ON A GENERALIZATION OF TRANSITIVITY FOR DIGRAPHS

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In this paper we investigate the following generalization of transitivity: A digraph D is \((m, n)\)-transitive whenever there is a path of length \(m\) from \(x\) to \(y\) there is a subset of \(n + 1\) vertices of these \(m + 1\) vertices which contain a path of length \(n\) from \(x\) to \(y\).

Here we study various properties of \((m, n)\)-transitive digraphs. In particular, \((m, 1)\)-transitive tournaments are characterized. Their similarities to transitive tournaments are analyzed and discussed.

Various other results pertaining to \((m, 1)\)-transitive digraphs are given.

Introduction

The study of transitive digraphs and their underlying properties has been for some time a "completed" topic, though new applications and variations occur. It is the purpose of this paper to discuss a generalization of transitivity for digraphs. For a digraph D, we denote by \(V(D)\) and \(A(D)\) the vertex and arc set respectively. For convenience \(V\) and \(A\) will be used when no confusion results. Let \(U \subseteq V(D)\), then the subdigraph \(\langle U \rangle\) of \(D\) induced by \(U\) is the subdigraph of \(D\) with \(V(\langle U \rangle) = U\) and \(A(\langle U \rangle)\) consisting of all the arcs of \(D\) joining vertices of \(U\). A path of \(D\) is formed by a sequence \(x_0, x_1, x_2, \ldots, x_m\) of vertices, all distinct, such that for \(i = 1, 2, \ldots, m, x_{i-1}x_i \in A(D)\). The length of a path is the number of arcs it contains. We will refer to such a path as an \(x_0-x_m\) path of length \(m\) or alternatively a path of length \(m\) from \(x_0\) to \(x_m\). A cycle is an \(x_0-x_m\) path together with arc \(x_mx_0\). A subdigraph \(D'\) of \(D\) is strongly connected if for each pair \(x, y\) of distinct vertices in \(D'\), \(D'\) contains an \(x-y\) path and a \(y-x\) path. A strong component in \(D\) is a maximal strongly connected subdigraph. Finally, a tournament is a digraph in which, for each pair \(x, y\) of distinct vertices, exactly one of the arcs \(xy\) and \(yx\) is in \(A(D)\).

We now introduce a generalization of transitivity presented by Harary through McMorris. A digraph \(D\) is \((m, n)\)-transitive if whenever there is an \(x_0-x_m\) path of

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length \( m \) there is a subset of \( n + 1 \) vertices (of the path), including \( x_0 \) and \( x_m \), which induces a digraph containing an \( x_0-x_m \) path of length \( n \). Note that the usual transitivity is \((2, 1)\)-transitivity in this notation.

In this paper we consider this generalization and various implications. Our main result characterizes \((m, 1)\)-transitive tournaments. We go on to discuss in detail \((3, 1)\)-transitive tournaments, how many there are as well as their structure displayed. We then discuss relations between such transitive digraphs. For example we show that a tournament is \((3, 1)\)-transitive if and only if it is \((3, 2)\)-transitive but that this is not the case for \((m, 1)\) transitivity with \( m \geq 4 \). Finally, we discuss various properties of \((m, n)\)-transitive digraphs and conclude with a number of problems and directions of pursuit.

**On \((m, 1)\)-transitive tournaments**

It is well known that \((2, 1)\)-transitive (or just transitive) asymmetric digraphs are acyclic. If one considers this as not containing a cycle of length 3 or more then in this context \((m, 1)\)-transitivity is a direct generalization of transitivity, at least for tournaments. We exhibit this in the following theorem.

**Theorem 1.** A tournament \( T \) is \((m, 1)\)-transitive if and only if it contains no cycles of length \( m + 1 \) or more.

**Proof.** Clearly, if \( T \) contains no cycles of length \( m + 1 \), then \( T \) must be \((m, 1)\)-transitive, since every \( x-y \) path of length \( m \) implies \( yx \notin A \), so that \( xy \in A \).

Now suppose \( T \) is \((m, 1)\)-transitive and does contain a cycle of length greater than \( m \). Clearly, \( T \) must contain a cycle of length greater than \( m + 1 \).

