ZASTOSOWANIA MATEMATYKI
APPLICATIONES MATHEMATICAE
XIX, 3–4 (1987), pp. 413–441

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PROBLEMS FROM THE WORLD SURROUNDING PERFECT GRAPHS

Abstract. A family $\mathscr{G}$ of graphs is called *χ-bound with binding function f* if $\chi(G') \leqslant f(\omega(G'))$ holds whenever G' is an induced subgraph of $G \in \mathscr{G}$. Here $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of G, respectively. The family of perfect graphs appears in this setting as the family of χ-bound graphs with binding function $f(x) = x$. The paper exposes open problems concerning χ-bound families of graphs.

CONTENTS

0. INTRODUCTION

Our aim is to introduce and propose a systematic study of χ-bound (and θ-bound) families of graphs and their binding functions. These families are natural extensions of the world of perfect graphs. Recall that the family $\mathscr{P}$ of perfect graphs contains the graphs G which satisfy $\chi(G') = \omega(G')$ for all induced subgraphs G' of G. Here $\chi(G)$ and $\omega(G)$ denote the chromatic number and the clique number of a graph G, respectively.

A family $\mathscr{G}$ of graphs is called *χ-bound with binding function f* if $\chi(G') \leqslant f(\omega(G'))$ holds whenever $G \in \mathscr{G}$ and G' is an induced subgraph of G.

Without restricting generality, we may assume that a binding function is an $N \to N$ function, where N denotes the set of positive integers; moreover, $f(1) = 1$ and $f(x) \geqslant x$ for all $x \in N$. Under these natural assumptions the smallest binding function is $f(x) = x$ and the family of graphs which is χ-bound with binding function $f(x) = x$ is the family of perfect graphs. The complementary notion of χ-bound families is the notion of θ-bound families. A family $\mathscr{G}$ of graphs is *θ-bound with binding function f* if $\bar{\mathscr{G}}$ is a χ-bound family with binding function f (here $\bar{\mathscr{G}}$ denotes the family containing the complements of the graphs of $\mathscr{G}$).

Section 1 introduces the notion of χ-bound and θ-bound families of graphs with several examples. The most frequently occurring problems concerning binding functions are formulated and illustrated there, namely:

1. Does there exist a binding function for a given family $\mathscr{G}$ of graphs?
2. What is the smallest binding function for $\mathscr{G}$?
3. Does there exist a linear binding function for $\mathscr{G}$?
4. Does there exist a polynomial binding function for $\mathscr{G}$?

The examples in 1.2 (e.g., circular arc graphs, multiple interval graphs, box graphs, polyomino graphs, overlap graphs) show that the behaviour (or at least the known properties) of these families concerning their binding functions are quite different. Although these families are usually χ-bound and θ-bound (the exception is the family of box graphs for more than two dimensions), in most cases the order of magnitude or linearity of their smallest binding function is not known.

The significance of binding functions from algorithmic point of view is discussed in 1.3. The idea is that families having "small" χ-binding functions (θ-binding function) are natural candidates for approximation algorithms with a "good" performance ratio for the coloring problem (clique cover problem). The smaller is a binding function of a family, the better performance ratio is to be expected from an approximation algorithm operating on the graphs of the family.

Perfect families of graphs are often characterized by a set of forbidden induced subgraphs. The family of P_4-free graphs, split graphs, threshold graphs, triangulated graphs, Meynel graphs are examples of such families. Analogous questions are discussed in Sections 2, 3 and 4 for χ-bound families of graphs: which forbidden induced subgraphs make a family χ-bound? Section 2 presents problems and results concerning the following conjecture: the family of graphs which does not contain a fixed forest as an induced subgraph is χ-bound. In Section 3 we discuss problems when the set of forbidden induced subgraphs is infinite. The Strong Perfect Graph Conjecture fits into this problem area. It is surprising that a much weaker conjecture, namely that the family of graphs without odd holes and their complements is χ-bound, seems to be difficult. We should call this conjecture the Weakened

Strong Perfect Graph Conjecture. In Section 4 we consider the case where the set of forbidden subgraphs is closed under taking complementary graphs.

In Section 5 we study the effect of taking the union and intersection of graphs on binding functions. It is straightforward that the union of χ-bound families is again a χ-bound family. However, the intersection of two χ-bound families (even the intersection of two perfect families) is not necessarily χ-bound.

The situation of having the notion of χ-bound and θ-bound families resembles the time B.P.G.T. (Before Perfect Graph Theorem) when two types of perfectness had to be defined. It is easy to construct families which are χ-bound but not θ-bound although "natural" graph families are usually both χ-bound and θ-bound. In Section 6 we try to find analogons of the Perfect Graph Theorem for certain χ-bound families of graphs. Let $\mathcal{G}_f$ denote the family of graphs θ-bound with θ-binding function f. If $\mathcal{G}_f$ is χ-bound, then the smallest χ-binding function of $\mathcal{G}_f$ is called the *complementary binding function* of f. It turns out that the only self-complementary binding function is $f(x) = x$, that is the Perfect Graph Theorem is stable in a certain sense. Only "small" binding functions may have complementary binding functions: if f has a complementary binding function, then $\inf f(x)/x = 1$. However, it remains an open problem even to prove that $f(x) = x+1$ has a complementary binding function.

All results appearing here with proofs are unpublished elsewhere. They are expository in nature and serve mainly as background material and status information for the open problems. In fact, the main motivation of the author for writing this paper is his desire to see some of these 44 problems to be solved. I am indebted to my friend and colleague J. Lehel for several discussions which helped these ideas to take shape.

1. χ-BOUND AND θ-BOUND FAMILIES AND THEIR BINDING FUNCTIONS

1.1. Basic concepts. Let $\omega(G)$ and $\chi(G)$ denote the *clique number* and the *chromatic number* of a graph G, i.e., $\omega(G)$ is the maximal number of pairwise adjacent vertices of G, and $\chi(G)$ is the minimal number k such that the vertices of G can be partitioned into k stable sets. A subset of vertices in a graph is called *stable* if it contains pairwise non-adjacent vertices.

A function f is a *χ-binding function* for a family $\mathcal{G}$ of graphs if

$$\chi(G') \leqslant f(\omega(G'))$$

holds for all induced subgraphs G' of $G \in \mathcal{G}$. We shall always assume that $f\colon N \to N$, where N denotes the set of positive integers; moreover, $f(1) = 1$, $f(x) \geqslant x$ for all $x \in N$.

A family $\mathscr{G}$ of graphs is *χ-bound* if there exists a χ-binding function for $\mathscr{G}$.

The above definitions can be formulated for the complementary parameters of graphs. Let $\alpha(G)$ and $\theta(G)$ denote the *stability number* and the *clique-cover number* of a graph G, i.e., $\alpha(G)$ is the maximal number of vertices in a stable set of G, and $\theta(G)$ is the minimal number k such that the vertices of G can be partitioned into k cliques.

A function f is a *θ-binding function* for a family $\mathscr{G}$ of graphs if

$$\theta(G') \leqslant f(\alpha(G'))$$

holds for all induced subgraphs G' of $G \in \mathscr{G}$. A family $\mathscr{G}$ of graphs is θ-bound if there exists a θ-binding function for $\mathscr{G}$.

Since $\omega(G) = \alpha(\bar{G})$ and $\chi(G) = \theta(\bar{G})$ hold for any graph G by definition (where $\bar{G}$ denotes the complement of G), we observe that

f is a χ-binding function for $\mathscr{G}$ if and only if f is a θ-binding function for $\bar{\mathscr{G}}$;

$\mathscr{G}$ is χ-bound if and only if $\bar{\mathscr{G}} = \{\bar{G}\colon G \in \mathscr{G}\}$ is θ-bound.

If a family $\mathscr{G}$ is χ-bound, then it has obviously a *smallest χ-binding function* defined by

$$f^*(x) = \max\{\chi(G')\colon G' \subset G \in \mathscr{G},\ \omega(G') = x\}.$$

Similarly, a θ-bound family has a smallest θ-binding function.

Due to the assumptions on binding functions, the smallest binding function a family may have is the identity function $f(x) = x$. The family of graphs with χ-binding function $f(x) = x$ is the important family of perfect graphs. The family of perfect graphs is denoted by $\mathscr{P}$. The Perfect Graph Theorem of Lovász [27] states that $\mathscr{P} = \bar{\mathscr{P}}$, which implies that $\mathscr{P}$ can be equivalently defined as the family of graphs with θ-binding function $f(x) = x$.

The basic problems in our approach concerning a family $\mathscr{G}$ of graphs are:

Is $\mathscr{G}$ a χ-bound (or θ-bound) family?

What is the order of magnitude of the smallest χ-binding (or θ-binding) function for $\mathscr{G}$?

Determine the smallest χ-binding (or θ-binding) function for $\mathscr{G}$.

Before looking at some examples of χ-bound or θ-bound families, let us have a glance at the outside world. Let G_i be a graph such that $\omega(G_i) = 2$ and $\chi(G_i) = i$ for each integer $i \geqslant 2$. The existence of G_i is well known (see, e.g., [30]). Now the family $\{G_2, G_3, \ldots\}$ is obviously not χ-bound since it is impossible to define the value of a χ-binding function $f(x)$ for $x = 2$. A more surprising example of a family which is not χ-bound is provided by the intersection graphs of boxes in the three-dimensional Euclidean space (see 1.2).

