

Cycles in graphs without proper subgraphs of minimum degree 3

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1. Introduction.

Let $G(n,m)$ denote the set of graphs with n vertices and m edges. It is well-known that each $G \in G(n, 2n - 2)$ contains a subgraph of minimum degree 3 but there exists a $G \in G(n, 2n - 3)$ with no subgraphs of minimum degree 3 (see [1] p. xvii).

It was proved in [2] that each $G \in G(n, 2n - 1)$ contains a *proper* subgraph of minimum degree 3, but there exists $G \in G(n, 2n - 2)$ without this property. In fact, a stronger result was proved in [2], namely that $G \in G(n, 2n - 1)$ must contain a subgraph of minimum degree 3 with at most $n - c\sqrt{n}$ vertices for some $c > 0$. It was conjectured in [2] that each $G \in G(n, 2n - 1)$ contains a subgraph of minimum degree 3 with at most cn vertices for some absolute constant $c < 1$.

In this paper we study cycle lengths of graphs which have no proper subgraphs of minimum degree 3. For ease of reference, let $G^*(n, m)$ denote the set of graphs with n vertices, m edges and with the property that no proper subgraph has minimum degree 3. The results mentioned so far show that $G \in G^*(n, m)$ implies $m \leq 2n - 2$, and if $G \in G^*(n, 2n - 2)$ then G has minimum degree 3. Throughout the paper we investigate the cycle structure of graphs G , with $G \in G^*(n, 2n - 2)$. In fact we give the following conjecture.

CONJECTURE: If $G \in G^*(n, 2n - 2)$, then G contains all cycles of length at most k where k tends to infinity with n .

Our results are all related to this conjecture. We have several examples to demonstrate the role of $2n - 2$ in this conjecture. For example for each n there exists graphs G , $G \in G^*(n, 2n - 3)$, such that G has no triangle (Examples 1 and 2). It is also true that there are $G \in G^*(n, 2n - 3)$ such that G has no cycles of length 5 or more (Example 3). For every r , we construct a graph $G \in G^*(n, 2n - c(r))$ such that G has no cycles of length less than or equal to r (Theorem 4). In fact, the minimum value of $c(r)$ is determined precisely for $r = 3, 4$.

On one hand, our conjecture says that the graphs in $G^*(n, 2n - 2)$ contain small cycles. We prove that these graphs contain C_3, C_4 and C_5 (Theorem 2.) On the other hand, our conjecture says that the graphs in $G^*(n, 2n - 2)$ contain long cycles. Our main result is that $G \in G^*(n, 2n - 2)$ contains a cycle of length at least $\lfloor \log n \rfloor$ (Theorem 5.). However,

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graphs in $G^*(n, 2n - 2)$ does not always contain very long cycles (as large as $c\sqrt{n}$ for some $c > 0$, Example 7).

2. Properties of Graphs without proper subgraphs of minimum degree 3.

In this section we give a lemma and a theorem which we shall use frequently in sections 3 and 4. We first introduce some terminology.

Consider an ordering x_1, x_2, \dots, x_n of the vertex set of a graph. An edge $x_i x_j, i > j$ of the graph is called a *forward* edge on x_i and a *backward* edge on x_j . The forward (backward) degree of x_i is the number of forward (backward) edges incident to x_i . We shall let $d^+(x_i), d^-(x_i)$ denote the forward and backward degree of x_i , respectively.

For any graph G we formally define an ordering of the vertices of G as follows: x_1 is a vertex of minimum degree in G . If x_1, x_2, \dots, x_t are already defined and $t < |V(G)|$, then let x_{t+1} be a vertex of minimum degree in $G - \{x_1, x_2, \dots, x_t\}$. If G has no proper subgraph of minimum degree 3, then $d^+(x_i) \leq 2$ for $2 \leq i \leq |V(G)|$. Since we shall use this ordering often, we formulate this statement as lemma.

