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CLUMSY PACKING OF DOMINOES

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One places dominoes on a chessboard. If there is no room for further dominoes, the board is said to be full. Here we investigate the question: what is the minimum number $d(n)$ of dominoes lying on an $n \times n$ full board. We prove that $d(n) = n^3/3$ if 3 divides n and $n^3/3 + n/111 < d(n) < n^2/3 + n/12 + 1$ for $n = 3k \pm 1$ if n is large. The same question is discussed also for triangulated and hexa boards.

1. Introduction

One places dominoes on an $n \times n$ chessboard. Each domino must cover exactly two squares of the board and no two dominoes may overlap. The board is said to be *full* if there is no room for placing further dominoes (Fig. 1). The problem of clumsy packing is formulated as follows: what is the minimum number $d(n)$ of dominoes lying on an $n \times n$ full board?

In fact, the clumsy packing of dominoes can be viewed as the worst case of the 'greedy' packing.

That kind of domineering is known under various names as a two person game (Cram, Plugg, Dots and pairs). Either player may place his dominoes in either direction and the player who finds no room for his next domino, loses (c.f. [1]). For this game the obvious meaning of the function $d(n)$ is that the game should not be finished in less than $d(n)$ moves. Here we prove that $d(n) = n^3/3$ if $3 \mid n$ and $d(n) > n^2/3 + n/111$ if n is large and not divisible by 3.

Let G be a graph. A *matching* of G is a set of independent edges. The ratio of the minimum and maximum size of a maximal matching, $r(G)$, measures the worst case behaviour of the greedy matching algorithm. It was proved in [2] that $r(G) \geq 1/2$ for any graph G.

If we consider the graph $G(n)$ whose vertices are the squares of the board and whose edges correspond to squares having a common side, then the packing of dominoes corresponds to a matching, i.e. a set of independent edges of $G(n)$. The meaning of the function $d(n)$ is obviously the minimum size of a maximal matching. The maximum size of a maximal matching of $G(n)$ is about $n^2/2$, therefore, our result shows that $r(G)$ is about 2/3 for chessboard graphs.

In Section 3, the problem of clumsy packing, or equivalently, the ratio $r(G)$ is

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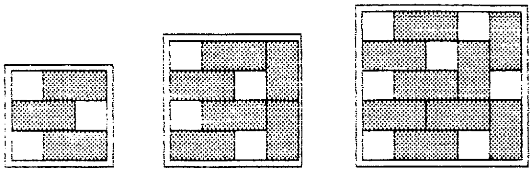

Fig. 1. Minimum full boards for $n=3$, 4 and 5.

investigated also for triangulated and hexa boards. In both cases, $r(G)$ is again about 2/3. The maximum matching in graphs corresponding to triangulated boards has been discussed in [3].

2. Packing of the chessboard

Squares of a full board which are not covered by dominoes are called *holes*. Let us denote by H and D, respectively, the number of holes and dominoes on a full board.

Proposition 2.1. *A full board contains at least as many dominoes as holes, i.e.* $H \leqslant D$.

Proof. Denote by $D(\text{top})$ and $D(\text{bot})$ the number of dominoes which meet the top row and the bottom row of the board, respectively. Let $W(\text{top})$ and $W(\text{bot})$ be the number of dominoes whose top side and whose bottom side touches only domino(es), respectively. Then, obviously,

$$D = W(\text{top}) + D(\text{top}) + D' \tag{1}$$

and

$$D = W(\text{bot}) + D(\text{bot}) + D'', \tag{2}$$

where D' and D'' denote the number of dominoes not yet counted.