1.2. Some examples of χ-bound and θ-bound families. Now let us have a look at some well-known families of graphs and their binding functions. We start with three classical subfamilies of $\mathscr{P}$ which we frequently need later.

Interval graphs: the intersection graphs of closed intervals on a line.

Triangulated graphs: the graphs containing no C_k (a cycle of k vertices) for $k \geqslant 4$ as an induced subgraph.

Comparability graphs: the graphs G whose edges can be oriented transitively (ab, $bc \in E(\vec{G})$ implies $ac \in E(\vec{G})$).

The proof of the perfectness of the above families can be found in [16]. We continue with some well-known non-perfect families of graphs defined as intersection graphs of geometrical objects. Proof techniques and results concerning their binding functions have been surveyed in [22].

Circular arc graphs (see [16], p. 188): the intersection graphs of closed arcs of a circle. The family of circular arc graphs is θ-bound, its smallest θ-binding function is $f(x) = x+1$. The family is χ-bound as well, the function $f(x) = 2x$ is a suitable χ-binding function for $x \geqslant 2$. Both of these statements follow immediately from the perfectness of interval graphs. It is easy to construct circular arc graphs G_k for all k, satisfying $\omega(G_k) = k$, $\chi(G_k) = \lfloor 3k/2 \rfloor$. Tucker conjectured (see [37]) and Karapetian [25] proved that

$$\chi(G) \leqslant \lfloor (3/2)\,\omega(G) \rfloor$$

holds for all circular arc graphs G. In our terminology, this result states:

THEOREM 1.1. *The smallest χ-binding function for the family of circular arc graphs is $f(x) = \lfloor (3/2)\,x \rfloor$.*

Multiple (or t-) interval graphs: intersection graphs of sets which are the union of t closed intervals on a line. In the special case where $t = 1$, we get interval graphs. These graphs were introduced in [17] and [24]. The results of [21] imply that the family of t-interval graphs is θ-bound for all fixed t. The order of magnitude of the smallest θ-binding function is not known even for $t = 2$.

PROBLEM 1.2. Determine the order of magnitude of the smallest θ-binding function for double interval graphs. In particular, does there exist a linear θ-binding function for double interval graphs?

It was proved in [20] that the family of t-interval graphs is χ-bound with a linear binding function $2t(x-1)$ for $x \geqslant 2$.

Box graphs (introduced in [34]): intersection graphs of sets of boxes in the d-dimensional Euclidean space. A *box* is a parallelepiped with sides parallel to the coordinate axes. For $d = 1$ we have the family of interval graphs.

It is easy to see that the family of d-dimensional box graphs is θ-bound with θ-binding function x^d (see Proposition 5.5). The order of magnitude of the smallest θ-binding functions is not known even for $d = 2$.

PROBLEM 1.3. Determine the order of magnitude of the smallest θ-binding function for two-dimensional box graphs.

Concerning χ-binding functions, it was proved by Asplund and Grünbaum in [1] that two-dimensional box graphs are χ-bound with an $O(x^2)$ χ-binding function. The order of magnitude of the smallest χ-binding function is not known, its value at $x = 2$ is 6 as proved in [1].

PROBLEM 1.4. Determine the order of magnitude of the smallest χ-binding function for two-dimensional box graphs. In particular, decide whether it is linear or not.

A surprising construction of Burling [4] shows that the family of three-dimensional boxes in not χ-bound.

Polyomino graphs. This subfamily of two-dimensional box graphs has received some attention in the last few years. A *polyomino* is a finite set of cells in the infinite planar square grid. With a polyomino P we may associate a hypergraph $H(P)$ whose vertices are the cells of P and whose edges are the set of cells in maximal boxes contained in P. The intersection graph $G(P)$ of $H(P)$ may be called a *polyomino graph.* Obviously, $G(P)$ is a subfamily of two-dimensional boxes, thus it is both θ-bound and χ-bound. Answering a question of Berge et al. [3], Shearer [36] proved that $G(P)$ is perfect if P is simply connected. It would be interesting to know whether the family of polyomino graphs has linear binding functions; these questions are attributed to P. Erdös.

PROBLEM 1.5. Does there exist a linear θ-binding function for polyomino graphs?

PROBLEM 1.6. Does there exist a linear χ-binding function for polyomino graphs?

Overlap graphs (alias circle graphs, stack sorting graphs; see [16], p. 242). These graphs are defined by closed intervals on a line as follows: the vertices are the intervals and two vertices are joined by an edge if the corresponding intervals overlap, i.e., they are intersecting but neither contains the other. An equivalent definition is obtained by considering the intersection graphs of chords of a circle. Golumbic calls these graphs "not so perfect" (see [16], p. 235). A measure of "non-perfectness" can be the order of magnitude of the smallest binding functions. It is easy to give an $O(x^2)$ θ-binding function for the family of overlap graphs (see Proposition 5.4). It is harder to prove that the family is χ-bound, the smallest known χ-binding function is exponential (see [20]).

PROBLEM 1.7. Does there exist a linear θ-binding function for the family of overlap graphs?

PROBLEM 1.8. Does there exist a linear χ-binding function for the family of overlap graphs?

Intersection graphs of straight-line segments in the plane. This family of graphs was introduced in [7]. The problem whether this family is χ-bound (θ-bound) arose during a conversation with P. Erdös. Denote this family by $\mathcal{G}_{\mathrm{SLS}}$.

PROBLEM 1.9. Is $\mathcal{G}_{\mathrm{SLS}}$ a χ-bound family?

PROBLEM 1.10. Is $\mathcal{G}_{\mathrm{SLS}}$ a θ-bound family?

1.3. Algorithmic aspects of binding functions. For various classes of perfect graphs there are fast polynomial algorithms to determine a largest stable set (of size $\alpha(G)$), a largest clique (of size $\omega(G)$), a good coloring of $V(G)$ with $\chi(G)=\omega(G)$ colors or a vertex-cover by $\theta(G)=\alpha(G)$ cliques. Many examples of such algorithms can be found in [16]. It turned out (see [18]) that all of these problems can be solved by polynomial algorithms for the family $\mathcal{P}$ of perfect graphs.

Families of χ-bound graphs are natural candidates for polynomial *approximation algorithms* for the vertex coloring problem. Similarly, polynomial approximation algorithms may work for the clique-cover problem in case of classes of θ-bound graphs. It is typical that the proof of the existence of a χ-binding function f for a family $\mathcal{G}$ of graphs provides a polynomial algorithm for a good coloring of the vertices of $G \in \mathcal{G}$ with at most $f(\omega(G))$ colors. In this case we have a polynomial approximation algorithm with performance ratio at most $f(\omega(G))/\omega(G)$, which may or may not be statisfactory in a particular situation. A very favourable case occurs when a family $\mathcal{G}$ has a *linear* χ-binding function. Then the performance ratio of the algorithm is constant. The polynomial approximation algorithm can be useful if the coloring problem is known to be NP-complete for the family $\mathcal{G}$ which is again a typical case. A similar reasoning shows the role of θ-binding functions in approximation algorithms for the clique-cover problem. (The basic notions on computational complexity are used here as defined in [14].)

To see some examples, consider the coloring problem for circular arc graphs. This problem is NP-complete (see [15]); on the other hand, it is easy to give a polynomial approximation algorithm with performance ratio at most 2. The algorithm comes from the proof of the fact that $2x$ is a χ-binding function for the family of circular arc graphs. It is possible to color better, the proof of Theorem 1.1 yields a polynomial approximation algorithm with performance ratio 3/2.

The situation is similar if the coloring problem is considered for multiple interval graphs. The problem is NP-complete since the family of 2-interval graphs contains the family of circular arc graphs and the latter is NP-complete. The proof of the existence of the χ-binding function $2t(x-1)$ for the family of t-intervals ($x \geqslant 2$) provides a very simple polynomial approximation algorithm with performance ratio less than $2t$ (see [20]).

The above reasoning might convince the reader of the importance of the following vaguely formulated problem:

PROBLEM 1.11. Find some applicable sufficient condition which implies that a family has a linear χ-binding function.

The existence of a linear binding function is an open problem for many χ-bound and/or θ-bound families. Problems 1.2–1.8 provide examples and we shall see others later.

Concerning potential applications, we note that the coloring problem of circular arc graphs and multiple interval graphs occurs in scheduling problems (see [39], [24], [16]), applications of the coloring problem of overlap graphs are discussed in [16]. The clique-cover problem of polyomino graphs is motivated by the problem of picture processing as noted in [3].

2. BINDING FUNCTIONS ON FAMILIES WITH ONE FORBIDDEN SUBGRAPH

Let H be a fixed graph and consider the family $\mathscr{G}(H)$ of graphs which does not contain H as an induced subgraph:

$$\mathscr{G}(H) = \{G\colon\ H \not\prec G\}.$$

What choices of H guarantee that $\mathscr{G}(H)$ is a χ-bound family? Assume that H contains a cycle, say of length k. Let G_i be a graph of chromatic number i and of girth at least $k+1$. The existence of such graphs was proved by Erdös and Hajnal in [10]. Clearly, $G_i \in \mathscr{G}(H)$ for $i = 1, 2, \ldots$, showing that $\mathscr{G}(H)$ is not χ-bound. I conjectured that $\mathscr{G}(H)$ is χ-bound in all other cases, i.e., the following holds:

CONJECTURE 2.1 ([19]). *$\mathscr{G}(F)$ is χ-bound for every fixed forest F.*

Let S_n denote the star on n vertices and let $R(p, q)$ be the *Ramsey function*, that is the smallest $m = m(p, q)$ such that all graphs on m vertices contain either a stable set of p vertices or a clique of q vertices. The following result shows that $\mathscr{G}(S_n)$ is χ-bound and its smallest χ-binding function is close to the Ramsey function.