LEMMA 1. *Let G have n vertices and contain no proper subgraph of minimum degree 3. Then, the vertices of G can be ordered so that $d^+(x_1)$ is the minimum degree of G and $d^+(x_i) \leq 2$ for $i \geq 2$.*

THEOREM 1. *If $G \in G^*(n, 2n - 2)$, then the vertices of G can be ordered so that $d^+(x_1) = 3, d^+(x_i) = 2$ for $2 \leq i \leq n - 2$, and $d^+(x_{n-1}) = 1$. Moreover $d^-(x_i) \geq 1$ for $2 \leq i \leq n$.*

PROOF: In the ordering of the vertices described in Lemma 1 observe that

$$2n - 2 = |E(G)| = \sum_{i=1}^{n-1} d^+(x_i) \leq d(x_1) + 2(n - 3) + 1 \leq 2n - 2.$$

Since $d(x_1) \leq 3$ (otherwise G has at least $2n$ edges), $d^+(x_i) \leq 2$ for $i = 2, 3, \dots, (n - 2)$ and $d^+(x_{n-1}) \leq 1$, all the inequalities are equalities. Thus, $d^+(x_1) = 3, d^+(x_i) = 2$ for $2 \leq i \leq n - 2$, and $d^+(x_{n-1}) = 1$. Since $d(x_i) \geq d(x_1) = 3$ and $d^+(x_i) \leq 2$ for $1 < i \leq n$, $d^-(x_i) \geq 1$ follows. ■

COROLLARY 1. *If $G \in G^*(n, 2n - 2)$ then G has minimum degree 3.*

3. Small Cycles in $G^*(n, 2n - 2)$.

THEOREM 2. *If $G \in G^*(n, 2n - 2)$ then for $n \geq 5$, G contains a C_3 and a C_5 . If $G \in G^*(n, 2n - 3)$ and $n \geq 6$, then G contains C_4 .*

PROOF: For $G \in G^*(n, 2n - 2)$ consider the ordering of vertices given in Theorem 1. Clearly,

x_{n-2}, x_{n-1} and x_n determine a C_3 . Without loss of generality we may assume that x_{n-3} is adjacent to x_{n-1} and x_n .

Assume that i is the largest index such that x_i is adjacent to x_j for some $j, i < j < n-1$. There exists such an index since $i = 1$ is a suitable choice. If x_i is adjacent to x_{n-1} or to x_n , say to x_n , then select any $k > i$ such that $k \neq j, k \neq n, k \neq n-1$. This gives the $C_5, x_i x_n x_k x_{n-1} x_j x_i$ in G .

If x_i is not adjacent to either x_{n-1} or to x_n , then (since $d^+(x_i) = 2$) x_i is adjacent to some x_k , with $i < k, j \neq k, n \neq k, n-1 \neq k$. But then $x_i x_k x_{n-1} x_n x_j x_i$ is a C_5 in G .

To see that $G \in G^*(n, 2n-3)$ contains a C_4 , observe that Theorem 1 almost holds in that we can order the vertices of G as x_1, x_2, \dots, x_n so that at most one of the equalities $d^+(x_i) = 2$ for $2 \leq i \leq n-2, d^+(x_1) = 3$, and $d^+(x_{n-1}) = 1$ fails to hold. Moreover if equality does not hold for some i then $d^+(x_i)$ is just one less than the value shown above. If each of the equalities $d^+(x_{n-1}) = 1, d^+(x_{n-2}) = d^+(x_{n-3}) = 2$ hold then the subgraph of G induced by $X = \{x_{n-3}, x_{n-2}, x_{n-1}, x_n\}$ has five edges and there is a C_4 in G . Therefore, we assume that there is no C_4 in the subgraph induced X . Also, by a suitable permutation of the vertices in X , we may assume that $x_{n-3}x_{n-2}, x_{n-3}x_{n-1}, x_{n-3}x_n$ and $x_{n-2}x_{n-1}$ are edges in X . But $d^+(x_{n-4}) = 2$ and the only way to avoid a C_4 in G is to assume x_{n-4} to be adjacent to x_{n-3} and to x_n . Since $n \geq 6, x_{n-5}$ exists and $d^+(x_{n-5}) \geq 2$. Thus, there exists a C_4 in G containing x_{n-5} and three vertices of $\{x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}\}$. ■

With more work it is possible to show that $G \in G^*(2n-2)$ always contains C_6 for $n \geq 6$. The following constructions show that Theorem 2 is sharp.