Denote by $H(\text{top})$ and $H(\text{bot})$ the number of holes in the top row and in the bottom row of the board, respectively, and put

$$H = H(\text{bot}) + H' \tag{3}$$

and

$$H = H(\text{top}) + H''. \tag{4}$$

Observe that any hole not in the bottom row of the board touches the top side of some domino; furthermore, there is no domino whose top side touches two holes. Consequently, there is a bijection from the set of holes not in the bottom row into the set of all dominoes counted in D', that is

$$H' = D'. \tag{5}$$

The same argument shows that

$$H'' = D''. \tag{6}$$

Then, using equalities (1)–(6), we obtain:

$$2(D - H) = W(\text{top}) + W(\text{bot}) + D(\text{top}) - H(\text{top}) + D(\text{bot}) - H(\text{bot}). \tag{7}$$

Now, clearly,

$$D(\text{top}) - H(\text{top}) \geqslant -1 \tag{8}$$

and

$$D(\text{bot}) - H(\text{bot}) \geqslant -1 \tag{9}$$

Therefore

$$D - H \geqslant \frac{(W(\text{top}) + W(\text{bot}))}{2} - 1 \tag{10}$$

and equality occurs if and only if there is equality in (8), (9), furthermore,

$$W(\text{top}) = W(\text{bot}) = 0. \tag{11}$$

An alternating chain of holes and dominoes, starting with a hole or with a domino, is called *vertical* (*horizontal*) if neighbouring elements have a common side and distinct elements meet distinct rows (columns) of the board.

Then (11) implies that the set of dominoes and holes, called *elements,* is partitioned into vertical alternating chains. Supposing equality in (8) and (9) it follows that the top row and the bottom row of the board are covered by horizontal alternating chains starting and terminating with holes.

Clearly, all of the above observations are valid if the leftmost and rightmost columns are considered as the top and the bottom of the board, respectively. In particular, the set of elements on the board can be partitioned into horizontal alternating chains and the leftmost and rightmost columns should be covered by vertical alternating chains starting and terminating with holes.

In case of $n = 3k$, $D - H \geqslant 0$ follows from (10) and the obvious equality $H + 2D = n^2$.

In case of $n = 3k \pm 1$ first we prove that any domino meeting the top or bottom row is in horizontal position. Suppose on the contrary that $s1$, $s2$ and $s3$ are three consecutive squares of the top row such that $s1$ and $s3$ are holes and $s2$ is covered by a domino in vertical position. It is easy to see that the three vertical alternating chains started at this "3-pattern" should contain only vertical dominoes which is impossible if n is not divisible by 3. The same argument shows that any domino meeting the leftmost or rightmost columns is in vertical position. This is possible only if $n-1$ is divisible by 3.

Consequently, $D-H \geqslant 0$ for $n=3k-1$. Moreover, if $n=3k+1$ then the boundary rows and columns containing $4k$ dominoes and $4k$ holes surround a $(3k-1)\times(3k-1)$ full board, therefore, by the previous observation, $D-H=(D-4k)-(H-4k)\geqslant 0$ follows. □

Since $H+2D=n^2$, Proposition 2.1 yields the lower bound:

$$\left[\frac{n^2+2}{3}\right] \leqslant d(n).$$

This bound is sharp for every n divisible by 3. Indeed, form alternating chains of holes and horizontal dominoes starting with holes in every second row. Figs 1 and 2 show that the lower bound is sharp also for small values of n. So we obtain

Theorem 2.2. *$d(n)=[(n^3+2)/3]$ if $2 \leqslant n \leqslant 12$ and for every n divisible by* 3.

For $n=3k\pm 1$ and $n>12$, $d(n)$ is not known. The best upper bound we know is given by the following proposition.