THEOREM 2.2. *The family $\mathscr{G}(S_n)$ is χ-bound and its smallest χ-binding function f^* satisfies*

$$\frac{R(n-1,\, x+1)-1}{n-2} \leqslant f^*(x) \leqslant R(n-1,\, x)$$

for all fixed n, $n \geqslant 3$.

Proof. Let G be a graph on $R(n-1, x+1)-1$ vertices such that G contains neither a stable set of $n-1$ vertices nor a clique of $x+1$ vertices. Clearly,

$$G \in \mathscr{G}(S_n) \quad \text{and} \quad \chi(G) \geqslant |V(G)|/(n-2),$$

which gives the lower bound for f^*.

To show the upper bound, let $G \in \mathscr{G}(S_n)$, $\omega(G) = x$. We claim that the degree of any vertex of G is less than $R(n-1, x)$. If some vertex $P \in V(G)$ has at least $R(n-1, x)$ neighbours, then the neighbourhood of P contains either a stable set of $n-1$ vertices or a clique of x vertices. The first possibility contradicts $G \in \mathscr{G}(S_n)$ and the second contradicts $\omega(G) = x$, and the claim follows. Therefore, the chromatic number of G is at most $R(n-1, x)$.

Note that for $n = 3$ the lower and upper bounds are the same showing that $f^*(x) = x$, i.e., $\mathscr{G}(S_3)$ is a perfect family. It is easy to see that $\mathscr{G}(S_3)$ consists of graphs which can be written as the union of disjoint cliques.

PROBLEM 2.3. Improve the estimates of Theorem 2.2 for the smallest χ-binding function of $\mathscr{G}(S_4)$.

The next special case where Conjecture 2.1 is solved occurs if the underlying forest is a path.

THEOREM 2.4. *Let P_n denote a path on n vertices, $n \geq 2$. Then $\mathscr{G}(P_n)$ is χ-bound and $f_n(x) = (n-1)^{x-1}$ is a suitable χ-binding function.*

Proof. Considering $n \geq 1$ fixed, we prove by induction on $\omega(G)$. To launch the induction, note that the theorem trivially holds for graphs G with $\omega(G) = 1$. Suppose that $(n-1)^{x-1}$ is a binding function for all $G' \in \mathscr{G}(P_n)$ such that $\omega(G') \leq t$ for some $t \geq 1$.

Let $G \in \mathscr{G}(P_n)$ and $\omega(G) = t+1$. Assuming that $\chi(G) > (n-1)^t$, we shall reach a contradiction by constructing a path $(Q_1, Q_2, \ldots, Q_n)$ induced in G. Technically, we define nested vertex sets

$$V(G) \supset V(G_1) \supset \ldots \supset V(G_i)$$

and vertices

$$Q_1 \in V(G_1), \quad Q_2 \in V(G_2), \quad \ldots, \quad Q_i \in V(G_i)$$

for all i satisfying $1 \leq i \leq n$ with the following properties:

(i) G_i is a connected subgraph of G;

(ii) $\chi(G_i) > (n-i)(n-1)^{t-1}$;

(iii) if $1 \leq j < i$ and $Q \in V(G_i)$, then $Q_j Q$ is an edge of G if and only if $j = i-1$ and $Q = Q_i$.

For $i = 1$ we choose G_1 as a connected component of G with $\chi(G_1) > (n-1)^t$ because $\chi(G) > (n-1)^t$ was assumed. Let Q_1 be any vertex of G_1.

Assume that $G_1, G_2, \ldots, G_i$ and $Q_1, Q_2, \ldots, Q_i$ are already defined for some $i < n$; moreover, (i)–(iii) are satisfied. Define G_{i+1} and Q_{i+1} as follows.

Let A denote the set of neighbours of Q_i in G_i. Let

$$B = V(G_i) - (A \cup \{Q_i\}).$$

The graph G_A induced by A in G satisfies $\omega(G_A) \leq t$ because the presence of a $(t+1)$-clique in G_A would give a $(t+2)$-clique in the subgraph induced by $A \cup \{Q_i\}$. Now the inductive hypothesis implies $\chi(G_A) \leq (n-1)^{t-1}$.

Assume that $B \neq \emptyset$. Now $\chi(G_i) \leqslant \chi(G_A)+\chi(G_B)$ since a good coloring of G_A with $\chi(G_A)$ colors, a good coloring of G_B with $\chi(G_B)$ new colors and an assignment of any color used on $V(G_B)$ to Q_i define a good coloring of G_i. Therefore

$$\begin{aligned}\chi(G_B) &\geqslant \chi(G_i)-\chi(G_A) > (n-i)(n-1)^{t-1}-(n-1)^{t-1}\\ &= \big(n-(i+1)\big)(n-1)^{t-1},\end{aligned}$$

which allows us to choose a connected component H of G_B satisfying $\chi(H) > \big(n-(i+1)\big)(n-1)^{t-1}$. Since G_i is connected by (i), there exists a vertex $Q_{i+1} \in A$ such that $V(H) \cup \{Q_{i+1}\}$ induces a connected subgraph which we choose as G_{i+1}. It is easy to check that $G_1, G_2, \ldots, G_{i+1}$ and $Q_1, Q_2, \ldots, Q_{i+1}$ satisfy the requirements (i)–(iii).

Assume that $B = \emptyset$. Now $\chi(G_i) \leqslant \chi(G_A)+1$, which implies

$$(n-i)(n-1)^{t-1} < (n-1)^{t-1}+1.$$

Consequently, $i = n-1$. Since $A \neq \emptyset$ by properties (i) and (ii) of G_i, Q_n can be defined as any vertex of A, $G_n = \{Q_n\}$.

The proof of Theorem 2.4 shows that for triangle-free graphs a stronger statement holds.

Corollary 2.5. *If G is a connected triangle-free graph of chromatic number n, then every vertex of G is an endpoint of an induced P_n in G.*

Let $f_n^*(x)$ denote the smallest χ-binding function of $\mathscr{G}(P_n)$. Then

$$\frac{R(\lceil n/2 \rceil, x+1)-1}{\lceil n/2 \rceil-1} \leqslant f_n^*(x) \leqslant (n-1)^{x-1}, \tag{1}$$

where the upper bound comes from Theorem 2.4 and the lower bound follows easily from the observation that an induced P_n in a graph G contains a stable set of size $\lceil n/2 \rceil$. The truth is probably close to the lower bound. For example, for $n = 4$ the lower bound is sharp, since the family $\mathscr{G}(P_4)$ is known to be perfect (see [35]).

Problem 2.6. Improve the lower or the upper bound of (1) for the smallest χ-binding function $f_n^*(x)$ of $\mathscr{G}(P_n)$.

Problem 2.7. What is the order of magnitude of $f_5^*(x)$?

Problem 2.8. Determine

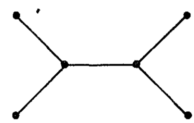
$$c = \lim_{n\to\infty} f_n^*(2)/n.$$

(It is easy to see that $1/2 \leqslant c \leqslant 1$.)

Combining the ideas of the proofs of Theorems 2.2 and 2.4, it is possible to prove that $\mathscr{G}(B)$ is χ-bound, where B denotes a broom. A *broom* is a tree defined by identifying an endvertex of a path with the center of a star. The broom is the maximal forest for which Conjecture 2.1 is known to be true in the following sense: if F is a forest which is not an induced subgraph of a broom, then Conjecture 2.1 is open. In particular, the following three special cases of Conjecture 2.1 are open problems:

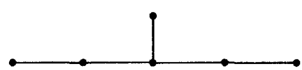
PROBLEM 2.9. Prove that $\mathscr{G}(F)$ is χ-bound for

$F =$



PROBLEM 2.10. Prove that $\mathscr{G}(F)$ is χ-bound for

$F =$

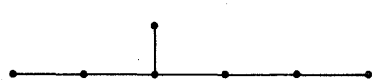


PROBLEM 2.11. Prove that $\mathscr{G}(2K_{1,3})$ is χ-bound.

It seems hard to attack the following special case of Conjecture 2.1: a χ-binding function $f(x)$ for $\mathscr{G}(F)$ can be defined at $x = 2$ if F is a forest. To settle this problem it is clearly enough to consider the case where F is a tree since every forest is an induced subgraph of some tree. Thus we have

CONJECTURE 2.12. *Let T be a tree and let G be a triangle-free graph which does not contain T as an induced subgraph. Then $\chi(G) \leqslant c$, where c is a constant depending only on T.*

Conjecture 2.12 was proved for trees of radius two in [23]. The smallest tree for which Conjecture 2.12 is open looks like:



PROBLEM 2.13. Prove Conjecture 2.12 for the tree above.

In what follows we consider problems concerning the smallest χ-binding functions of some special forests. The first example is mK_2, the union of m disjoint edges. Note that mK_2 is an induced subgraph of P_{3m-1}, therefore $\mathscr{G}(mK_2)$ is χ-bound by Theorem 2.4. Theorem 2.4 gives an exponential χ-binding function for $\mathscr{G}(mK_2)$. The methods used in [40] give better results.

