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On Graphs Of Irregularity Strength 2

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1. Introduction

We consider undirected graphs without loops or multiple edges. A *weighting* of a graph G is an assignment of a positive integer $w(e)$ to each edge of G. For a vertex $x \epsilon V(G)$, the (weighted) degree $d(x)$ is the sum of weights on the edges of G incident to x. The irregularity strength $s(G)$ of a graph G was introduced by Chartrand et al. in [1] as the minimum integer t such that G has a weighting with the following two properties:

(i) $w(e) \leq t$ for all $e \epsilon E(G)$

(ii) $d(x) \neq d(y)$ if $x, y \epsilon V(G)$, $x \neq y$

Since every graph has two vertices of the same degree, $s(G) \geq 2$ for all graphs G. Some results and problems concerning the irregularity strength of graphs appear in [1] and [2].

Assume that G is a graph with $|V(G)| = n$ and $s(G) = 2$. We will determine the minimum and maximum number of edges in a graph of irregularity strength 2. We prove that $|E(G)| \geq \lceil (n^2 - 1)/8 \rceil$ and establish this bound is sharp (Theorem 1). Concerning the upper bound of $|E(G)|$, Jacobson and Lehel conjectured that

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$|E(G)| \le \binom{n}{2} - f(n)$, where $f(n)$ tends to infinity with n. We prove this conjecture with $f(n) = (n-1)/4$ (Theorem 2). We also show that this upper bound is best possible in that there exist graphs with n vertices, $\binom{n}{2} - \lceil (n-1)/4 \rceil$ edges of irregularity strength 2. Perhaps unexpected, we show that *all* but one graph with n vertices, and with $\binom{n}{2} - \lceil (n-1)/4 \rceil$ edges are of irregularity strength 2 (Corollary 4). This is a corollary of the following more general result: Let m, n and α be positive integers such that $2m \le n \le 4\alpha + 1$ and let M be a fixed m-element subset of the vertex set of K_n. If G is a graph obtained from K_n by deleting α edges of the complete graph induced by M, then $s(G) = 2$ (Theorem 3).

Another corollary of Theorem 3 is the following: if we delete α edges of K_n in such a way that the deleted edges form a connected graph and $\alpha \le n/2 - 1$, then the resulting graph is of irregularity strength 2 (Corollary 5). Another special case of Theorem 3 occurs when the missing edges form a complete subgraph. In this case we get a necessary and sufficient condition for $s(K_n - K_m) = 2$, namely $s(K_n - K_m) = 2$ if and only if $2m - 1 \le n \le 2m^2 - 2m + 1$ (Corollary 6). This solves a problem which is also due to Jacobson and Lehel.

2. Results

Theorem 1. *Let G be a graph such that $s(G) = 2$ and $|V(G)| = n$. Then, $|E(G)| \ge \lceil (n^2-1)/8 \rceil$, and for $n = 3 \pmod 4$, $|E(G)| \ge (n^2-1)/8 + 1$. Furthermore there exist graphs for which equality holds.*

Proof. Assume that $|V(G)| = n, s(G) = 2$, and consider a weighting of G with 1 and 2 such that all degrees of G are different. Set $|E(G)| = e_1 + e_2$ where e_i denotes the number of edges in G with weight i ($i = 1$ or 2). Assume that G has p odd degrees so that p is even. The sum of the odd degrees of G is at least $1 + 3 + \ldots + 2p - 1 = p^2$, and the sum of the even degrees of G is at least $0 + 2 + \ldots + 2(n-p-1) = (n-p)(n-p-1)$. It is clear that at least one edge of weight 1 is incident to each vertex of odd degree, and therefore $e_1 \ge (p/2)$. Hence,

$$p^2 + (n-p)(n-p-1) \le \sum_{x \in V(G)} d(x) = 2 \sum_{e \in E(G)} w(e) = 4|E(G)| - 2e_1 \le 4|E(G)| - p,$$

which implies

$$(p^2 + (n-p)(n-p-1) + p)/4 \le |E(G)| \tag{1}$$

The left hand side of (1) is minimum for $p = (n-1)/2$ and for this value of p inequality (1) reduces to

$$(n^2 - 1)/8 \le |E(G)|. \tag{2}$$

Since p is even, inequality (2) is strict for $n = 4m + 3$. For this case $(n^2 - 1)/8$ is an integer and $(n^2 - 1)/8 < |E(G)|$. For all other cases $\lceil (n^2 - 1)/8 \rceil \le |E(G)|$. This proves the first part of the theorem.