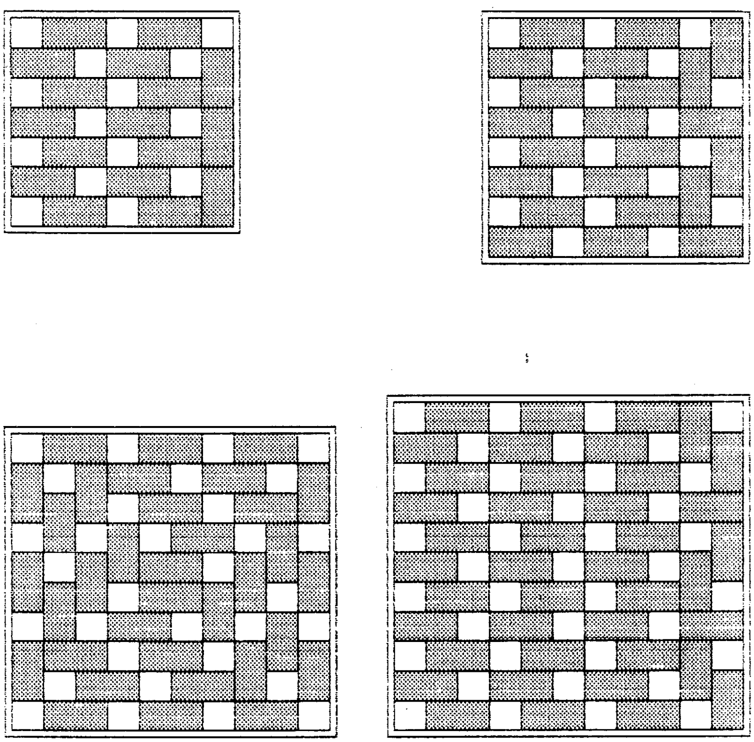

Fig. 2. Minimum full boards for $n=7$, 8, 10 and 11.

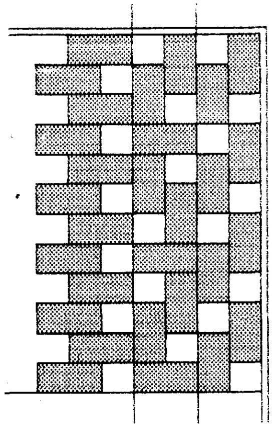

Fig. 3. Packing of the last columns for $n = 3k \pm 1$.

Proposition 2.3. $d(n) < n^2/3 + n/12 + 1$.

Proof. For $n = 3k - 1$, let us start with an $n \times (n-2)$ full board containing $n(n-2)/3$ dominoes and complete it by adding two columns packed with at most $(3n+2)/4$ dominoes as it is shown in Fig. 3. For $n = 3k + 1$, we start with an $n \times (n-2)$ full board packed as above and we complete the last two columns as it is shown in Fig. 3. (In both cases the packing of the bottom part of the last columns should be completed according to $n \bmod 12$.) $\square$

One may think, however, that the term $O(n)$ is superfluous in Proposition 2.3, that is there must be constructions of full boards containing at most $n^2/3 + c$ dominoes with some constant c. Surprisingly, this is not the case. We show that, in fact, $D - n^3/3$ tends to infinity with n if $n = 3k \pm 1$.

Theorem 2.4. *If n is large and $n = 3k \pm 1$ then $d(n) > n^2/3 + n/111$.*

Proof. Dominoes on a "sparse" full board are normally surrounded by holes. First we give a lower bound on $D - H$, as a function of the number of so called wrong dominoes which have a side touching domino(es) only. Then we show that in case of $n = 3k \pm 1$ there exist at least $n/9 - o(n)$ wrong dominoes.

Consider a full board containing $D = d(n)$ dominoes and H holes. By Proposition 2.3, there are $D + H = 2n^2/3 - O(n)$ elements (dominoes and holes) on the board. Call a domino *wrong* if it has a side not incident to a border line of the board and touching no hole. Denote by W the number of all wrong dominoes on the board.

Claim 1. $D - H \geq W/4 - 1$.

Proof. Denote by $W(\text{top})$ and $W(\text{bot})$ the number of wrong dominoes which have a top side and which have a bottom side touching domino(es) only. W.l.o.g. one can assume that

$$W(\text{top}) + W(\text{bot}) \geqslant \frac{W}{2}. \tag{12}$$

By using (12), the lower bound (10) obtained in the proof of Proposition 2.1 proves the claim. □

We call a horizontal or vertical alternating chain *perfect* if

(i) the chain contains no wrong dominoes;

(ii) the first and last elements of the chain meet two opposite (left and right or top and bottom) borderlines of the board.

Let a and b denote the number of vertical and horizontal perfect chains, respectively. W.l.o.g. one can assume that $a \geqslant b$. Obviously, there are exactly ab dominoes and holes, called *perfect elements,* belonging to both horizontal and vertical perfect chains.

An example is shown in Fig. 4, where the starting elements of the perfect chains are indicated by arrows, the perfect dominoes are black and the perfect holes are filled with dots.