THEOREM 2.14 (Wagon [40]). *The family $\mathscr{G}(mK_2)$ has an $O(x^{2(m-1)})$ χ-binding function.*

THEOREM 2.15 (Wagon [40]). *The function*

$$\binom{x+1}{2}$$

is a χ-binding function for $\mathscr{G}(2K_2)$.

Problem 2.16. What is the order of magnitude of the smallest χ-binding function for $\mathscr{G}(2K_2)$?

Problem 2.16 was posed in [40] and arose again in connection with a problem of Erdös and El-Zahar [9]. Wagon notes in [40] that $3x/2$ is a lower bound for the smallest χ-binding function of $\mathscr{G}(2K_2)$. A much better lower bound is

$$\frac{R(C_4, K_{x+1})}{3},$$

where $R(C_4, K_{x+1})$ denotes the smallest k such that every graph on k vertices contains either a clique of size $x+1$ or the complement of the graph contains C_4 (a cycle on four vertices). The above lower bound is non-linear because $R(C_4, K_t)$ is known to be at least $t^{1+\varepsilon}$ for some $\varepsilon > 0$ as proved by Chung in [5]. Concerning particular values of the smallest χ-binding function f^* for $\mathscr{G}(2K_2)$, it is easy to see that $f^*(2) = 3$. Erdös offered 20$ to decide whether $f^*(3) = 4$. The prize went to Nagy and Szentmiklóssy who proved [31] that $f^*(3) = 4$.

Now we turn our attention to the smallest χ-binding function of $\mathscr{G}(F)$, where F is a forest on four vertices. The number of such forests is six and three of them (P_4, S_4 and $2K_2$) have been discussed before. The smallest χ-binding function of $\mathscr{G}(4K_1)$ is asymptotically $\frac{1}{3}R(4, x+1)$ as the next proposition shows.

Proposition 2.17. *Let $f^*(x)$ be the smallest χ-binding function for $\mathscr{G}(4K_1)$. Then*

$$\frac{R(4, x+1)-1}{3} \leqslant f^*(x) \leqslant \frac{R(4, x+1)+2R(3, x+1)}{3} - 1.$$

Proof. The lower bound is obvious. Let p be the maximal number of disjoint three-vertex stable sets in $G \in \mathscr{G}(4K_1)$. Let $|V(G)| = 3p+q$; then $q \leqslant R(3, x+1)-1$ and

$$\chi(G) \leqslant p+q = \frac{|V(G)|+2q}{3} \leqslant \frac{R(4, x+1)-1+2\big(R(3, x+1)-1\big)}{3}$$

$$= \frac{R(4, x+1)+2R(3, x+1)}{3} - 1.$$

The smallest χ-binding function of the family $\mathscr{G}(P_3 \cup K_1)$ is asymptotically $\frac{1}{2}R(3, x+1)$.

Theorem 2.18. *Let $f^*(x)$ be the smallest χ-binding function of $\mathscr{G}(P_3 \cup K_1)$. Then*

$$\frac{R(3, x+1)-1}{2} \leqslant f^*(x) \leqslant \frac{R(3, x+1)+x-2}{2}.$$

The lower bound is obvious. The proof of the upper bound is based on the following lemma:

LEMMA 2.19. *Assume that* $G \in \mathscr{G}(P_3 \cup K_1)$ *and* $\alpha(G) \geqslant 3$. *Let* S *be a largest stable set of* G, *i.e.*, $|S| = \alpha(G)$. *Then* $\omega(G-S) = \omega(G)-1$.

Proof. Let

$$S = \{s_1, s_2, \ldots, s_\alpha\} \quad \text{and} \quad v \in V(G)-S.$$

Since $G \in \mathscr{G}(P_3 \cup K_1)$, v is adjacent either to exactly one vertex of S or to all vertices of S. Therefore, $V(G)-S = V_1 \cup V_2$, where $v \in V_1$ is adjacent to exactly one vertex of S and $v \in V_2$ is adjacent to all vertices of S. Let W be a clique of $V(G)-S$. Assume that $w_1, w_2 \in W \cap V_1$, $w_1 \neq w_2$, and $w_1 s_i \in E(G)$, $w_2 s_j \in E(G)$, $i \neq j$. Since $|S| \geqslant 3$, we can choose $s_k \in S$ such that $k \neq i$, $k \neq j$. Now $\{w_1, w_2, s_i, s_k\}$ (or $\{w_1, w_2, s_j, s_k\}$) induces $P_3 \cup K_1$ in G, which contradicts $G \in \mathscr{G}(P_3 \cup K_1)$. We conclude that all vertices of $W \cap V_1$ are adjacent to the same vertex, say $s_i \in S$. Clearly, s_i is adjacent to all vertices of $W \cap V_2$. Therefore, any clique of $V(G)-S$ can be augmented to a larger clique by adding a suitable vertex of S.

Proof of Theorem 2.18. The theorem is trivial if $\alpha(G) = 1$. Assume that $\alpha(G) = 2$ and let $x_1 y_1, x_2 y_2, \ldots, x_p y_p$ be a largest matching of $\bar{G}$. Let $q = |V(G)|-2p$; then $\chi(G) \leqslant p+q$ and $\omega(G) \geqslant q$. Thus

$$\chi(G) \leqslant p+q = \frac{|V(G)|+q}{2} \leqslant \frac{R(3, \omega(G)+1)-1+q}{2}$$

as stated in the theorem.

Now we can proceed by induction on $\omega(G)$. The case $\omega(G) = 1$ is trivial. The inductive step follows from Proposition 2.17 and from the fact that the Ramsey function $R(x, 3)$ is strictly increasing. Let $\alpha(G) \geqslant 3$ and let S be a stable set of size $\alpha(G)$. The inductive hypothesis can be applied to $G' = G-S$. Thus

$$\chi(G) \leqslant \chi(G')+1 \leqslant \frac{R(3, x)+x-2}{2}+1 \leqslant \frac{R(3, x+1)+x-1}{2}.$$

The sixth four-vertex forest which was not discussed yet is $P_2 \cup 2K_1$.

PROBLEM 2.20. What is the order of magnitude of the smallest χ-binding function for $\mathscr{G}(P_2 \cup 2K_1)$? The lower bound

$$\frac{R(3, x+1)-1}{2}$$

is obvious and it is easy to prove that

$$\binom{x+1}{2}+x-1$$

is an upper bound.

3. BINDING FUNCTIONS ON FAMILIES WITH AN INFINITE SET OF FORBIDDEN SUBGRAPHS

Let $\mathscr{H}$ be a set of graphs and let $\mathscr{G}(\mathscr{H})$ denote the family of graphs containing no graph of $\mathscr{H}$ as an induced subgraph:

$$\mathscr{G}(\mathscr{H}) = \{G\colon\ H \not\subset G \text{ for all } H \in \mathscr{H}\}.$$

In Section 2 we have dealt with χ-binding functions of $\mathscr{G}(\mathscr{H})$ for the case $|\mathscr{H}| = 1$. Now we are concerned with the case

$$\mathscr{H} = \{H_1, H_2, \ldots, H_i, \ldots\}.$$

If $H_i \in \mathscr{H}$ were acyclic for some i, then Conjecture 2.1 would imply that $\mathscr{G}(\mathscr{H})$ is a χ-bound family. Assume that, for some fixed k, $g(H_i) \leqslant k$ for all i, where $g(H_i)$ denotes the girth (the length of the smallest cycle) of H_i. By the basic result of Erdős and Hajnal (see [10]), one can define G_i as a graph of chromatic number i and girth of at least $k+1$ for all i. Consequently, the family

$$\mathscr{G} = \{G_1, G_2, \ldots, G_i, \ldots\}$$

is not χ-bound. Since $G_i \in \mathscr{G}(\mathscr{H})$ for all i, we observe that

PROPOSITION 3.1. *If $\mathscr{G}(\mathscr{H})$ is χ-bound, then*

$$\operatorname*{Sup}_{H \in \mathscr{H}} g(H) = \infty.$$

The most challenging open problem concerning perfect graphs is the *Strong Perfect Graph Conjecture.* Let us define $\mathscr{H}_0$ as

$$\{C_5, C_7, \ldots, C_{2i+1}, \ldots\}.$$

The Strong Perfect Graph Conjecture states that $\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}}_0)$ is the family of perfect graphs, i.e.,

$$\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}}_0) = \mathscr{P}.$$

Using our terminology, the Strong Perfect Graph Conjecture is equivalent to the statement that $\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}})$ is a χ-bound family with χ-binding function $f(x) = x$. Surprisingly, it is not even known if $\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}}_0)$ is χ-bound.

CONJECTURE 3.2 (*Weakened Strong Perfect Graph Conjecture*). *The family $\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}}_0)$ is χ-bound.*

The Strong Perfect Graph Conjecture gives a necessary and sufficient condition for perfectness in terms of forbidden subgraphs. To state similar conjectures for families having binding functions different from $f(x) = x$ seems to be difficult. Consider, for example, the family of graphs with θ-binding function $f(x) = x+1$. Graphs of that family do not contain the (disjoint) union of G_1 and G_2 as an induced subgraph, where $G_1, G_2 \in \mathscr{H}_0 \cup \bar{\mathscr{H}}_0$. The following proposition shows that "critical" graphs can be much more complicated. Since its proof is based on case analysis, we state it without proof.

