can be covered by $(s/2)\ell_1$ disjoint edges matching its vertices by adjacent pairs of the type $x, x+\ell_1$. Also by Lemma 1 the subgraph L spanned by $B_i (1 \leq i \leq q - 1)$ is a union of cycles. Hence the subgraph

$$< \bigcup_{i=1}^{q-1} B_i \cup B''_u >$$

spanned by these B_i and B''_u satisfies

$$\beta(< \bigcup_{i=1}^{q-1} B_i \cup B''_u >) \leq | \bigcup_{i=1}^{q-1} B_i \cup B''_u|/2.$$

In addition by Lemma 2 $H =< B_q \cup B'_u >$ consists of a union of cycles and a nonempty collection of sunflower graphs. Hence by Lemma 4 $\beta(H) < |H|/2$ implying

$$\beta(G_N) < (|\bigcup_{i=1}^{q-1} B_i \cup B''_u| + |H|)/2 \leq N/2 \leq n.$$

Subcase 2. $u \neq 0$ and s is odd.

In this case split B_{q+1} into four sets C_1, B'_u, C_2, and B''_u, placing the first u elements of B_{q+1} in C_1, the next $\ell_1 - u$ into B'_u, followed by the next u into C_2. Place the remaining $(s-1)\ell_1$ into B''_u. Note that each vertex x in C_1 can be matched with the vertex $x + \ell_1$ in C_2, so that $< C_1 \cup C_2 >$ can be covered by u disjoint edges. Also, as in Subcase 1, (since $s-1$ is even)

$$\beta(< \bigcup_{i=1}^{q-1} B_i \cup B''_u >) \leq |\bigcup_{i=1}^{q-1} B_i \cup B''_u|/2.$$

Thus letting $H =< B_q \cup B'_u >$ it follows similar to Subcase 1 that

$$\beta(G_N) \leq \beta(< \bigcup_{i=1}^{q-1} B_i \cup B''_u >) + \beta(< C_1 \cup C_2 >) + \beta(H) < N/2 \leq n.$$

Subcase 3. $u = 0$ and s is odd.

Again split B_{q+1} into two sets B'_u and B''_u, the first ℓ_1 elements placed in B'_u and the remaining ones in B''_u. Then $|B''_u| = (s-1)\ell_1$ so that the elements of B''_u can be covered by $[(s-1)/2]\ell_1$ disjoint edges as was done in Subcase 1. The rest of the argument for this subcase parallels that of Subcase 1. The graph $H =< B_q \cup B'_u >$ satisfies $\beta(H) < |H|/2$ by Lemma 2 and

$$\beta(< \bigcup_{i=1}^{q-1} B_i \cup B''_u >) \leq |\bigcup_{i=1}^{q-1} B_i \cup B''_u|/2$$

so that $\beta(G_N) < N/2 \leq n$.

Subcase 4. $u = 0$ and s is even.

In ths case split B_{q+1} into sets B'_u and B''_u letting $B'_u = \{v, v + \ell_1\}$, where v is the first element in B_{q+1}, and $B''_u = B_{q+1} - B'_u$. Clearly B''_u has an even number of elements which can be paired, pairing adjacent vertices of the type $x, x + \ell_1$. Hence $\beta(< B''_u >) \leq |B''_u|/2$. Also $H = < B_q \cup B'_u >$ satisfies the conditions of Lemma 3 so that it is the disjoint union of cycles and a fan subgraph. Hence by Lemma 4, $\beta(H) < |H|/2$. Thus

$$\beta(G_N) \leq \beta(< \bigcup_{i=1}^{q-1} B_i \cup B''_u >) + \beta(H) < N/2 \leq n$$

completing the proof.

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