Elements meeting the top row and belonging to vertical perfect chains are called *a-holes, a-dominoes* or *a-elements.*

Claim 2. If $n = 3k \pm 1$ then $a < 2n/3 + 1$.

Proof. Let $e1$, $e2$ and $e3$ be a-elements meeting three consecutive squares $s1$, $s2$, $s3$ of the top row; furthermore, suppose that $e1$ and $e3$ are a-holes. Then by the definition of perfect chains, any domino which belongs to the vertical perfect chains starting at $e1$, $e2$ and $e3$ should be in vertical position (see Fig. 5a). Obviously, this is possible only if $3 \mid n$. We say that $s1$, $s2$ and $s3$ form a 3-*pattern.*

Let fi, $1 \leqslant i \leqslant 5$, be elements meeting five consecutive squares of the top row

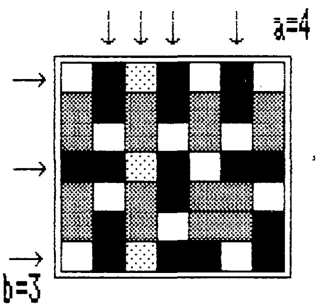

Fig. 4. Perfect chains and elements.

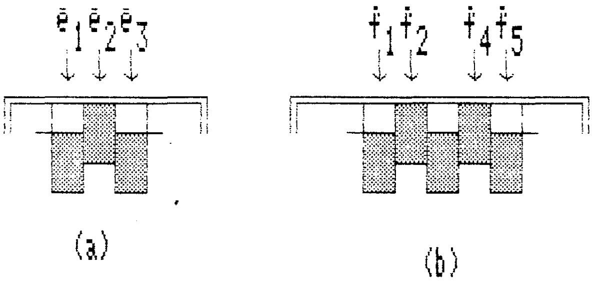

Fig. 5. The 3-pattern and the 5-pattern.

and suppose that $f1$, $f5$ are a-holes and $f2$, $f4$ are a-dominoes placed in vertical position (see Fig. 5b). The same argument as above shows that this can occur only if $3 \mid n$. We say that the five squares of the top row covered by fi, $1 \leqslant i \leqslant 5$, define a *5-pattern*.

Thus we conclude that 3-patterns and 5-patterns are excluded when $n = 3k \pm 1$.

Now we define two further patterns which can occur for any n.

Five consecutive squares, si, $1 \leqslant i \leqslant 5$, in the top row are called an *s-sequence* if $s2$ and $s4$ are covered by a-dominoes in vertical position and $s3$ is an a-hole (see Fig. 6a). Since 5-patterns are excluded, s-sequences are disjoint.

Consecutive squares covered by an alternating chain of dominoes in horizontal position and holes lying in the top row are called a *t-sequence* if the first and last elements of the chain are a-dominoes and the alternating chain is maximal with this property (see Fig. 6b). Note that t-sequences are disjoint by definition.

The maximal sequences of consecutive squares of the top row which do not belong to any s- and t-sequences are called *u-sequences*. Clearly s-, t- and u-sequences define a partition of the top row.

Observe that a-elements meeting the u-sequences are either holes or dominoes in vertical position. (Any horizontal a-domino belongs to a t-sequence.) Furthermore, among these a-elements there are no three which meet consecutive squares, since 3-patterns are excluded.

Thus a u-sequence of length $l = 3p + q$ ($p \geqslant 0$ and $q = 0$, 1 or 2) meets $a(l) \leqslant 2p + q$ a-elements.

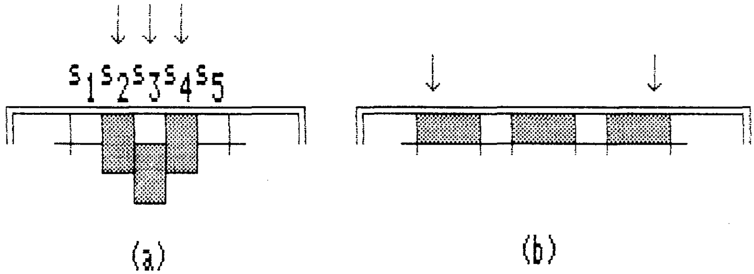

Fig. 6. The s-sequence and the t-sequence.