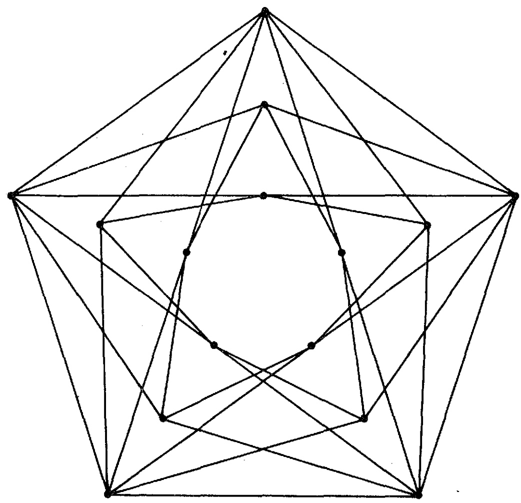


Fig. 1

PROPOSITION 3.3. *Let G be a graph shown in Fig. 1. Then $\theta(G) = \alpha(G) + 2$ and every induced proper subgraph $G' \subset G$ satisfies $\theta(G') \leqslant \alpha(G') + 1$.*

A natural way to show Conjecture 3.2 is to prove the following stronger conjecture:

CONJECTURE 3.4. *The family $\mathscr{G}(\mathscr{H}_0)$ is χ-bound.*

Perhaps Conjecture 3.4 can be strengthened further:

CONJECTURE 3.5. *The family $\mathscr{G}(\mathscr{H}_0^m)$ is χ-bound for all $m \geqslant 2$, where*

$$\mathscr{H}_0^m = \{C_{2m+1}, C_{2m+3}, \ldots\}.$$

A weaker version of Conjecture 3.5 seems to be also interesting:

CONJECTURE 3.6. *The family $\mathscr{G}(\mathscr{C}_l)$ is χ-bound for all $l \geqslant 4$, where*

$$\mathscr{C}_l = \{C_l, C_{l+1}, C_{l+2}, \ldots\}.$$

Note that $\mathscr{G}(\mathscr{C}_4)$ is the family of triangulated graphs which is perfect. However, for $l \geqslant 5$ the conjecture is open.

Special cases of the Strong Perfect Graph Conjecture are known to be true. Some of these results say that $\mathscr{G}(\mathscr{H})$ is perfect if

$$\mathscr{H} = \mathscr{H}_0 \cup \bar{\mathscr{H}}_0 \cup \{H\},$$

where H is a four-vertex graph. J. Lehel was curious about the four-vertex graphs H for which the perfectness of $\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}}_0 \cup \{H\})$ is not known. The

Perfect Graph Theorem reduces the eleven cases to six. The perfectness of $\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}}_0 \cup \{H\})$ is known in the following cases:

$H = K_4$ (Tucker [38]);

$H = K_4 - e$ (Parthasarathy and Ravindra [33]);

$H = K_{1,3}$ (Parthasarathy and Ravindra [32]);

$H = K_3 \cup e$, where e is an edge which has exactly one vertex in common with K_3 (a consequence of Meyniel's theorem [29], and a direct proof follows from Lemma 2.19);

$H = P_4$ (Seinsche proved [35] that $\mathscr{G}(P_4)$ is perfect).

It remains to solve

CONJECTURE 3.7 (J. Lehel). *The family*

$$\mathscr{G}(\mathscr{H}_0 \cup \bar{\mathscr{H}}_0 \cup \{C_4\}) = \mathscr{G}(\mathscr{H}_0 \cup \{C_4\})$$

is perfect.

4. BINDING FUNCTIONS ON FAMILIES HAVING A SELF-COMPLEMENTARY SET OF FORBIDDEN SUBGRAPHS

A family $\mathscr{G}$ of graphs is *self-complementary* if $\mathscr{G} = \bar{\mathscr{G}}$, i.e., $G \in \mathscr{G}$ if and only if $\bar{G} \in \mathscr{G}$. A self-complementary family $\mathscr{G}$ is χ-bound if and only if $\mathscr{G}$ is θ-bound. Moreover, if $\mathscr{G}$ is χ-bound, then the smallest χ-binding function of $\mathscr{G}$ is the same as the smallest θ-binding function of $\mathscr{G}$. Therefore, we can speak about binding functions of $\mathscr{G}$ without referring to χ or to θ. We mention two well-known families of perfect self-complementary graphs.

Permutation graphs (see [16]): graphs G such that both G and $\bar{G}$ are comparability graphs.

Split graphs (see [16]): graphs G such that both G and $\bar{G}$ are triangulated graphs. Equivalently, split graphs are graphs whose vertices can be partitioned into a clique and a stable set.

Let $\mathscr{H}$ be a family of graphs. Obviously, $\mathscr{G}(\mathscr{H})$ is self-complementary if and only if $\mathscr{H}$ is self-complementary. In what follows, we investigate binding functions of $\mathscr{G}(\mathscr{H})$ for self-complementary $\mathscr{H}$. To see some perfect families first, note that $\mathscr{G}(P_4)$ is perfect [35], $\mathscr{G}(C_4, 2K_2, C_5)$ is perfect and coincides with the family of split graphs as proved by Földes and Hammer [13]. A slightly more general result is in [21] (Theorem 3). The family $\mathscr{G}(C_4, 2K_2, P_4)$ is a subfamily of both previous families, thus it is perfect. The family contains the so-called threshold graphs (see [16]).

Concerning the existence of binding functions, the main open problem is a special case of Conjecture 2.1.

CONJECTURE 4.1. *The family $\mathscr{G}(F, \bar{F})$ has a binding function for every fixed forest F.*

It seems useful to look at some special cases of Conjecture 4.1. A straightforward attempt is to settle the following weaker versions of Problems 2.9–2.11:

PROBLEM 4.2. Prove Conjecture 4.1 for F from Problem 2.9.

PROBLEM 4.3. Prove Conjecture 4.1 for F from Problem 2.10.

PROBLEM 4.4. Prove Conjecture 4.1 for $F = 2K_{1,3}$.

Another problem is to determine or estimate the smallest binding function of $\mathscr{G}(F, \bar{F})$ when $\mathscr{G}(F)$ is known to be χ-bound. The rest of the section is devoted to problems and results of this kind.

PROBLEM 4.5. Estimate the smallest binding function of $\mathscr{G}(S_n, \bar{S}_n)$. ($S_n$ is a star on n vertices.)

Concerning special cases of Problem 4.5, note that the case $n = 3$ is trivial since $\mathscr{G}(S_3, \bar{S}_3)$ contains only cliques and their complements. The case $n = 4$ is settled by the following theorem (cf. Theorem 2.2):

THEOREM 4.6. *The smallest binding function of $\mathscr{G}(S_4, \bar{S}_4)$ (the claw and co-claw free graphs) is*

$$f(x) = \lfloor 3x/2 \rfloor.$$

Proof. Let G be a non-perfect member of $\mathscr{G}(S_4, \bar{S}_4)$. The result of Parthasarathy and Ravindra [32] implies that G contains an induced odd cycle or its complement. By symmetry we may assume that

$$C_{2k+1} = \{v_1, v_2, \ldots, v_{2k+1}\}$$

is an induced subgraph of G for some $k \geqslant 2$.

We claim that any vertex $x \in V(G) - V(C_{2k+1})$ is adjacent to all or to no vertices of C_{2k+1}.

To prove the claim assume that x is adjacent to v_i. If x is not adjacent to v_{i-1} and x is not adjacent to v_{i+1} (indices are taken modulo $2k+1$), then $\{v_{i-1}, v_i, v_{i+1}, x\}$ induces S_4 in G, a contradiction. We may assume that v_i and v_{i+1} are both adjacent to x. If there exists a vertex v_j in

$$C' = \{v_{i+3}, v_{i+4}, \ldots, v_{i-3}, v_{i-2}\}$$

such that v_j and x are not adjacent, then $\{v_j, v_i, v_{i+1}, x\}$ induces $\bar{S}_4$ in G, a contradiction. Thus x is adjacent to all vertices of C'. Assume that x is not adjacent to v_{i-1} or to v_{i+2}, say x and v_{i-1} are not adjacent. If $k = 2$, then x and v_{i+2} are adjacent (otherwise, $\{v_{i-1}, v_{i+1}, v_{i+2}, x\}$ would induce S_4); therefore $\{v_{i-1}, v_{i+1}, v_{i+2}, x\}$ induces $\bar{S}_4$. If $k \geqslant 3$, then $\{v_{i-1}, v_{i-3}, v_{i-4}, x\}$ induces $\bar{S}_4$. In all cases we have obtained a contradiction. Therefore x is adjacent to all vertices of C_{2k+1} and the claim is proved.

Let $V(G) - V(C_{2k+1}) = A \cup NA$, where A (NA) denotes the set of vertices adjacent (non-adjacent) to C_{2k+1}. We claim that either A or NA is empty.

Assume that $a \in A$, $b \in NA$ and $ab \in E(G)$. Let $v_i v_j \notin E(G)$; now $\{a, b, v_i, v_j\}$ induces S_4. Similarly, if $ab \notin E(G)$, then we choose i and j such that $v_i v_j \in E(G)$, and now $\{a, b, v_i, v_j\}$ induces $\bar{S}_4$. Thus the claim is true.

The theorem follows by induction on the number of vertices of $G \in \mathcal{G}(S_4, \bar{S}_4)$. The inductive step goes as follows.