To prove the second part of the theorem, we define the *half complete* graph H_m on the vertex set $\{a_0, a_1, \ldots, a_m\}$ as follows. For $0 \le i < j \le m$, a_i and a_j are adjacent if and only if $i + j \ge m + 1$. It is easy to check that $d(a_i)$, the degree of a_i in a half complete H_m satisfies

$$d(a_i) = \begin{cases} i & \text{if} \quad 0 \le i \le \lfloor m/2 \rfloor \\ i - 1 & \text{if} \quad \lfloor m/2 \rfloor \le i \le m \end{cases} \tag{3}$$

From (3) it follows that H_m has $(\binom{m}{2} + \lfloor m/2 \rfloor)/2$ edges. We are now able to describe the graphs which show that equality can hold in Theorem 1. The construction varies slightly depending upon the remainder of n (mod 4).

Case 1: $n = 4m$.

Take two disjoint copies of H_{2m-1}, one with vertex set $A = \{a_0, a_1, \ldots, a_{2m-1}\}$ and the other with vertex set $B = \{b_0, b_1, \ldots, b_{2m-1}\}$. Add edges $a_i b_i$ for $m \le i \le$ $\le 2m - 1$. Assign weight 2 to all edges of this graph. Using (3), the (weighted) degree sequence of this graph is $0, 0, 2, 2, 4, 4, \ldots, 4m - 2, 4m - 2$. Finally, add the edges $a_{2i} b_{2i+1}$ for $i = 0, 1, \ldots, m - 1$ and assign weight 1 to each of them. The degree sequence of the constructed graph is $0, 1, 2, \ldots, 4m - 2, 4m - 1$, and it is easy to check that the graph has $2|E(H_{2m-1})| + 2m = \binom{2m-1}{2} + (m - 1) + 2m =$ $= 2m^2 = \lceil (n^2 - 1)/8 \rceil$ edges.

Case 2: $n = 4m + 1$

Again take two disjoint copies of H_{2m-1}, one with vertex set A and one with vertex set B. Add a new vertex c and make it adjacent to a_i and to b_i for $m \le i \le 2m - 1$. If all of these edges are of weight 2 and we add edges $a_{2i} b_{2i+1}$ for $0 \le i \le m - 1$ with weight 1, the derived graph is obtained. Simply observe that the weighted graph constructed has degree sequence $0, 1, 2, \ldots, 4m - 1, 4m$ and the number of edges is $2|E(H_{2m-1})| + 3m = 2m^2 + m = (n^2 - 1)/8$.

Case 3: $n = 4m + 2$

Take two disjoint copies of H_{2m} one with vertex set A and the other with vertex set B. Insert edges $a_i b_i$ for $m + 1 \le i \le 2m$ and assign weight 2 to all edges of this graph. Next insert additional edges $a_{2i} b_{2i+1}$, for $i = 0, 1, \ldots,$ $m - 2$, and $a_{2m} b_{2m-1}, a_{2m} b_{2m-2}$ with each of these edges assigned weight 1. We have the desired graph, since the degrees are all distinct (the degree sequence is $0, 1, 2, \ldots, 4m$ and $4m + 2$) and the number of edges is $2|E(H_{2m})| + m + (m-1) + 2 =$ $= 2m^2 + 2m + 1 = \lceil (n^2 - 1)/8 \rceil$.

Case 4: $n = 4m + 3$

Take two half complete graphs with vertex sets A and B. Add all edges $a_i b_i$ for $1 \le i \le 2m - 1$. Take three new vertices x, y and z and add the edge yz

and the edges a_iz, b_iz for $m \le i \le 2m-1$. Assign weight 2 to all edges of this graph. At this point $d(x) = 0, d(a_o) = d(b_o) = 0, d(a_i) = d(b_i) = 2i+2$ for $i = 1, 2, \ldots, 2m-1, d(y) = 2$, and $d(z) = 4m+2$. Next, to the constructed graph add edges $xy, xa_o, a_{2m-1}z$ and $a_{2i+1}b_{2i+2}$ with each of these assigned weight 1, $(0 \le i \le m-2)$. The graph which results has the following degrees: $d(b_o) = = 0, d(a_o) = 1, d(x) = 2, d(y) = 3$,$d(z) = 4m+3$, and $b_1, a_1, a_2, b_2, \ldots, b_{2m-1}$, a_{2m-1} have degrees $4, 5, \ldots, 4m, 4m+1$. Therefore, all the degrees of the graph are distinct and the number of edges is $2|E(H_{2m-1})| + 2m - 1 + 1 + 2m + 3 + m - 1 = = 2m^2 + 3m + 2 = (n^2-1)/8 + 1$. ∎