It is easy to see that if $q = 0$ or $a(l) < 2p + q$ then $a(l) \leqslant 2l/3$ holds. Thus if U denotes the number of squares of the top row covered by such u-sequences and $a(U)$ is the number of a-elements meeting these u-sequences then

$$a(U) \leqslant 2U/3. \tag{13}$$

Suppose now that $a(l) = 2p + q$, $q = 1$ or 2. Then any domino meeting the u-sequence is in vertical position and the first and last q squares of the u-sequence are a-elements (since any three consecutive positions may contain at most two a-elements). Inspection shows that $q = 2$ is possible only if no s- and t-sequences occur and then $n = 3p + 2$, $a = 2p + 2 < 2n/3 + 1$.

Let us suppose now that a u-sequence of length $l = 3p + 1$ is followed by an s-sequence, and consider the number $a(l + 5)$ of a-elements meeting the 'us-sequence' obtained by their concatenation. Then, obviously, $a(l + 5) = 2p + 4 = 2(l + 5)/3$ and therefore,

$$a(US) = \frac{2US}{3}, \tag{14}$$

where $a(US)$ is the number of all a-elements meeting that type of concatenated sequences having total length US.

If the s-sequences which are not concatenated with u-sequences cover S squares then for the number $a(S)$ of a-elements meeting them, we obtain:

$$a(S) = 3S/5 \leqslant 2S/3. \tag{15}$$

Suppose now that a u-sequence of length $l = 3p + 1$ which covers $2p + 1$ a-elements is followed by a t-sequence of length $m = 3r + 2$. Then for the number $a(l + m)$ of all a-elements meeting the "ut-sequence" obtained by their concatenation we have $a(l + m) \leqslant 2p + 1 + 2r + 1 = 2(l + m)/3$. Therefore,

$$a(UT) \leqslant \frac{2UT}{3}, \tag{16}$$

where $a(UT)$ denotes the number of all a-elements meeting the ut-sequences having total length UT.

If the t-sequences which are not concatenated with u-sequences cover T squares of the top row, then for the number $a(T)$ of all a-elements meeting those squares we obtain

$$a(T) < \frac{2T}{3}. \tag{17}$$

Finally, if the rightmost sequence of the top row is a u-sequence of length $L = 3p + 1$ which meets $a(L) = 2p + 1$ a-elements, then

$$a(L) < \frac{2L}{3} + 1. \tag{18}$$

Since $U + US + S + UT + T + L = n$, $a < 2n/3 + 1$ follows by adding inequalities (13)–(18). □

Claim 3. $W \geqslant n/9 - o(n)$ for $n = 3k \pm 1$.

Proof. By definition, an element is non-perfect only if the horizontal or vertical alternating chain containing it terminates in a wrong domino. Thus any wrong domino excludes at most $2n$ elements from the set of perfect elements. Therefore, $ab \geqslant D + H - 2Wn$, and since $D + H = 2n^2/3 - O(n)$, we obtain:

$$ab \geqslant \frac{2n^2}{3} - 2Wn - O(n). \tag{19}$$

On the other hand, by Claim 2,

$$ab \leqslant \frac{4n^2}{9} + O(n). \tag{20}$$

The upper bound on the number of perfect elements in (20) should be at least of the same order of magnitude as the lower bound in (19), which means that $W \geqslant n/9 - o(n)$. □

To finish the proof of Theorem 2.3, observe that

$$2D + H = n^2, \tag{21}$$

and by using Claims 1 and 3,

$$D - H \geqslant W/4 - 1 \geqslant \frac{n}{36} - o(n) > \frac{n}{37} \tag{22}$$

holds if n is large. From (21) and (22), $D = d(n) > n^2/3 + n/111$ follows for large n. □

The smallest unknown value for the clumsy packing is $d(13)$. For $n = 13$ we know full boards containing at least 58 dominoes, perhaps this is the first case when the lower bound $(n^2 + 2)/3 = 57$ is not tight.