EXAMPLE 1: Let $n \geq 6$ be even. Consider the graph on n vertices defined as follows. Let $x_1 x_2 \dots x_{n-2}$ be a cycle of length $n-2$. Let y and w be two new vertices with y adjacent to all x_i of even index and w adjacent to all x_i of odd index. Finally place an edge between y and w . The graph obtained contains no triangles, (in fact, is bipartite) has no proper subgraph of minimum degree 3, and has $2n-3$ edges. ■

EXAMPLE 2: Let $n = 2k+1 \geq 9$ and consider a cycle of length k with vertices $x_1 x_2 \dots x_k$. For $i = 1, 2, \dots, k-1$ place new vertices y_i in the graph with each y_i adjacent to x_i . Finally, let v and w be two additional vertices of the graph such that each are adjacent to y_1, y_2, \dots, y_{k-1} and x_k . The resulting graph has $2n-3$ edges, no triangle and no proper subgraph of minimum degree 3. ■

EXAMPLE 3: Consider the graph obtained from $K_{2, n-2}$ by placing an edge between the two vertices of the two-vertex color class. This graph has no cycles of length 5 or more, has $2n-3$ vertices, and contains no proper subgraph of minimum degree 3. ■

EXAMPLE 4: Assume that $n-2$ is divisible by 4, $n \geq 10$, and consider a cycle of length $n-2$ with vertices $x_1, x_2, x_3, \dots, x_{n-2}$. Let y and w be two new vertices. Join vertex y to

x_i for $i \equiv 1$ or $i \equiv 2 \pmod{4}$ and join w to x_i for $i \equiv 0$ or $i \equiv 3 \pmod{4}$. This graph has no C_4 , has $2n - 4$ vertices, and has no proper subgraphs of minimum degree 3. It is easy to modify this example for $n \equiv 0, 1, 3 \pmod{4}$. ■

Based on these examples, we conclude that Theorem 2 is sharp: there exists $G \in G^*(n, 2n - 3)$ without C_3 (Example 1 and 2); there exists $G \in G^*(n, 2n - 3)$ without C_5 (Example 1 and 3); there exist $G \in G^*(n, 2n - 4)$ without C_4 (Example 4).

Up to now we've only considered the existence of C_k (for $k = 3, 4, 5$) in $G \in G^*(n, 2n - 2)$. We continue by looking for the minimum m that $G \in G^*(n, m)$ contains a cycle of length less than r . Theorem 2 and Examples 1 and 2 show that $m = 2n - 2$ when $r = 4$. The upper bound for m in cases $r = 5$ and $r = 6$ are given in the next result.

THEOREM 3. *Let $g(G)$ denote the girth of G . If $n \geq 6$ and $G \in G^*(n, 2n - 4)$, then $g(G) \leq 4$. If $n \geq 8$ and $G \in G^*(n, 2n - 6)$ then $g(G) \leq 5$.*

PROOF: Assume $G \in G^*(n, 2n - 4)$ and apply Lemma 1. Clearly $d^+(x_1) \leq 3$, otherwise G has at least $2n$ edges. If $n \geq 6$ the subgraph H induced by $x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}$ in G has at least $(2n - 4) - 3 - 2(n - 6) = 5$ edges. We may assume that H is a cycle of length 5, otherwise H contains C_3 or C_4 and $g(G) \leq 4$ follows. Therefore $d^+(x_1) = 3, d^+(x_i) = 2$ for $i = 2, 3, \dots, n - 5$. But x_{n-5} is adjacent to two vertices of the five-cycle H giving a C_3 or C_4 .