Theorem 2. *Let G be a graph such that $|V(G)| = n$, and $s(G) = 2$. Then, $|E(G)| \le \binom{n}{2} - (n-1)/4$.*

Proof. Assume that G has a weighting with 1 and 2 such that all the (weighted) degrees of G are different. Let $\alpha = \binom{n}{2} - |E(G)|$, the number of edges which need to be deleted from K_n to obtain the graph G. Set $p = \lceil n/2 \rceil$ and $q = \lfloor n/2 \rfloor$, and partition the vertices of G into two sets A and B such that $|A| = p, |B| = q$, and $d(x) \le d(y)$ for all $x \epsilon A$ and $y \epsilon B$. Thus A contains the p vertices of smallest degree and B contains the q vertices of the largest degree.

We first estimate the weighted edge sum $\sum\limits_{x\epsilon A, y\epsilon B} w(x,y) = W(A,B)$. On one hand, a vertex $y \epsilon B$ contributes at least $d(y) - 2(q-1)$ to $W(A,B)$, so that

$$W(A,B) \ge \sum_{y\epsilon B}(d(y) - 2(q-1)) = \sum_{y\epsilon B} d(y) - 2q(q-1) \tag{4}$$

On the other hand, a vertex $x \epsilon A$ contributes at most $d(x) - (p - 1 - \overline{d}_A(x))$ to $W(A,B)$, where $\overline{d}_A(x)$ denotes the degree of x in the subgraph induced by A in $\overline{G}$.

Therefore

$$W(A,B) \le \sum_{x\epsilon A}(d(x) - (p-1-\overline{d}_A(x))) = \sum_{x\epsilon A} d(x) - p(p-1) + \sum_{x\epsilon A} \overline{d}_A(x)$$

Since $\sum\limits_{x\epsilon A} \overline{d}_A(x)$ is equal to twice the number of edges of $\overline{G}$ in A, $\sum\limits_{x\epsilon A} \overline{d}_A(x) \le 2\alpha$, and

$$W(A,B) \le \sum_{x\epsilon A} d(x) - p(p-1) + 2\alpha. \tag{5}$$

Let t denote $\min\limits_{y\epsilon B} d(y)$. Then since the vertices of G have distinct degrees, $\sum\limits_{y\epsilon B} d(y) \ge t + (t+1) + \ldots + (t+q-1) = q(2t+q-1)/2$. Similarly, $\sum\limits_{x\epsilon A} d(x) \le \le (t-1) + (t-2) + \ldots + (t-p) = p(2t-p-1)/2$. Using these inequalities, we obtain from (4) and (5) the inequality $p(2t - p - 1)/2 - p(p-1) + 2\alpha \geq q(2t + q - 1)/2 - 2q(q-1)$. From this it follows that

$$2\alpha \geq (p - q)((3n - 1)/2 - t) + q. \tag{6}$$

If n is even then $p = q = n/2$ and (6) gives $4\alpha \geq n$. If n is odd then $p = (n+1)/2, q = (n-1)/2$ and we have $2\alpha \geq 2n - 1 - t$. Since the maximum degree of G is at most $2(n-1)$, $t \leq 2(n-1) - ((n-1)/2 - 1) = 3(n-1)/2 + 1$. Therefore $2\alpha \geq 2n - 1 - t \geq (n-1)/2$, which implies in this case that $4\alpha + 1 \geq n$. Therefore,in general, $\alpha \geq (n-1)/4$, and the theorem is proved. ■

Theorem 3. *Let m, n, and α be fixed positive integers with $2m \leq n \leq 4\alpha + 1$, and let M be a fixed m-element subset of the vertex set of K_n. If G is a graph obtained from K_n by deleting α edges of the complete subgraph induced by M, then $s(G) = 2$.*

Proof. Assume G is defined as described in the theorem. We shall prove that for even n, $(2m \leq n \leq 4\alpha)$ G has a weighting with 1 and 2 such that all the degrees are distinct and $d(x) < 2(n-1)$ for all $x \epsilon V(G)$. From this the theorem follows.