3. Packing of triangulated and hexa boards

A triangulated n-board is an equilateral triangle with side of n units decomposed into n^2 unit triangles. One places rombus-shaped "dominoes" on the board which cover just two unit triangles having a common side. A board is said to be full if there is no room for further dominoes. One can see that the maximum number of dominoes on a full n-board is $n(n-1)/2$. Denote by $\Delta(n)$ the clumsy packing function for triangulated n-boards, which is defined as the minimum number of dominoes on a full n-board.

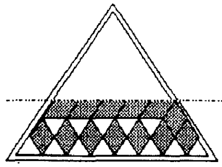

Fig. 7. Packing the last three rows.

Theorem 3.1. $n(n-1)/3 \leqslant \Delta(n) \leqslant [n^2/3]$.

Proof. To prove the upper bound, we have to construct full n-boards containing $[n^2/3]$ dominoes. For $n = 1$, 2 and 3 the statement is obvious. If $n > 3$ then we can pack the last three rows with $2n-3$ dominoes as shown in Fig. 7. Thus the upper bound follows by induction from $n-3$ to n.

Now we prove the lower bound. Assume that the unit triangles are painted in black and white so that neighbouring triangles have distinct colors and suppose that the number of black and white triangles is $n(n-1)/2$ and $n(n+1)/2$, respectively. Consider a full n-board containing D dominoes and $H = H(b) + H(w)$ holes, where $H(b)$ and $H(w)$, respectively, denotes the number of black and white unit triangles not covered by dominoes. Then, obviously,

$$2D + H = n^2 \tag{23}$$

and

$$H(w) - H(b) = \frac{n(n+1)}{2} - \frac{n(n-1)}{2} = n \tag{24}$$

since each domino covers exactly one black and one white triangle. We shall show (in Claim 2 below) that

$$D \geqslant 2H(b). \tag{25}$$

Assuming that (25) is true, the lower bound $\Delta(n) \geqslant n(n-1)/3$ can be obtained by using (24) and (23) as follows:

$$3D \geqslant 2D + 2H(b) = 2D + H(b) + H(w) - n = 2D + H - n = n^2 - n.$$

To prove (25), first we describe the structure of black holes in terms of their adjacency graph. Let G be the graph whose vertex set is the set of all black holes of the triangulated full board and xy is an edge of G if and only if the black holes x and y share a common corner.

Claim 1. Any disconnected component of G is the union of at most three edge disjoint paths either having a common endvertex or starting from distinct vertices of a cycle of length three.

Proof. Dominoes have three possible positions on the triangulated board: left

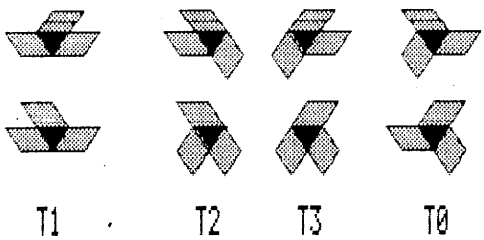

Fig. 8. The four types of black holes.

horizontal (l), right horizontal (r) or vertical (v). Denote by $p(A)$ the position of a domino A ($p(A)=l$, r or v).

Any black hole x on a full board is surrounded by just three dominoes. Let T, L and R be the neighbouring dominoes touching x at its top, left and right side, respectively. We distinguish between four types of black holes according to the position of the neighbouring dominoes (see also Fig. 8).

In case of $p(L)=l$ and $p(R)=r$, x is of type $T1$;
in case of $p(T)=l$ and $p(R)=v$, x is of type $T2$;
in case of $p(T)=l$ and $p(R)=v$, x is of type $T3$;
in any other cases x is of type $T0$.

Let V be the vertex set of a connected component of G and denote by Vi the set of all black holes of type Ti (clearly, $V=V0\cup V1\cup V2\cup V3$).

Suppose now that $V1$ is not empty. One can check easily that the set $V1$ induces a path in G starting at the bottom-most black hole $x1\in V1$; moreover, vertices of $V1$ different from $x1$ have no neighbours in $V\setminus V1$. By symmetry reasons, the same is true for the subgraphs induced by $V2$ and induced by $V3$. Consequently, the subgraph of G induced by the vertex set $V\setminus V0$ is the union of at most three pairwise disjoint paths starting at xi, eventually with edges between the xi's ($i=1, 2, 3$).