To prove the second part of the Theorem, assume $G \in G^*(n, 2n - 6)$ and apply Lemma 1. Again, $d^+(x_1) \leq 3$. Since $n \geq 8$, we consider the subgraph H induced by $\{x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}, x_{n-6}\}$ in G . Thus, H contains at least $(2n - 6) - 3 - 2(n - 8) = 7$ edges. Let C be a cycle of H with minimum length, so that C is a cycle without a diagonal. If $|C| = 7$ then $H = C$ and $d^+(x_1) = 3, d^+(x_i) = 2$ for $i = 2, 3, \dots, n - 7$. In particular, x_{n-7} is adjacent to at least two vertices of C giving a cycle of length at most 5. If $|C| = 6$, then without loss of generality assume $x_n, x_{n-1}, x_{n-2}, x_{n-3}, x_{n-4}, x_{n-5}, x_n$ is a 6 - cycle and x_{n-6} is adjacent to x_{n-5} . If x_{n-6} is adjacent to any vertex x_i for $n - 4 \leq i \leq n$ then we have a C_3, C_4 or C_5 . Therefore, H has 7 edges and again $d^+(x_1) = 3, d^+(x_i) = 2$ for $2 \leq i \leq n - 7$. In particular $d^+(x_{n-7}) = 2$, and it is easy to check that the only case when $x_{n-7}, x_{n-6}, \dots, x_n$ does not induce a cycle of length at most 5 in G occurs if x_{n-7} is adjacent to x_{n-6} and x_{n-2} (see Figure 4). It is easy to see that $d^+(x_{n-8}) = 2$ implies the existence of a cycle of length at most 5. Thus $|C| \leq 5$ completing proof of the theorem. ■

To show that the first part of Theorem 3 is best possible we give the following example.

EXAMPLE 5: Assume n is divisible by 5 and $n \geq 10$. Let $x_1 x_3 x_5 x_2 x_4 x_1$ be a five-cycle and $y_1 y_2 \dots y_{n-5} y_1$ is a $n - 5$ cycle. Vertex x_i is adjacent to y_j if and only if $j \equiv i \pmod{5}$ (for all $i, 1 \leq i \leq 5$). This graph has $2n - 5$ edges, has no proper subgraph of minimum degree 3 and contains no C_3 or C_4 .

We do not know examples of $G \in G^*(n, 2n - 7)$ with $g(G) \geq 6$ for infinitely many n . However, it is possible to find $G \in G^*(n, 2n - 8)$ with $g(G) = 6$ for infinitely many n .

The next theorem shows that graphs in $G^*(n, 2n - c)$ do not always contain small cycles.

THEOREM 4. *For every positive integer r there exists $c = c(r)$ and a graph $G \in G^*(n, 2n - c(r))$ such that $g(G) > r$.*

PROOF: Let k be a natural number and let C_1, C_2, \dots, C_k be vertex disjoint cycles of length $t = 2 \cdot 5^{r+1} - 1$. We shall define the graph G_k by adding edges to the graph $C_1 \cup C_2 \cup \dots \cup C_k$. Assume that the vertices of C_i are $x_1^i, x_2^i, \dots, x_t^i$ (indexed in the natural order of the cycle). The definition of G_k is recursive. Set $G_1 = C_1$. If G_1, G_2, \dots, G_{k-1} are already defined we shall define G_k by adding edges xy to $G_{k-1} \cup C_k$ such that $x \in C_k, y \in C_{k-1}$. The definition will preserve the following properties (for each $i, 1 \leq i \leq k$):

- (i) each cycle of G_i is longer than r
- (ii) the maximum degree of G_i is at most 5, and
- (iii) $d_{G_i}(x_1^i) = 4, d_{G_i}(x_t^i) = 2, d_{G_i}(x_j^i) = 3$ for $2 \leq j \leq t - 1$ and $i \geq 2$.

Note that properties (i), (ii) and (iii) trivially hold for $i = 1$, since $G_1 = C_1$.