Let $G \in \mathcal{G}(S_4, \bar{S}_4)$. If G is perfect, then

$$\chi(G) = \omega(G) \leqslant \left\lfloor \frac{3\omega(G)}{2} \right\rfloor.$$

Otherwise $G = C_{2k+1} \cup A$ or $G = C_{2k+1} \cup NA$ as was proved above. In the first case we use the inductive hypothesis for A:

$$\chi(G) = \chi(A)+3 \leqslant \left\lfloor \frac{3\omega(A)}{2} \right\rfloor + 3 = \left\lfloor \frac{3(\omega(G)-2)}{2} \right\rfloor + 3 = \left\lfloor \frac{3\omega(G)}{2} \right\rfloor.$$

In the second case we use the inductive hypothesis for NA: $\chi(G) = \chi(NA)$ and $\omega(G) = \omega(NA)$, so

$$\chi(G) = \chi(NA) \leqslant \left\lfloor \frac{3\omega(NA)}{2} \right\rfloor = \left\lfloor \frac{3\omega(G)}{2} \right\rfloor.$$

We have proved that $f(x) = \lfloor 3x/2 \rfloor$ is a binding function for $\mathcal{G}(S_4, \bar{S}_4)$. To see that it is the smallest one, let G_m be defined as follows. Consider K_m and remove the edges of $\lfloor m/5 \rfloor$ vertex disjoint C_5. Now it is easy to see that $G_m \in \mathcal{G}(S_4, \bar{S}_4)$ for all m; moreover,

$$\omega(G_{5k}) = 2k, \quad \chi(G_{5k}) = 3k, \quad \omega(G_{5k+1}) = 2k+1, \quad \chi(G_{5k+1}) = 3k+1.$$

Problem 4.7. Estimate the smallest binding function of $\mathcal{G}(P_n, \bar{P}_n)$ (cf. Theorem 2.4 and Problem 2.6).

Problem 4.8. What is the order of magnitude of the smallest binding function for $\mathcal{G}(P_5, \bar{P}_5)$? (Cf. Problem 2.7.)

Problem 4.9. What is the order of magnitude of the smallest binding function for $\mathcal{G}(mK_2, \overline{mK_2})$? (Cf. Theorem 2.14.)

The case $m = 2$ in Problem 4.9 is settled by the following theorem:

Theorem 4.10. *The smallest binding function for* $\mathcal{G}(2K_2, \overline{2K_2})$ *is* $f(x) = x+1$. (Cf. Problem 2.16.)

Proof. Let $G \in \mathcal{G}(2K_2, \overline{2K_2})$ and let S be a stable set of G such that $|S| = \alpha(G)$. Assume that $x, y \in V(G)-S$, $xy \notin E(G)$. The definition of S and $2K_2 \not\subset G$ imply that $\Gamma(x) \cap S$ and $\Gamma(y) \cap S$ are non-empty sets and one contains the other, say

$$\Gamma(x) \cap S \subseteq \Gamma(y) \cap S$$

($\Gamma(p)$ denotes the set of neighbours of $p \in V(G)$). Now $\overline{2K_2} \not\subset G$ implies $|\Gamma(x) \cap S| = 1$.

Let

$$K_1 = \{x:\ x \in V(G)-S,\ |\Gamma(x) \cap S| > 1\};$$

then K_1 is a clique in G by the argument above. We proceed to show that $V(G)-(S \cup K_1)$ is again a clique of G. Assume that $p, q \in V(G)-(S \cup K_1)$ and $pq \notin E(G)$. By definition,

$$|\Gamma(p) \cap S| = |\Gamma(q) \cap S| = 1.$$

However, $\Gamma(p) \cap S = \Gamma(q) \cap S$ contradicts the maximality of S, and $\Gamma(p) \cap S \neq \Gamma(q) \cap S$ contradicts the assumption $2K_2 \not\subset G$.

We have shown that the deletion of a stable set S of G results in a perfect graph (the complement of a bipartite graph). Thus

$$\chi(G) \leqslant \chi(G-S)+1 = \omega(G-S)+1 \leqslant \omega(G)+1,$$

showing that $f(x) = x+1$ is a binding function for $\mathscr{G}(2K_2, \overline{2K_2})$. To see that $f(x) = x+1$ is the smallest binding function, it is enough to consider complete graphs from which the edges of a C_5 are deleted.

The proof of Theorem 4.10 gives

COROLLARY 4.11. *If $G \in \mathscr{G}(2K_2, \overline{2K_2})$, then $V(G)$ can be partitioned into two cliques and a stable set. By symmetry, $V(G)$ can be also partitioned into two stable sets and a clique.*

Using Lemma 2.19 it is easy to prove

THEOREM 4.12. *Let F denote the forest $P_3 \cup K_1$. Then $\mathscr{G}(F, \bar{F})$ contains complete multipartite graphs and their complements, and moreover the graph C_5.*

Using the result of Parthasarathy and Ravindra [33] which proves the Strong Perfect Graph Conjecture for $\mathscr{G}(K_4-e)$ (or, equivalently, for $\mathscr{G}(K_2 \cup 2K_1)$), it is easy to derive

THEOREM 4.13. *Let F denote the forest $K_2 \cup 2K_1$. Then the non-perfect members of $\mathscr{G}(F, \bar{F})$ are*

1. *the graph of Fig. 2 and its non-perfect subgraphs;*
2. *a clique K whose vertices are adjacent to two consecutive vertices of a C_5;*
3. *the complements of the graphs defined in 1 and 2.*

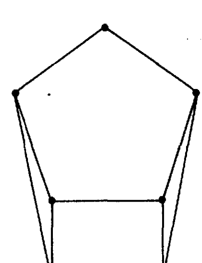


Fig. 2

Putting together the previous two theorems, we have the following corollary:

COROLLARY 4.14. *Let F denote either $P_3 \cup K_1$ or $K_2 \cup 2K_1$. Then the smallest binding function of $\mathscr{G}(F, \bar{F})$ is*

$$f(x) = \begin{cases} 3 & \text{if } x = 2, \\ x & \text{if } x > 2. \end{cases}$$

Before completing this section, note that the smallest binding function of $\mathscr{G}(F, \bar{F})$ was found for four-vertex forests F with one exception. The exceptional case occurs when $F = \bar{K}_4$, i.e., F is a stable set of four vertices. The family $\mathscr{G}(\bar{K}_4, K_4)$ is very eccentric since it is finite (like $\mathscr{G}(\bar{K}_m, K_m)$ in general for fixed m). Its smallest binding function $f^*(x)$ is determined by the values $f^*(2)$ and $f^*(3)$. It is easy to deduce that $f^*(2) = 3$ from the facts that $R(3, 4) = 9$ and that a graph G with $\omega(G) = 2$, $\chi(G) \geqslant 4$ satisfies $|V(G)| \geqslant 9$. (In fact, $|V(G)| \geqslant 11$ is true as proved by Chvatal in [6].) Is it possible to determine $f^*(3)$ without brute force?

5. BINDING FUNCTIONS ON UNION AND INTERSECTION OF GRAPHS

For graphs $G_1, G_2, \ldots, G_k$, the graphs $\bigcup_{i=1}^{k} G_i$ and $\bigcap_{i=1}^{k} G_i$ are usually defined as follows:

$$V\left(\bigcup G_i\right) = \bigcup V(G_i), \quad E\left(\bigcup G_i\right) = \bigcup E(G_i),$$
$$V\left(\bigcap G_i\right) = \bigcap V(G_i), \quad E\left(\bigcap G_i\right) = \bigcap E(G_i).$$

If $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_k$ are families of graphs, then their union is the family $\{\bigcup G_i\colon G_i \in \mathscr{G}_i\}$ and their intersection is the family $\{\bigcap G_i\colon G_i \in \mathscr{G}_i\}$. By definition, $\bigcap \mathscr{G}_i$ is a χ-bound family if and only if $\bigcup \bar{\mathscr{G}}_i$ is a θ-bound family. This fact combined with $\chi(G_1 \cup G_2) \leqslant \chi(G_1)\chi(G_2)$ gives the following obvious observation:

PROPOSITION 5.1. (a) *If $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_k$ are χ-bound families with binding functions $f_1, f_2, \ldots, f_k$, then $\bigcup \mathscr{G}_i$ is a χ-bound family and $\prod_{i=1}^{k} f_i$ is a suitable χ-binding function.*

(b) *If $\mathscr{G}_1, \mathscr{G}_2, \ldots, \mathscr{G}_k$ are θ-bound families with binding functions $f_1, f_2, \ldots, f_k$, then $\bigcap \mathscr{G}_i$ is a θ-bound family and $\prod_{i=1}^{k} f_i$ is a suitable θ-binding function.*

Proposition 5.1, trivial as it is, can sometimes be conveniently applied to prove the existence of binding functions.

COROLLARY 5.2. *Let $\mathscr{P}$ denote the family of all perfect graphs. The union (intersection) of k copies of $\mathscr{P}$ is χ-bound (θ-bound) with binding function x^k.*

PROBLEM 5.3. What is the smallest χ-binding function for $\mathscr{P} \cup \mathscr{P}$?

PROPOSITION 5.4. *The family of overlap graphs is θ-bound with θ-binding function x^2.*

Proof. Let $\mathscr{G}_1$ denote the family of co-interval graphs, and let $\mathscr{G}_2$ denote the family of interval inclusion graphs. Since $\mathscr{G}_1$ and $\mathscr{G}_2$ are perfect families, x^2 is a χ-binding function for $\overline{\mathscr{G}_1 \cup \mathscr{G}_2}$ by Corollary 5.2. The family of overlap graphs is a subfamily of $\overline{\mathscr{G}_1 \cup \mathscr{G}_2}$.

