THE IRREGULARITY STRENGTH OF $K_n - mK_2$

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Abstract. The irregularity strength of $K_n - mK_2$ is 3 unless $n = 4m$, $n = 4m + 1$ or $n = 4m - 1$.

An *irregular weighting* of a graph G is an assignment of positive integers to the edges of G so that the weighted degrees of G are all distinct. The *irregularity strength*, $s(G)$, of G is the minimum integer s such that G has an irregular weighting with maximum weight s. This concept has been introduced in [1], followed by several subsequent papers ([2], [3], [4], [5], [6]).

This note answers a question raised in [4], to determine $s(K_n - mK_2)$, where K_n is the complete graph on n vertices and mK_2 denotes m disjoint edges. The proof method applied here has been used earlier in [2] and in [5] to show that $s(G) > 2$ for certain graphs G.

Theorem. *$s(K_n - mK_2) = 3$, unless $n = 4m$, $n = 4m + 1$, $n = 4m - 1$. In the exceptional cases, the irregularity strength is 2.*

Proof: Assume that we have an irregular weighting of $K_n - mK_2$ using weights 1 or 2. The minimum (weighted) degree is at least $n - 2$ and the maximum degree is at most $2n - 2$. Irregularity implies that the degrees are either $n - 2, \ldots, 2n - 3$ (lower segment) or $n - 1, \ldots, 2n - 2$ (upper segment).

Case 1. $n = 4k + 2$. This is impossible since the lower or upper segment would consist of $2k + 1$ odd numbers.

Case 2. $n = 4k + 1$. The lower segment is impossible (odd number of odd degrees). Assume we have the upper segment. Removing the vertex of degree $8k$, we get an irregularly weighted graph to be considered at Case 4.

Case 3. $n = 4k - 1$. The upper segment is impossible (odd number of odd degrees). Assume that we have the lower segment. Adding a new vertex with weight 2 edges to all other vertices, we get an irregular weighting leading to Case 4.

Case 4. (Essential case) $n = 4k$. Let A denote the set of vertices with the smallest $2k$ degrees and let B denote the set of vertices with the largest $2k$ degrees. Set

$$W(A, B) = \sum_{\substack{x \in A \\ y \in B}} w(x, y),$$

where $w(x, y)$ is the weight of the edge xy. It is obvious that

$$W(A, B) \geq \sum_{y \in B} (d(y) - 2(2k - 1)) = \sum_{y \in B} d(y) - 4k(2k - 1) \quad (1)$$

with equality in (1) if and only if B induces a complete graph of weight 2 edges. Also,

$$W(A, B) \leq \sum_{x \in A} (d(x) - 2k + 2) = \sum_{x \in A} d(x) - 2k(2k - 2) \quad (2)$$

with equality in (2) if and only if A induces $K_{2k} - k\, K_2$ of weight 1 edges.

Combining (1) and (2) we have

$$\sum_{y \in B} d(y) - 4k(2k - 1) \leq \sum_{x \in A} d(x) - 2k(2k - 2) \quad (3)$$

Since (3) becomes equality by substituting either the lower or the upper segment for the degrees, equality must hold in (1) and in (2). Therefore, $n = 4m$.

At this point it is proved that $s(K_n - mK_2) \geq 3$, except, possibly when $n = 4m, n = 4m + 1, n = 4m - 1$. To finish the proof we have to show that irregular weighting with weights 1, 2 is possible in the exceptional cases and irregular weighting is possible with weights 1, 2, 3 in all other cases. These weightings have been given in [2] and in [4], respectively. To keep this note self contained, the required weightings are described below.

Let $X = \{x_1, x_2, \ldots, x_{2m}\}, Y = \{y_1, y_2, \ldots, y_{2m}\}$ and identify $K_{4m} - mK_2$ with the complete graph on vertices $X \cup Y$ from which the edges $x_{2i-1}x_{2i}$ are missing for $i = 1, 2, \ldots, m$. Define the set E as

$$E = \{(x_i, y_j): \ i + j \leq 2m + 1\}.$$

Consider two weightings of the edges of $K_{4m} - mK_2$. In weighting A, assign weight 2 to E and to pairs of Y, all other edges get weight 1. In weighting B, the pairs of X and E get weight 1 and all other edges get weight 2. It is easy to check that both A and B are irregular weightings, the weighted degree set under A is $\{4m - 1, 4m, \ldots, 8m - 2\}$ and under B is $\{4m - 2, 4m - 1, \ldots, 8m - 3\}$. To get an irregular weighting for $K_{4m-1} - mK_2$, use A on $K_{4m} - mK_2$ and delete y_1. To get an irregular weighting for $K_{4m+1} - mK_2$, use B on $K_{4m} - mK_2$ and add a new vertex adjacent to all other vertices with weight 2 edges. This argument shows that the irregularity strength is 2 in the exceptional cases.

To show that $s(K_n - mK_2) \leq 3$, define a weighting C on $K_{4t} - tK_2$ by modifying its weighting A by adding one to the weights of the edges within Y.

Assuming $t > 1$, $d(x_1) \leq d(y_{2t}) - 3$ under C. To get an irregular weighting for $K_{4t} - mK_2$, modify C as follows. If $m < t$, add $t - m$ edges $x_1 x_2$, $x_3 x_4, \ldots$, all with weight 1. If $m > t$, delete $m - t$ edges $y_{2t-1}\ y_{2t}$, $y_{2t-3}\ y_{2t-2}, \ldots$. It is obvious that the weighting we get for $K_{4t} - mK_2$ is irregular. It is straightforward to modify this construction for $n \not\equiv 0 \pmod 4$. This (and the case $t = 1$) is left to the reader.

References

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