To define G_k we add edges $e_0 = x_1^k y_0, e_1 = x_1^k y_1, e_2 = x_2^k y_2, e_3 = x_3^k y_3, \dots, e_{t-1} = x_{t-1}^k y_{t-1}$ to $G_{k-1} \cup C_k$, such that $y_j \in V(C_{k-1})$ for $j = 0, 1, \dots, t - 1$ and G_k satisfies properties (i), (ii), and (iii) for $i = k$. Observe that (iii) holds independent of the choice of each y_j , so that we need only select each y_j such that (i) and (ii) hold. The edge e_0 can be defined arbitrarily. Assume that e_0, e_1, \dots, e_s are defined for $0 \leq s < t - 1$ in such a way that properties (i) and (ii) hold for $G' = G_{k-1} \cup C_k \cup \{e_0, e_1, \dots, e_s\}$. We define e_{s+1} as follows. Let W denote the set of vertices in C_{k-1} which can be reached by a path of length at most r from x_{s+1}^k in the graph G' . Since (ii) holds for G' , $|W| < 5^{r+1}$ and therefore $|V(C_{k-1}) - W| > t - 5^{r+1} = 5^{r+1} - 1$. Let T be a subset of $V(C_{k-1}) - W$ such that $|T| = 5^{r+1}$. By definition, for any $y \in T$ the graph $G' \cup e_{s+1}$ satisfies (i) with $e_{s+1} = x_{s+1}^k y$. Using property (iii) for $i = k - 1$

$$\sum_{y \in T} d_{G_{k-1}}(y) \leq 3|T| + 1 = 3 \cdot 5^{r+1} + 1.$$

Since G' is obtained from G_{k-1} by adding $s + 1$ edges,

$$\sum_{y \in T} d_{G'}(y) \leq \sum_{y \in T} d_{G_{k-1}}(y) + s + 1 \leq 3 \cdot 5^{r+1} + 1 + t - 1 = 5^{r+2} - 1.$$

Thus there exists an $y_{s+1} \in T$ with $d_{G'}(y_{s+1}) < 5$. Thus with $e_{s+1} = x_{s+1}^k y_{s+1}$, the graph

$G'' = G' \cup e_{s+1}$ satisfies properties (i), (ii) and (iii). Therefore G_k is defined.

It is clear that $|V(G_k)| = kt$ and $|E(G_k)| = 2kt - t$. The proof is completed by showing that G_k has no proper subgraph of minimum degree 3. Assume to the contrary that G^* is such a proper subgraph. Since $d_{G_k}(x_t^k) = 2$, $x_t^k \notin V(G^*)$. However, $d_{G_k - x_t^k}(x_{t-1}^k) = 2$ implies $x_{t-1}^k \notin V(G^*)$. Repeating this argument we get that $x_j^k \notin V(G^*)$ for $1 \leq j \leq t$. But $d_{G_k - C_k}(x^{k-1}) = 2$ and by observations just like those made above, none of the vertices of C_{k-1} belong to G^* ; Continuing in this way we see that G^* is the empty graph, a contradiction. Hence $G_k \in G^*(tk, 2tk - t)$ for all k with $t = 2 \cdot 5^{r+1} - 1$, showing that $c(r) = 2 \cdot 5^{r+1} - 1$ is a suitable choice. ■

4. Long cycles in $G^*(n, 2n - 2)$.

In this section we prove one of the main results of the paper, that is $G \in G^*(n, 2n - 2)$ contains a long cycle. Note that $G \in G^*(n, 2n - 3)$ does not necessarily contain even a path of length 4 (see Example 3 in Section 2).

THEOREM 5: If $G \in G^*(n, 2n - 2)$, then G contains a cycle of length at least $\lfloor \log n \rfloor$.

PROOF: Consider the ordering of G of Theorem 1. Since $d^-(x_i) > 0$ for $i = 2, \dots, n$, we can find a spanning tree T recursively in G as follows. Place x_1 in T . If x_1, x_2, \dots, x_t are in T and $t < n$, then choose any edge $x_i x_{t+1}$ of G such that $1 \leq i \leq t$. Redefine T by adding vertex x_{t+1} and the edge $x_i x_{t+1}$ to the old T . By definition of the tree, $d^-(x_i) = 1$ in T for $2 \leq i \leq n$ and $d^+(x_i) \leq 2$ in T for $2 \leq i \leq n$. Since $d^+(x_1) = 3$ in G , T is a tree of maximum degree ≤ 3 . Therefore, T contains a path P of length at least $\lfloor \log n \rfloor$ starting with x_i .