Observing that any xi ($i=1, 2, 3$) has at most one neighbour in $V0$: furthermore, there are no edges between vertices belonging to $V0$, one can verify easily that $|V0|\leqslant 1$, i.e. either $V0=\emptyset$ or $V0=\{x0\}$.

If $x0$ and one of xi ($i=1, 2, 3$) do not exist then V spans a path in G (see Fig. 9a; in (a2), both $x1$ and $x2$ exist). If $x0$ does not exist but $Vi\neq\emptyset$ for every

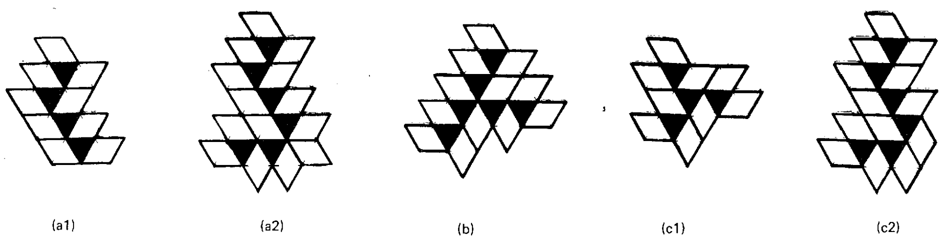

Fig. 9. Connected components of black holes.

$i=1, 2, 3$ then it is easy to check that $x1$, $x2$, $x3$ induce a cycle (see Fig. 9b). Finally, if $V0$ is not empty then $x0$ should be connected to xi by an edge for every existing xi (see Fig. 9c; in (c2), $x3$ does not exist). □

Claim 2. $D \geqslant 2H(b)$.

Proof. Observe that if a domino touches two black holes then they have a common corner, i.e. they belong to the same connected component of the adjacency graph G. Therefore, we have only to prove that black holes belonging to the vertex set V of some component touch at least $2|V|$ dominoes.

By using Claim 1, one can verify easily that just $2|V|$ dominoes are touched if the component contains a cycle and just $2|V|+1$ dominoes are touched otherwise. □

The exact value of $\Delta(n)$ is not known in general. However, if the question below has an affirmative answer then $\Delta(n) = [n^2/3]$ follows.

Problem 3.2. Does $D+1 \geqslant H$ hold for any triangulated full board? (D and H is the number of dominoes and holes, respectively.)

Now we turn to the clumsy packing of hexa boards which proves to be the easiest question among those investigated for different boards. Hexa boards which are in some sense the duals of triangulated boards consist of hexagonal cells dividing regularly a board having triangle-like shape (see Fig. 10).

Special "dominoes" designed for hexa boards cover two neighbouring hexagonal cells. Clearly, one can cover the whole hexa n-board (containing n rows) by packing $[n(n+1)/4]$ such dominoes onto it.

In our following result we completely answer the clumsy packing problem for hexa boards.

Theorem 3.3. *The minimum number of dominoes in a full hexa n-board is* $[n(n+1)/6]$.

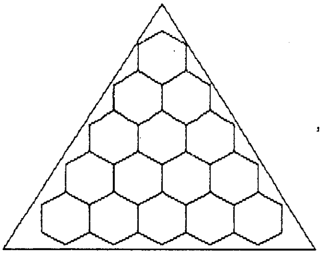

Fig. 10. The hexa 5-board.

Proof. Denote by D and H, respectively, the number of dominoes and uncovered cells, called holes, on the full board and put $C=n(n+1)/2$ for the number of all hexagonal cells. Obviously,

$$2D+H=C, \tag{26}$$

hence the theorem can be proved by determining the maximum number $h(n)$ of holes in a full n-board. Thus we have to show that

$$h(n)=\left[\frac{C+2}{3}\right], \tag{27}$$

which is obviously true for $n=1$, 2 and 3.