PROPOSITION 5.5. *The family of d-dimensional box graphs is θ-bound with θ-binding function x^d.*

Proof. The family in question is the intersection of d families of interval graphs and we can apply Corollary 5.2.

It is tempting to think that $\bigcap_{i=1}^{k} \mathscr{G}_i$ is χ-bound provided that $\mathscr{G}_i$ is χ-bound for $i = 1, 2, \ldots, k$. However, this is not the case. It may happen that $\mathscr{G}_1 \cap \mathscr{G}_2$ is not χ-bound although $\mathscr{G}_1$ and $\mathscr{G}_2$ are perfect families. A surprising construction of Burling [4] gives three-dimensional box graphs B_n for all positive integers n such that $\omega(B_n) = 2$ and $\chi(B_n) = n$. The result shows that $\mathscr{I} \cap \mathscr{I} \cap \mathscr{I}$ is not χ-bound, where $\mathscr{I}$ denotes the family of interval graphs. The analysis of Burling's construction shows moreover that $\mathscr{I} \cap \mathscr{J}$ is not χ-bound, where $\mathscr{J}$ is the family of "crossing graphs" of boxes in the plane. The vertices of crossing graphs are boxes in the plane and two vertices are adjacent if and only if the corresponding boxes cross each other. It is immediate to check that $\mathscr{J}$ is a subfamily of the family of comparability graphs. Note that $\mathscr{I} \cap \mathscr{I}$ is χ-bound with an $O(x^2)$ χ-binding function as proved by Asplund and Grünbaum [1]. Therefore the results in [4] and in [1] imply

THEOREM 5.6. *Let $\mathscr{I}$ and $\mathscr{C}$ denote the family of interval graphs and comparability graphs, respectively. Then*

(a) *$\mathscr{I} \cap \mathscr{I}$ is χ-bound;*

(b) *$\mathscr{I} \cap \mathscr{I} \cap \mathscr{I}$ is not χ-bound;*

(c) *$\mathscr{I} \cap \mathscr{C}$ is not χ-bound.*

Perhaps part (a) holds in a stronger form.

PROBLEM 5.7. Let $\mathscr{T}$ denote the family of triangulated graphs. Is $\mathscr{T} \cap \mathscr{T}$ χ-bound? In particular, is $\mathscr{T} \cap \mathscr{I}$ χ-bound?

Since the graphs of $\mathscr{T}$ can be represented as subtrees of a tree (see [16]), Problem 5.7 can be viewed as a geometrical problem.

The following result shows a pleasant property of comparability graphs.

PROPOSITION 5.8. *Let $\mathscr{C}$ denote the family of comparability graphs. The intersection of k copies of $\mathscr{C}$ is χ-bound and x^{2^k-1} is a suitable χ-binding function.*

Proof. Let $G_1, G_2, \ldots, G_k \in \mathscr{C}$ and assign a transitive orientation to the edges of G_i for all i $(1 \leqslant i \leqslant k)$. Assume that

$$xy \in E\Big(\bigcap_{i=1}^{k} G_i\Big).$$

The edge xy is oriented according to its orientation in G_k; moreover, we assign a type to it as follows. The type of xy is a 0-1 sequence of length $k-1$. For all j $(1 \leqslant j \leqslant k-1)$ the j-th element of the sequence is 0 if xy is oriented in G_j from x to y, and it is 1 otherwise. It is immediate to check that the edges of a fixed type of $\bigcap_{i=1}^{k} G_i$ define a transitively oriented graph. The number of possible types is at most 2^{k-1}, which implies that $\bigcap_{i=1}^{k} G_i$ can be written as the union of at most 2^{k-1} comparability graphs. Now the proposition follows from Corollary 5.2.

Problem 5.9. Estimate the smallest χ-binding function of $\mathscr{C} \cap \mathscr{C}$.

A subfamily of perfect graphs, the permutation graphs, occur in many applications. *Permutation graphs* can be defined as graphs G such that both G and $\bar{G}$ are comparability graphs. Corollary 5.2 and Proposition 5.8 give

Proposition 5.10. *Let k be fixed and consider the family $\mathscr{G}$ of graphs obtained by at most k applications of intersections and unions from permutation graphs. Then $\mathscr{G}$ is χ-bound and θ-bound.*

Now we want to determine the smallest θ-binding function of a family obtained as the union of k bipartite graphs. Observe that this family contains exactly the graphs of chromatic number at most 2^k. Therefore, we are interested in finding the smallest θ-binding function for the family $\mathscr{G}_m$ of at most m-chromatic graphs.

Proposition 5.11. *Let $f_m^*(x)$ denote the smallest θ-binding function for $\mathscr{G}_m$. Then*

(a) $f_m^*(x) \leqslant \lfloor (m+1)/2 \rfloor x$;

(b) $f_m^*(x) \geqslant (m/2)x$ *for* $x > x_0 = x_0(m)$.

Proof. It is trivial to cover the vertex set of an at most m-chromatic graph G by the vertices of at most $\lfloor (m+1)/2 \rfloor = s$ bipartite graphs $B_1, B_2, \ldots, B_s$. Now

$$\theta(G) \leqslant \sum_{i=1}^{s} \theta(B_i) = \sum_{i=1}^{s} \alpha(B_i) \leqslant s \cdot \alpha(G)$$

and (a) follows.

The lower bound is pointed out by Erdös, remarking that for $n \geqslant n_0$ and for arbitrary m there is a graph $G = G(n, m)$ on kn vertices satisfying $\alpha(G) = n$, $\omega(G) = 2$, and $\chi(G) = m$ (see [8]).

mn ?

PROPOSITION 5.12. *The smallest binding function $f_3^*(x)$ of $\mathscr{G}_3$ satisfies:*
(a) $f_3^*(x) \leqslant \frac{5}{3}x$;
(b) $f_3^*(x) \geqslant \frac{8}{5}x$ *if* x *is divisible by* 5.

Proof. First we prove (a). We may assume that $G \in \mathscr{G}_3$ is 3-chromatic. Let A_1, A_2, A_3 be the color classes of G in a good 3 coloring of $V(G)$. Let G_{12}, G_{13}, G_{23} be the subgraphs of G induced by $A_1 \cup A_2$, $A_1 \cup A_3$, $A_2 \cup A_3$, respectively. Since G_{ij} is a bipartite graph, $\theta(G_{ij}) = \alpha(G_{ij})$, which shows that $V(G_{ij})$ can be covered by at most $\alpha(G)$ cliques (vertices or edges) of G_{ij} for $1 \leqslant i < j \leqslant 3$.

We may assume that the clique cover of $V(G_{ij})$ covers all vertices of $V(G_{ij})$ exactly once. The cliques in the covers of $V(G_{12})$, $V(G_{13})$, $V(G_{23})$ form a clique cover of G with at most $3\alpha(G)$ elements and all vertices of G are covered exactly twice by these cliques. This cover can be partitioned into components where the cliques (edges and vertices) of each component are either the edges and the two endvertices of a path (allowing two identical vertices as a degenerate case) or the edges of a cycle of length divisible by 3. It is easy to check that the vertices of a component of m cliques can be covered by at most $5m/9$ cliques. These cliques are edges and vertices except for a component which forms a triangle; in this case the triangle is used instead of three edges. Therefore, we get a clique cover of $V(G)$ with at most

$$3\alpha(G) \cdot \frac{5}{9} = \frac{5\alpha(G)}{3}$$

cliques.

The lower bound (b) was guessed by Erdös who devised to find a graph G with $|V(G)| = 15$, $\alpha(G) = 5$, $\chi(G) = 3$, and $\omega(G) = 2$. Really, such a graph G exists as a subgraph of a 17-vertex graph H containing neither triangles nor six independent vertices (see H in [26]). The graphs containing disjoint copies of G form a family with θ-binding function $8x/5$ for the cases where x is divisible by 5.

PROBLEM 5.13. Let $f_3^*(x)$ be the smallest binding function of $\mathscr{G}_3$. Determine

$$\lim_{x \to \infty} f_3^*(x)/x.$$

(It is at least $\frac{8}{5}$ and at most $\frac{5}{3}$ by Proposition 5.12.)

6. COMPLEMENTARY BINDING FUNCTIONS AND STABILITY OF THE PERFECT GRAPH THEOREM

We say that a binding function f has a *complementary binding function* if the family $\mathscr{G}_f$ of graphs with θ-binding function f is χ-bound. The smallest χ-binding function of $\mathscr{G}_f$ is called the *complementary binding function* of f.

Note that θ and χ can change roles in the definitions. We are interested in the following general problem:

PROBLEM 6.1. Which binding functions have complementary binding functions and what are their complementary binding functions?

Using the notion of complementary binding function, the Perfect Graph Theorem states that $f(x)=x$ is a self-complementary binding function. (The converse statement is also true, see Theorem 6.7.)

One feels that only "small" functions may have complementary binding functions. This is really the case as the next theorem shows.

THEOREM 6.2. *If $f(x)$ has a complementary binding function, then* $\inf f(x)/x = 1$.