Let $x_1 = x_{i_1}, x_{i_2}, \dots, x_{i_k}$ denote the vertices of P in the natural order defined by P i.e. $x_{i_j} x_{i_{j+1}}$ is an edge of P for $1 \leq j \leq k - 1$. Notice that $i_1 < i_2 < \dots < i_k$ follows from the definition of T since $d^-(x_i) < 2$ in T for $2 \leq i \leq n$. We call a path $P = (x_{i_1}, x_{i_2}, \dots, x_{i_k})$ in G a *forward path* if $i_1 < i_2, \dots < i_k$. Note that the definition depends on the order x_1, x_2, \dots, x_n defined by Theorem 1. The discussion up to this point insures that G has a forward path of length at least $\lfloor \log n \rfloor$.

Let $P = (x_{i_1}, x_{i_2}, \dots, x_{i_k})$ be a forward path of G with a maximum length. Since $d^-(x_i) \geq 1$, and $d^+(x_i) \geq 1$ in G for $2 \leq i \leq n - 1$, it follows that $i_1 = 1, i_k = n$. Let t be any positive integer such that $1 \leq t < k$ and $i_t \neq n - 1$. Since $d^+(x_i) \geq 2$ in G for $1 \leq i \leq n - 2$, we can find a forward path P_t in G starting at x_{i_t} and ending at some vertex $x_{i_t'}$ of P such that

- (a) P_t and P are edge disjoint
- (b) $V(P_t) \cap V(P) = \{x_{i_t}, x_{i_t'}\}$

Note that (a) implies $t + 1 < t'$. We claim for $t < s$ that the paths P_t and P_s are vertex disjoint if $t' < s$ or if $s = t' - 1$. The case $t' < s$ is obvious since both P_t and P_s are forward

paths. Assume $s = t' - 1$. But $t + 1 < t'$ and $s + 1 < s'$ implies that P_t and P_s do not have common endpoints. Assume that x is the last common vertex of P_s and P_t we find on P_s by traveling along P_s from its starting point x_{i_s} . It is easy to see that the following edge sequence is a forward path: starting from $x_{i_1} = x_1$, travel along P to x_{i_s} ; continue on P_s to x ; from x travel to $x_{i_t} = x_{i_s+1}$ along P_t ; finally from x_{i_t} to $x_{i_k} = x_n$ travel along P . This path is longer than P . This contradiction proves the claim.

Choose a subset Q_1, Q_2, \dots, Q_r of the paths P_1, P_2, \dots, P_t as follows. Set $Q_1 = P_1$. If Q_1, Q_2, \dots, Q_s are defined and the endpoint x_{i_s} of Q_s is not x_n then $Q_{s+1} = P_{s'-1}$. If the endpoint x_{i_s} of Q_s is x_n then set $r = s$.

It is now easy to construct a cycle using all the vertices of $P \cup Q_1 \cup \dots \cup Q_r$. But P has at least $\lfloor \log n \rfloor$ vertices so that the cycle C has length at least $\lfloor \log n \rfloor$. ■

Finally, to see that $G \in G^*(n, 2n - 2)$ does not contain necessarily a very long cycle, (larger than $c\sqrt{n}$) consider the following example.

EXAMPLE 6: Let k be an integer, $k \geq 4$. Let C be a k -cycle with vertices x_1, x_2, \dots, x_k . Select a new vertex w and connect w to each x_i with vertex-disjoint paths of length $k - 1$. (The only common vertex of these paths is w). Select another new vertex y and let y be adjacent to all vertices except those of $\{x_2, x_3, \dots, x_k\}$. Let G_k be the graph just defined.

The graph G_k has $k(k - 1) + 2 = n$ vertices. Since $d(w) = k + 1$, $d(y) = n - k$, $d(x_1) = 4$ and all the other vertices are of degree 3, G_k has

$$\frac{k + 1 + n - k + 4 + (n - 3)3}{2} = 2n - 2$$

edges. It is easy to check that G_k has no proper subgraph of minimum degree 3. It is also easy to see that the longest path of $G_k - y$ is smaller than $5k$. Therefore the longest path of G_k is smaller than $10k \leq 10\sqrt{n + 1}$.

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