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NOTE

THE IRREGULARITY STRENGTH OF $K_{m,m}$ IS 4 FOR ODD m

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G. Chartrand et al. showed in [1] that for odd m, $m \geqslant 3$, the edges of $K_{m,m}$ can be labeled with $1, 2, 3, 4$ in such a way that the (weighted) degrees of the vertices are all different. They conjectured that no such labeling exists with labels $1, 2, 3$. In this note we prove this conjecture.

The irregularity strength $s(G)$ of a graph of G was introduced in [1] as the minimum integer t for which the edges of G can be weighted with $1, 2, \ldots, t$ in such a way that the weighted degrees of G are distinct numbers. The value of $s(G)$ has been determined for several graphs in [1] and [2].

For complete bipartite graphs $K_{m,m}$ it was proved that $s(K_{m,m}) = 3$ for even m (see in [1]) and in [2] $s(K_{m,n})$ was determined for all pairs m, n except for the case when $m = n = 2k + 1$. Concerning this case, it was shown in [1] that $s(K_{m,m}) \leqslant 4$ for odd m ($m \geqslant 3$) and it was conjectured that $s(K_{m,m}) = 4$. It was noted in [2] that the conjecture can be formulated as follows: if m is odd and A is an $m \times m$ matrix with entries -1, 0, $+1$, then the row and column sums of A cannot be all different. The purpose of this note is to prove this conjecture.

Theorem. *Let m be an odd positive integer. If the edges of $K_{m,m}$ are weighted with $1, 2, 3$ then there exist two vertices with the same (weighted) degree.*

Proof. Assume that for some weight-assignment (with weights $1, 2, 3$) all degrees of $K_{m,m}$ are different. Subtracting 1 from all edge weights we get a bipartite graph $G = (A, B)$, $|A| = |B| = m$ with the edge weights 1 and 2 such that all (weighted) degrees of G are different. Since the possible degrees of vertices of G are $0, 1, 2, \ldots, 2m$ and all the degrees are different, the degree set of G can be written as $\{0, 1, 2, \ldots, 2m\} - \{i\}$ for some i, $0 \leqslant i \leqslant 2m$. It is obvious that i must be odd otherwise G would contain an odd number of vertices of odd degree.

Let S denote the set of vertices with the smallest m degrees and set $L = V(G) - S$, i.e. L is the set of vertices with the largest m degrees.

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Denote by $W(S, L)$ the sum of weights over the subset of edges of G with one endpoint in S and the other endpoint in L. Set $A_L = A \cap L$, $B_L = B \cap L$, $a = |A_L|$, $b = |B_L|$. The degree of $x \in A_L$ in L is at most $2b$ and the degree of $x \in B_L$ in L is at most $2a$. Therefore

$$|W(S, L)| \geq \sum_{x \in A_L} (d(x) - 2b) + \sum_{x \in B_L} (d(x) - 2a) = \sum_{x \in L} d(x) - 4ab.$$

(Here and in what follows $d(x)$ denotes the weighted degree of x.) Since $a + b = m$ and m is odd, the maximum value for ab is $\frac{1}{2}(m-1) \cdot \frac{1}{2}(m+1)$. Therefore

$$|W(S, L)| \geq \sum_{x \in L} d(x) - (m^2 - 1). \tag{1}$$

We consider two cases.

Case 1. $i \leq m$. Since the degrees of the vertices of L are $m+1, m+2, \ldots, 2m$, (1) gives

$$|W(S, L)| \geq \frac{(3m+1)m}{2} - (m^2 - 1) = \frac{m^2 + m + 2}{2}. \tag{2}$$

On the other hand, the degrees of the vertices of S are $\{0, 1, 2, \ldots, m\} - \{i\}$ so their sum is maximum for $i = 1$. Therefore

$$|W(S, L)| \leq 0 + 2 + 3 + \cdots + m = \frac{m^2 + m - 2}{2},$$

contradicting (2).

Case 2. $i \geq m + 2$. The degree set of L is $\{m, m+1, \ldots, 2m\} - \{i\}$ so that the minimum sum of the degrees of vertices in L occurs for $i = 2m - 1$. Using (1) we have

$$|W(S, L)| \geq m + (m+1) + \cdots + (2m-2) + 2m - (m^2 - 1) = \frac{m^2 - m + 4}{2}. \tag{3}$$

On the other hand, the degree set of S is $\{0, 1, 2, \ldots, m-1\}$. Therefore

$$|W(S, L)| \leq 0 + 1 + 2, \ldots, m - 1 = \frac{m^2 - m}{2},$$

contradicting (3). □

References

[1] G. Chartrand, M. Jacobson, J. Lehel, O. Oellerman, S. Ruiz and F. Saba, Irregular networks, Submitted to Fort Wayne Conference Proceedings.

[2] R. Faudree, M. Jacobson, J. Lehel and R. H. Schelp, Irregular networks, Regular graphs and integer matrices with distinct row and column sums, Submitted to Discrete Math.