First, suppose there are no cycles of length \( m + 2 \), i.e., the shortest cycle greater than \( m + 1 \) is of length \( k \geq m + 3 \). Denote the vertices \( x_0x_1x_2, \ldots, x_{m+2}, x_{m+3}, \ldots, x_k \). It follows then that \( x_{i+2}x_i \) and \( x_{i+3}x_i \in A \) for \( i = 0, 1, \ldots, k \), where the addition of subscripts is taken modulo \( (k + 1) \).

Consider the following \( x_m-x_0 \) path of length \( m \):

\[
\begin{align*}
x_m & x_{m-2}x_{m-3}x_{m-5}x_{m-4}x_{m-6}x_{m-8} \ldots x_3x_1x_2x_0, & \text{if } m \equiv 0 \pmod{3}, \\
x_m & x_{m+2}x_{m-2}x_{m-4}x_{m-6}x_{m-5}x_{m-7} \ldots x_3x_1x_2x_0, & \text{if } m \equiv 1 \pmod{3}, \\
x_m & x_{m+1}x_{m-2}x_{m-4}x_{m-3}x_{m-5}x_{m-7}x_{m-6} \ldots x_3x_1x_2x_0, & \text{if } m \equiv 2 \pmod{3}.
\end{align*}
\]

Since \( T \) is \((m, 1)\)-transitive, this implies \( x_mx_0 \in A \), but this is a contradiction since \( x_0x_1 \ldots x_m \) is a path of length \( m \) from \( x_0 \) to \( x_m \), implying \( x_0x_m \in A \).

Hence we may assume there is a cycle of length \( m + 2 \), say \( x_0x_1 \ldots x_{m+1}x_0 \). It follows that \( x_{i+2}x_i \in A \) for \( i = 0, 1, \ldots, m + 1 \) where addition of subscripts is taken modulo \( (m+2) \). If for some \( i \), both \( x_ix_{i+3} \) and \( x_{i+1}x_{i+4} \in A \), then

\[
x_0x_1 \ldots x_ix_{i+3}x_{i+1}x_{i+4} \ldots x_{m+1}
\]
would be a path of length \( m \) from \( x_0 \) to \( x_{m+1} \) which would contradict \( x_{m+1} x_0 \in A \) since \( T \) is \((m, 1)\)-transitive. Thus, no such \( i \) exists. As above if \( m \equiv 0 \) or \( 2 \) (mod 3) a contradiction is quickly established by considering the given paths of length \( m \).

In the case \( m \equiv 1 \) (mod 3), since there does not exist \( i \) with both \( x_i x_{i+3} \) and \( x_{i+1} x_{i+4} \in A \), it must be the case that there exists an \( i \) with \( x_{i+1} x_{i-2} \) and \( x_{i+3} x_i \) both elements of \( A \). By relabeling \( i = m - 2 \) and the \((m - 2)\)-cycle appropriately, the \( x_m - x_0 \) path of length \( m \) given in the \( m \equiv 1 \) (mod 3) case above again yields a contradiction. With all cases exhausted, the proof is complete. \( \square \)

This theorem gives a complete characterization of \((m, 1)\)-transitive tournaments. For asymmetric \((m, 1)\)-transitive digraphs in general, restrictions on cycle length are much more subtle. While it is well known that \((2, 1)\)-transitivity implies no cycle of length three or more, and we prove a similar result when \( m = 3 \), no such restriction exists if \( m \geq 4 \). Examples are given in Fig. 1 and the construction following it.

The structure and existence of long cycles in \((m, 1)\)-transitive digraphs has been considered in [2]. It is easy to see that the following digraph is \((m, 1)\)-transitive and contains a cycle of length \( 2m - 2 \) for \( m \geq 4 \). The vertices are \( x_0, x_1, x_2, \ldots, x_{2m-3} \) and the edges are \( x_i x_{i+1} \) and \( x_i x_{i+m} \) for \( i = 0, 1, \ldots, 2m-3 \) (subscript addition taken modulo \((2m - 2)\)). Fig. 1 shows the case \( m = 4 \).