Proof. To prove the theorem, it is enough to show that $f_\varepsilon(x)=(1+\varepsilon)x$ has no complementary binding function if ε is a real number satisfying $0<\varepsilon\leqslant 1$. The proof is based on graphs defined by Erdös and Hajnal in [11]: for every $\varepsilon\in(0,1]$ and for every natural number k there exists a graph G_k^ε with the following properties:

$$(1) \qquad \chi(G_k^\varepsilon)=k,$$

$$(2) \qquad \frac{|V(G)|}{\alpha(G)}<2+\varepsilon \quad \text{for all induced subgraphs } G\subseteq G_k^\varepsilon.$$

Note that (2) implies that G_k^ε is a triangle-free graph. Therefore, (1) implies that the family $\mathcal{G}_\varepsilon=\{G_1^\varepsilon, G_2^\varepsilon, \dots\}$ is not χ-bound. We are going to prove that $\mathcal{G}_\varepsilon$ is a θ-bound family with θ-binding function $f_\varepsilon(x)$.

Let G be an induced subgraph of G_k^ε. We have to prove that $\theta(G)\leqslant(1+\varepsilon)\alpha(G)$. Since G is triangle-free, $\theta(G)=|V(G)|-\nu(G)$, where $\nu(G)$ is the cardinality of a maximal matching in G. We can express $\nu(G)$ by the Tutte–Berge formula (see [39] and [2]) as follows:

$$(3) \qquad \nu(G)=\min_{A\subseteq V(G)}\frac{|V(G)|+|A|-\sigma(H)}{2},$$

where H denotes the subgraph induced by $V(G)-A$ in G and $\sigma(H)$ denotes the number of odd components of H. Using (3) and $\theta(G)=|V(G)|-\nu(G)$, we can write $\theta(G)\leqslant(1+\varepsilon)\alpha(G)$ equivalently as

$$(4) \qquad \alpha(G)\geqslant\frac{|V(H)|+\sigma(H)}{2(1+\varepsilon)} \quad \text{for all } H\subseteq G.$$

In order to prove (4), let H be an induced subgraph of G with connected components $H_1, H_2, \dots, H_m$. Consider the partition of $\{1, 2, \dots, m\}$ into I_1, I_2, I_3 defined as follows:

$$(5) \qquad \begin{cases} i\in I_1 & \text{if } H_i \text{ is bipartite and } |V(H_i)| \text{ is even,}\\ i\in I_2 & \text{if } H_i \text{ is bipartite and } |V(H_i)| \text{ is odd,}\\ i\in I_3 & \text{if } H_i \text{ is not bipartite.}\end{cases}$$

We claim that

$$\text{(6)}\qquad \begin{aligned} \alpha(H_i) &\geq \frac{|V(H_i)|}{2} && \text{if } i \in I_1, \\ \alpha(H_i) &\geq \frac{|V(H_i)|+1}{2} && \text{if } i \in I_2, \\ \alpha(H_i) &> \frac{|V(H_i)|+1}{2(1+\varepsilon)} && \text{if } i \in I_3. \end{aligned}$$

The first two inequalities are obvious. To prove the third one, let C_{2t+1} be a minimal odd cycle of H_i for some $i \in I_3$. Using (2) for C_{2t+1}, we get

$$t = \alpha(C_{2t+1}) > \frac{2t+1}{2+\varepsilon}, \quad \text{i.e.,} \quad t > \frac{1}{\varepsilon},$$

which implies

$$\text{(7)}\qquad |V(H_i)| \geq 2t+1 > \frac{2}{\varepsilon}+1.$$

Observing that (7) is equivalent to

$$\frac{|V(H_i)|}{2+\varepsilon} > \frac{|V(H_i)|+1}{2(1+\varepsilon)},$$

and

$$\alpha(H_i) > \frac{|V(H_i)|}{2+\varepsilon}$$

by (2), we get the third inequality of (6).

Now we use (6) to estimate $\alpha(G)$. Clearly,

$$\begin{aligned} \alpha(G) \geq \sum_{i=1}^{m} \alpha(H_i) &= \sum_{i \in I_1} \alpha(H_i) + \sum_{i \in I_2} \alpha(H_i) + \sum_{i \in I_3} \alpha(H_i) \\ &\geq \frac{|V(H)| + |I_2 \cup I_3|}{2(1+\varepsilon)} \geq \frac{V(H)+\sigma(H)}{2(1+\varepsilon)} \end{aligned}$$

since $|V(H_i)|$ is even for $i \in I_1$ by (5). Thus we have proved (4) and the theorem follows.

Theorem 6.2 gives a necessary condition for the existence of complementary binding functions. Concerning sufficient conditions, the main open problem is the following

CONJECTURE 6.3. *The function $f(x) = x + c$ has a complementary binding function for any fixed positive integer c.*

Conjecture 6.3 is open even in the case $c = 1$. Probably this case already contains all the difficulties. An evidence supporting Conjecture 6.3 is the following result:

PROPOSITION 6.4. *If $\mathcal{G}$ is the family of graphs with θ-binding function $f(x) = x+c$, then, for all $G \in \mathcal{G}$, $\omega(G) = 2$ implies $\chi(G) \leqslant 6c+2$.*

Proof. Assume that $G \in \mathcal{G}$, $\omega(G) = 2$. Clearly,

$$\frac{|V(G)|}{2} \leqslant \theta(G) \leqslant \alpha(G)+c,$$

which implies

$$\alpha(G) \geqslant \frac{|V(G)|-2c}{2}. \tag{8}$$

Let C_1 be an odd cycle of minimal length in G, let C_2 be an odd cycle of minimal length in the subgraph induced by $V(G)-V(C_1)$ in G, etc. We continue to define $C_1, C_2, \ldots, C_m$ until the subgraph induced by

$$V(G) - \bigcup_{i=1}^{m} V(C_i)$$

in G does not contain odd cycles. Applying (8) to the subgraph C induced by $\bigcup_{i=1}^{m} V(C_i)$ in G, we get

$$\frac{|V(C)|-2c}{2} \leqslant \alpha(C) \leqslant \sum_{i=1}^{m} \alpha(C_i) = \frac{|V(C)|-m}{2}$$

from which $m \leqslant 2c$ follows. A good coloring of $V(G)$ can be defined by coloring $V(C)$ with $3m$ colors and using two additional colors for the bipartite graph induced by $V(G)-V(C)$. Therefore,

$$\chi(G) \leqslant 3m+2 \leqslant 6c+2.$$

By a deep result of Folkman [12] which answers a conjecture of Erdös and Hajnal, condition (8) implies $\chi(G) \leqslant 2c+2$. Therefore, Proposition 6.4 holds with $2c+2$ instead of $6c+2$.

The existence of complementary binding functions is known only for "very small" functions. We mention a modest result of this type.

PROPOSITION 6.5. *Let t be a fixed positive integer. If $f(x)$ is a binding function such that $f(x) = x$ for all $x \geqslant t$, then $f(x)$ has a complementary binding function.*

It does not seem to be a trivial problem to determine the complementary binding functions of *any* function different from $f(x) = x$. Perhaps the simplest problem of this type is

PROBLEM 6.6. Let f be the binding function defined as

$$f(x) = \begin{cases} x & \text{if } x \neq 2, \\ 3 & \text{if } x = 2. \end{cases}$$

What is the complementary binding function of f? Perhaps $\lfloor 3x/2 \rfloor$ is the truth.

The following result shows that the Perfect Graph Theorem is stable in a certain sense:

THEOREM 6.7. *If $f(x)$ is a self-complementary binding function, then $f(x) = x$ for all positive integers.*

Proof. Assume that f is self-complementary.

Case 1. Assume that $f(2) = 2$. If $f(x) \neq x$ for some $x \in N$, then we can choose $k \in N$ such that $k \geqslant 3$, $f(k) > k$ and $f(x) = x$ for $x < k$. Clearly, f is a θ-binding function for $\{C_{2k+1}\}$ but fails to be a χ-binding function for $\{C_{2k+1}\}$, i.e., f is not self-complementary. The contradiction shows that $f(x) = x$ for all $x \in N$.

Case 2. Assume that

$$f(2) > 2 \quad \text{and} \quad f(k) < \lceil (3k-1)/2 \rceil \text{ for some } k.$$

Consider the graph G_k whose complement is $\lfloor k/2 \rfloor$ disjoint C_5 and, for odd k, an additional isolated vertex. Now f is a θ-binding function for $\{G_k\}$ ($\alpha(G_k) = 2$, $\theta(G_k) = 3$) but fails to be a χ-binding function for $\{G_k\}$ ($\omega(G_k) = k$, $\chi(G_k) = \lceil (3k-1)/2 \rceil$).

Case. 3. $f(k) \geqslant \lceil (3k-1)/2 \rceil$ for all $k \in N$. In this case Theorem 6.2 implies that $f(x)$ has no complementary binding function, again a contradiction.

A generalization of the Perfect Graph Theorem (proved also by Lovász in [28]) states that a graph G is perfect if $\alpha(G') \cdot \omega(G') \geqslant |V(G')|$ holds for all induced subgraphs G' of G. The first step in searching analogous properties would be to settle

PROBLEM 6.8. Let $\mathcal{G}$ be a family of graphs G satisfying

$$\alpha(G') \cdot \omega(G') \geqslant |V(G')| - 1$$

for all induced subgraphs G' of G. Is it true that $\mathcal{G}$ is a χ-bound (or, equivalently, θ-bound) family? If yes, what is the smallest binding function for $\mathcal{G}$?

Acknowledgment. I should like to express my gratitude to the referee, to the technical editor and to all others whose work improved my manuscript.

Added in proof. Problems 1.7 and 1.8 are answered by A. Kostochka. Problems 2.11 and 4.4 are easy.

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