Observe first that among three pairwise neighbouring cells of a full board there is at most one hole. Thus it would be helpful to find a decomposition of the n-board into (non-overlapping) 2-boards. The decomposition can be given by induction on n.

Suppose that the cells of the n-board are labelled consecutively by 1, 2 and 3 so that neighbouring cells have distinct label (see Fig. 11).

We start with the case $n=3k+1$. For $k=1$ the decomposition of the 4-board pictured in Fig. 11(a) shows that $H \leqslant (C+2)/3=4$.

Assume now that $k>1$, and a decomposition has been defined for all $k'<k$. Then the boundary cells and their neighbours decomposed as shown in Fig. 11b and (c) are surrounding an $(n-6)$-board which has the desired decomposition by the induction hypothesis. Thus,

$$H \leqslant \frac{(C+2)}{3} \tag{28}$$

holds for $n=3k+1$.

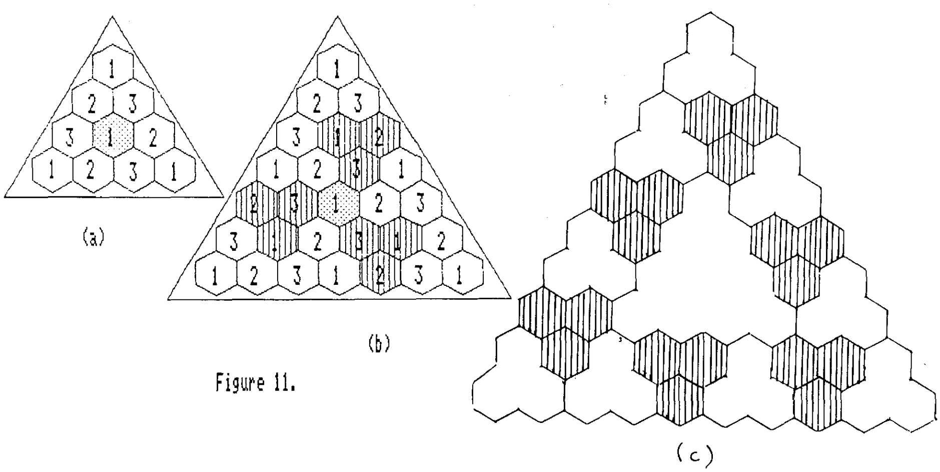

Fig. 11. The decomposition of $(3k+1)$-boards.

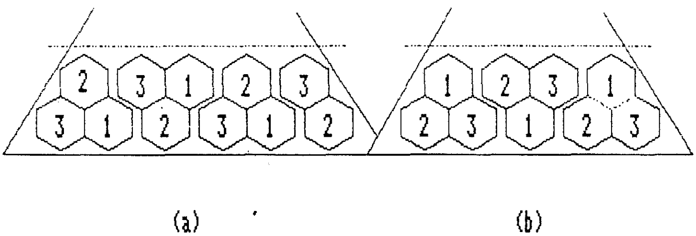

Fig. 12. The decomposition of the last two rows for $n = 3k + 2$.

When $n = 3k$ $(k > 1)$, consider the decomposition of the cells in the last two rows as in Fig. 12a. Since $n - 2 = 3(k - 1) + 1$, the decomposition of the remaining $(n - 2)$-board is known by induction. Consequently, $H \leqslant C/3 + 1$. Equality is not possible, however, which can be seen easily by using (26).

Assume now that (28) holds whenever n or $n - 1$ is divisible by 3 and suppose that $n = 3k + 2$. Again, decompose the cells in the last two rows as is shown in Fig. 12b. Then (28) immediately follows by induction also for $n = 3k + 2$.

To complete the proof of $h(n) = [(C + 2)/3]$ one has to give full n-boards containing $[(C + 2)/3]$ holes and $[C/3]$ dominoes. Observe that each 2-board of the decompositions described above contains exactly one label 1, 2 and 3. Thus one can pack a domino into the 2-boards in such a way that all uncovered cells are labeled by 1. (Note that the two rightmost cells in Fig. 12a have labels 2 and 3, so they also define a domino in case of $n = 3k + 1$.) $\square$

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