To see that this gives the result for n odd, simply delete a vertex x of maximum degree from G and assign the weighting of 1 and 2 to edges of $G - x$ as just mentioned for even order graphs. Extend this weighting to G by assigning weight 2 to all edges incident to x.

Assume that n is even, $2m \leq n \leq 4\alpha$, and set $k = n/2$. Assume that the vertex set of G is $\{a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$. We may also assume that $M = \{a_{k-m+1}, a_{k-m+2}, \ldots, a_k\}$. Let H be the graph induced by the vertex set $\{a_1, a_2, \ldots, a_k\}$. We may clearly assume that $d_H(a_i) \geq d_H(a_j)$ for $k - m + 1 \leq$ $\leq i < j \leq k$. Since $d_H(a_i) = k - 1$ for $1 \leq i \leq k - m$, $k - 1 = d_H(a_1) = \ldots$ $= d_H(a_{k-m}) \geq d_H(a_{k-m+1}) \geq \ldots \geq d_H(a_k)$.

We wish to assign labels $0, 1, 2, \ldots, k-1$ to the vertices $a_1, a_2, \ldots, a_k$ (with $L(i)$ the label of a_i) which satisfy the following properties:

(i) $L(i) \neq L(j)$ for $i \neq j$,

(ii) $d_H(a_i) + L(i) \neq d_H(a_j) + L(j)$ for $i \neq j$, and

(iii) if $L(i) = k - 1$, then $d_H(a_i) < k - 1$.

Before showing that such a labelling exists, we apply it to complete the proof of the theorem. Consider a bipartite graph B with color classes $X = \{x_1, x_2, \ldots, x_k\}$ and $Y = \{y_1, y_2, \ldots, y_k\}$ which satisfies the following condition:

$$d(x_i) = d(y_i) = i - 1 \qquad \text{for} \qquad i = 1, 2, \ldots, k.$$

Consider the following one-to-one mapping f from the vertices of B to the vertices of G. The vertex x_i of B is associated with the vertex a_j of G with label $i-1$, i.e., $f(x_i) = a_j$ if $L(j) = i - 1$. The vertex y_i of B corresponds to b_i, $i.e. f(y_i) = b_i$. The required f exists by property (i).

We are now ready to define the weighting of G. All edges in the subgraph induced by $\{a_1, a_2, \dots a_k\}$ get weight 1. All edges in the complete subgraph induced by $\{b_1, b_2, \dots, b_k\}$ get weight 2. An edge $a_i b_j$ gets weight 2 if and only if $a_i b_j$ is the image of an edge of B under f, otherwise, $a_i b_j$ gets weight 1.

We claim that all the (weighted) degrees of G are different under this weighting. From the definition of the weighting

$$d(a_i) = d_H(a_i) + L(i) + k \text{ for } 1 \le i \le k, \quad \text{and} \tag{7}$$

$$d(b_i) = i - 1 + k + 2(k-1) \text{ for } 1 \le i \le k \tag{8}$$

Since property (ii) holds for the labelling L, it follows from (7) that $d(a_i) \neq d(a_j)$ for $i \neq j$. Clearly, from (8), $d(b_i) \neq d(b_j)$ for $i \neq j$. If $d(a_i) = d(b_j)$, then from (7) and (8) we have

$$d_H(a_i) + L(i) + k = j - 1 + k + 2(k-1),$$

so that

$$d_H(a_i) + L(i) = j + 2k - 3.$$

Since

$$d_H(a_i) \le k - 1, \quad L(i) \le k - 1 \text{ and } j \ge 1,$$

this equality can hold if and only if

$$d_H(a_i) = k - 1, \quad L(i) = k - 1.$$

However, property (iii) excludes this possibility.

Finally, we must verify the maximum degree of G is less than 2(n-1). Obviously, the maximum degree of G is the degree of b_k, and by (8), $d(b_k) = k - 1 + k + 2(k-1) = 4k - 3 = 2n - 3$. Thus the theorem follows from the existence of the required labelling.

The labelling we describe first assigns labels sequentially to the vertices $a_{k-m+t}, a_{k-m+t+1}, \dots, a_k$ where t is the smallest nonnegative integer such that $d_H(a_{k-m+t}) < k - 1$. This assignment is as follows: let $L(k - m + t) = k - 1$, and if $L(k - m + l)$ has been defined for $t \le l \le k - 1$, then define $L(k - m + l + 1) = \max\{L(k - m + l) + d_H(a_{k-m+l}) - k + 1, m - l - 1\}$. The remaining labels $L(i)$, for $1 \le i \le k - m + t - 1$, can be assigned arbitrarily using each of the unused labels exactly once.