**Proposition 2.** If \( D \) is a \((3, 1)\)-transitive asymmetric digraph, then \( D \) contains no cycles of length 4 or more.

**Proof.** Suppose the result is false and let \( C \) be a shortest cycle of length 4 or more. Clearly, since \( D \) is \((3, 1)\)-transitive it can contain no cycles of length 4 and if \( C \) was a cycle of length 6 or more a shorter such cycle would result by the presence of arc \( x_0 x_3 \). Hence, we may assume \( C \) is a cycle of length 5. Label the vertices of \( C \): \( x_0, x_1, x_2, x_3 \) and \( x_4 \). Since \( D \) is \((3, 1)\)-transitive, \( x_0 x_3 \) and \( x_4 x_2 \in A \). By considering the paths \( x_2 x_3 x_4 x_0 \) and \( x_0 x_3 x_4 x_2 \) a contradiction results. Thus \( D \) can contain no cycles of length 4 or more. \( \square \)

![Fig. 1. A (4, 1)-transitive digraph with a cycle of length 6.](image-url)
Next we use Theorem 1 to count the number of \((m, 1)\)-transitive tournaments.

Let \(S_1, S_2, \ldots, S_n\) be the strong components of \(D\). The condensation \(D^*\) of \(D\) is the digraph with vertex set \(\{S_1, S_2, \ldots, S_n\}\) and \(S_iS_j\) is an arc of \(D^*\) if and only if \(i \neq j\) and for some vertex \(u_i \in S_i\) and \(u_j \in S_j\), \(u_i u_j \in A(D)\). It is well known that in the case of a tournament \(T\), \(T^*\) is transitive. Also, for a tournament \(T\), any strong component is Hamiltonian, i.e., there is a directed cycle containing each vertex of the component. Using these facts, we can more accurately describe the structure of \((m, 1)\)-transitive tournaments.

For convenience, we will call the order of \(D^*\) the height of \(D\) and denote it by \(h(D)\). The concept of height is exhibited in Fig. 2.

By the previous observations and by Theorem 1, if \(T\) is \((m, 1)\)-transitive, then \(S_i\) is a Hamiltonian tournament of order at most \(m\). Furthermore, to construct an \((m, 1)\)-transitive tournament of height \(h\), we can select any \(h\) strong (i.e., Hamiltonian) tournaments and give the structure of \(T^*\) as shown and the resulting tournament is \((m, 1)\)-transitive.

Moon [4] determined the number of strong tournaments. If we let \(t(m)\) be the number of strong tournaments of order \(m\) or less than the following result holds.

**Theorem 3.** There are \((t(m))^h\) \((m, 1)\)-transitive tournaments of height \(h\).

To count the number of \((m, 1)\)-transitive tournaments of a particular order is a bit more difficult since it would include the number of partitions of \(n\) such that no part is 2. For the case of \((3, 1)\)-transitive tournaments the problem is somewhat less difficult since there are only 2 possible strong components for \(T^*\); a single vertex or a 3-cycle. In this case we get the following result.

![Fig. 2. The transitive tournament \(T^*\), the condensation of a tournament \(T\).](image)
Theorem 4. Let $n$ be a positive integer and $r$ be the largest integer $\leq \frac{1}{3} n$. The number of $(3, 1)$-transitive tournaments of order $n$ is

$$\sum_{i=0}^{r} \binom{n-2i}{i}.$$ 

Proof. At most $\frac{1}{3} n$ of the $h(T)$ strong components are triangles. If $i$ triangles occur, $T$ has height $n - 2i$. Thus, there are $\binom{n-2i}{i}$ $(3, 1)$-transitive tournaments of order $n$ and height $n - 2i$. \hfill \Box

We note that the degree sets and sequences for $(3, 1)$-transitive tournaments are studied in [2].

On relations between transitivities

In this section we consider the connection between tournaments that are $(m, 1)$-transitive and $(m, i)$-transitive. There are some obvious relations between transitivities of digraphs. For example, $(m, 1)$-transitivity implies $(t(m - 1) + 1, 1)$-transitivity, $(m + t, 1 + t)$-transitivity and $(tm, f)$-transitivity for all integers $t \geq 1$. In tournaments, $(m, 1)$-transitivity implies $(n, 1)$-transitivity for $n \geq m$ according to Theorem 1, and there are connections between $(m, 1)$-transitivity and $(m, i)$-transitivity as the next theorem shows:

Theorem 5. If a tournament $T$ is $(m, 1)$-transitive, then $T$ is $(m, k)$-transitive for $k = 1, 2, \ldots, m$.

Proof. Let $x_0, x_1, \ldots, x_m$ be an $m$-path in a $(m, 1)$-transitive tournament $T$ and let $k$ be an integer, $1 \leq k \leq m$. The tournament $T'$ induced by $\{x_0, x_1, \ldots, x_m\}$ in $T$ is clearly not Hamiltonian; it has a directed cut, i.e., $V(T') = V(T_1) \cup V(T_2)$, where $V(T_1) \cap V(T_2) = \emptyset$ and $y_1y_2$ is an arc of $T'$ for all $y_1 \in V(T_1), y_2 \in V(T_2)$. Let $p = |V(T_1)|$. Since $x_0x_1 \ldots x_m$ is a (Hamiltonian) path of $T'$, $\{x_0, x_1, \ldots, x_{p-1}\} = V(T_1)$ and $\{x_p, \ldots, x_m\} = V(T_2)$ follow.

To see that $T$ is $(m, k)$-transitive, we exhibit a suitable $k$-path $P_k$ from $x_0$ to $x_m$:

$$P_k = \begin{cases} x_0x_1 \ldots x_kx_m, & \text{if } k \leq p - 1, \\ x_0x_1 \ldots x_{p-1}x_{q-1}x_{q+1} \ldots x_m, & \text{if } k < p - 1, \end{cases}$$

where $q = m - k + p - 1$. \hfill \Box

In case $m = 3$, a stronger result holds.

Theorem 6. A tournament $T$ is $(3, 1)$-transitive if and only if $T$ is $(3, 2)$-transitive.

Proof. Suppose $T$ is $(3, 2)$-transitive but not $(3, 1)$-transitive. Then it must be the
case that $T$ contains a 4-cycle, say $x_0 x_1 x_2 x_3$. Since $T$ is $(3, 2)$-transitive either $x_1 x_3$ or $x_0 x_2 \in A(T)$. Without loss of generality, suppose $x_0 x_2 \in A(T)$. By considering the path $x_2 x_3 x_0 x_1$ it must be the case that $x_3 x_1 \in A(T)$. Now consider the path $x_1 x_2 x_3 x_0$; a contradiction results since neither $x_2 x_0$ nor $x_1 x_3$ can be an arc of $T$. Thus $T$ can contain no cycles of length 4 and therefore $T$ is $(3, 1)$-transitive. □

Generally $(m, k)$-transitivity does not imply $(m, 1)$-transitivity for tournaments. In fact, the only case when it does appears in Theorem 6. For $k = 2$ and $m = 4$ the following tournament demonstrates our assertion. The tournament of Fig. 3 is not $(4, 1)$-transitive since it is Hamiltonian. There is a 2-path from $x_i$ to $x_j$ if $i \neq j$, $0 \leq i \leq 4$, $0 \leq j \leq 4$, except for the following pairs: $i = 0$, $j = 3$; $i = 1$, $j = 2$; $i = 3$, $j = 4$; $i = 4$, $j = 0$. However, for these exceptional pairs $(i, j)$, there are no 4-paths from $x_i$ to $x_j$. Therefore the tournament is $(4, 2)$-transitive.

If $k = 2$ and $m \geq 4$, the construction of $(m, 2)$-transitive tournaments that are not $(m, 1)$-transitive is very complex and will appear in [2].

**Conclusion**

This generalization of transitivity seems like a fruitful area of research. There seems to be a number of directions to be pursued.

(1) Completely characterize the cycle structure of $(m, 1)$-transitive digraphs.
(2) Study the cycle structure, or maximum cycle length of $(m, n)$-transitive digraphs, and/or tournaments.
(3) Consider the problem of what graphs can be given $(m, n)$-transitive orientations?
(4) Is there a reasonable way to define and study the \((m, n)\)-transitive closure of a digraph?

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References