We need to establish that the labelling just described satisfies properties (i) and (ii), since property (iii) clearly holds. First observe that $L(k - m + l) + d_H(a_{k-m+l}) - k + 1 < L(k - m + l)$ for $t \le l$, and if for some j, $j \ge$

$\geq t+1$, $L(k-m+j)=m-j$, then $L(k-m+i)=m-i$ for all $j\leq i\leq m$. Thus in particular we have $L(k-m+t)=k-1>L(k-m+t-1)>\ldots>L(k)\geq 0$, so that property (i) holds and property (ii) holds when $i,j\leq k-m+t-1$ or when $k-m+t\leq i,j$.

It only remains to show that property (ii) holds when $i\leq k-m+t-1<j$. To do this note that in the labelling described $L(k)=0$. If this were not the case, then $\max\{L(k-m+l)+d_H(a_{k-m+l})-k+1; m-l-1\}=L(k-m+l)+d_H(a_{k-m+l})-k+1$ for all $t\leq l\leq m-1$. From this it would follow that

$$\begin{aligned}L(k) &= \sum_{i=k-m+t}^{k-1} d_H(a_i)-(m-t)(k-1)\\ &= (m-t+1)(k-1)-2\alpha-d_H(a_k)-(m-t)(k-1)\\ &= (k-1)-2\alpha-d_H(a_k)<0, \text{since } k\leq 2\alpha,\end{aligned}$$

which is impossible. Further the recursive definition of $L(j)$ for $k-m+t\leq j$, gives for $1\leq i\leq k-m+t-1$, $L(i)\neq L(j)+d_H(a_j)-(k-1)=L(j)+d_H(a_j)-d_H(a_i)$. Thus property (ii) also holds for $i\leq k-m+t-1$ and $j\geq k-m+t$ and the labelling satisfies all required conditions. ■

It is worth mentioning some special cases of Theorem 3. If we remove $\alpha = \lceil(n-1)/4\rceil$ edges of K_n and the removed edges do not form a matching then the conditions of Theorem 3 are satisfied with $m=2\alpha-1$. Therefore we have

Corollary 4. *If G has $\binom{n}{2}-\lceil(n-1)/4\rceil$ edges and $\bar{G}$ is not a matching, then $s(G)=2$.*

Concerning the case not covered by Corollary 4 we note without proof that $s(K_n-M)=2$ for $n\equiv 0,1,3 \pmod 4$ but $s(K_n-M)=3$ for $n\equiv 2 \pmod 4$. (M is a matching with $\lceil(n-1)/4\rceil$ edges.)

Notice that Theorem 2 and Corollary 4 together imply that $\lceil(n-1)/4\rceil$ is the minimum number of edges which need to be removed from K_n such that the resulting graph has strength 2.

Let G be a graph on n vertices with α "missing edges" such that $\lceil(n-1/4\rceil \leq \alpha \leq n/2-1$ and the missing edges determine a connected subgraph. Since the missing edges are incident to at most $\alpha+1\leq n/2$ vertices, Theorem 3 gives the following.

Corollary 5. *If G is a graph obtained from K_n by deleting α edges, $\lceil(n-1)/4\rceil \leq \alpha\leq n/2-1$, such that the removed edges induce a connected graph, then $s(G)=2$.*

Finally, consider the case when $G=K_n-K_m$ and $2m\leq n\leq 2m^2-2m+1$. We can apply Theorem 3 with $\alpha=\binom{m}{2}$ to obtain the following result.

Corollary 6. *If* $G = K_n - K_m$ *and* $2m \leq n \leq 2m^2 - 2m + 1$, *then* $s(G) = 2$.

Remark: Theorem 2 implies that for $n \geq 2m^2 - 2m + 2$, $s(K_n - K_m) \geq 3$. In fact, it is easy to show that in this case that $s(K_n - K_m) = 3$. It is also easy to show that $s(K_n - K_m) = 2$ for $n = 2m - 1$ and that $s(K_n - K_m) \geq 3$ for $n \leq 2m - 2$. These remarks show that $s(K_n - K_m) = 2$ if and only if $2m - 1 \leq n \leq 2m^2 - 2m + 1$.

References

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