Graphs Which Have an Ascending Subgraph Decomposition

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Abstract. The graph G on $\binom{n+1}{2}$ edges is shown to have an ascending subgraph decomposition when either G is of bounded degree and sufficiently large order or when G is a star forest.

I. Introduction.

In [1] the authors give the following decomposition conjecture.

CONJECTURE: Let G be a graph with $\binom{n+1}{2}$ edges. Then the edge set of G can be partitioned into n sets generating graphs $G_1, G_2, \ldots, G_n$ such that $|E(G_i)| = i$ (for $i = 1, 2, \ldots, n$) and G_i is isomorphic to a subgraph of G_{i+1} for $i = 1, 2, \ldots, n-1$.

A graph G which can be decomposed as described in the conjecture will be said to have an *ascending subgraph decomposition* (abbreviated ASD). The graphs $G_1, G_2, \ldots, G_n$ are said to be members of such a decomposition.

We establish that the conjecture holds for certain classes of graphs. In particular we show the conjecture holds if G is of bounded degree and of sufficiently large order or if G is a star forest. Surprisingly the latter of these, when G is a star forest, is the most difficult to prove. This could indicate that the conjecture (if true) is a difficult one to prove.

The ascending subgraph decomposition of a graph is also closely related to the packing problem considered in [2]. There the authors conjecture that the graph K_{n+1} can be decomposed into any n edge disjoint trees $T_1, T_2, \ldots, T_n$ where each T_i has i edges. It should be emphasized that this decomposition does not require that each T_i be isomorphic to a subgraph of T_{i+1}. This suggests the weaker conjecture that $G = K_{n+1}$ has an ASD where each member G_i is any tree on i edges. It is easy to see that K_{n+1} has an ASD when each G_i is a star $K_{1,i}$ or each G_i is a path P_{i+1} on i edges, but even the weaker form of the conjecture is unsolved.

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II Results.

The first theorem we present is a graph less special than K_{n+1}, but which still has the special ASD into stars.

THEOREM 1. *Let H_{n-1} be any $n-1$ edge graph with at most n vertices. Then $K_n - H_{n-1}$ has an ASD with $K_{1,1}, K_{1,2}, \ldots, K_{1,n-2}$ the members of the decomposition.*

PROOF: The proof is by induction on n, being trivial when n is small. Assume the result holds for the graph $K_{n-1} - H_{n-2}$, i.e. this graph has an ASD into $K_{1,1}, K_{1,2}, \ldots, K_{1,n-3}$. Let v be the vertex of largest degree in $K_n - H_{n-1}$. Clearly its degree, $\deg v = n-1$ or $n-2$. We consider these two possible cases separately.

CASE 1: $\deg(v) = n-2$.

The graph $K_n - H_{n-1} - v$ which results by deleting v from $K_n - H_{n-1}$ can be written as $K_{n-1} - H_{n-2}$. By assumption $K_{n-1} - H_{n-2}$ has an ASD into the stars $K_{1,1}, K_{1,2}, \ldots, K_{1,n-3}$. But v is incident to precisely $n-2$ edges of $K_n - H_{n-1}$. Therefore this star $K_{1,n-2}$ with center at v together with the decompositon $K_{1,1}, K_{1,2}, \ldots, K_{1,n-3}$ of $K_n - H_{n-1} - v$ yield the desired ASD of $K_n - H_{n-1}$.

CASE 2: $\deg(v) = n-1$

We again consider the graph $K_n - H_{n-1} - v$. Let w and z be any two nonadjacent vertices in $K_n - H_{n-1} - v$. Modify the graph $K_n - H_{n-1} - v$ by inserting edge wz. The resulting graph $K_n - H_{n-1} - v + wz$ can be written as $K_{n-1} - H_{n-2}$ and has by assumption an ASD into stars $K_{1,1}, K_{1,2}, \ldots, K_{1,n-3}$. Since wz is an edge of one of these stars, we may assume it is the star $K_{1,i}$ with center at vertex w. Replace the edge wz of $K_{1,i}$ by the edge wv in $K_n - H_{n-1} - v$ obtaining an ASD of $K_n - H_{n-1} - v + wv$ into stars $K_{1,1}, K_{1,2}, \ldots, K_{1,n-3}$. The remaining $n-2$ edges of $K_n - H_{n-1}$ incident to v (other than wv) form a star $K_{1,n-2}$ with center at v. This star together with the ASD of $K_n - H_{n-1} - v + wv$ into $K_{1,1}, K_{1,2}, \ldots, K_{1,n-3}$ give the desired decomposition of $K_n - H_{n-1}$.

In order to prove the next theorem we need the following lemma.

LEMMA 2: Let $M_1, M_2, \ldots, M_\ell$ be a partition of a matching of the graph G into ℓ sets and let $e_1, e_2, \ldots, e_m$ be any collection of m edges in $G - \bigcup_{i=1}^{\ell} M_i$. If $\ell \geq m+2$, then the set of edges $(\bigcup_{i=1}^{\ell} M_i) \cup \{e_1, e_2, \ldots, e_m\}$ can be partitioned into sets of matchings $M_1', M_2', \ldots, M_\ell', M_{\ell+1}'$ such that $|M_i| = |M_i'|$ for $i = 1, 2, \ldots, \ell$ and $|M_{\ell+1}'| = m$.

PROOF: Form a bipartite graph G' whose vertex set $A \cup B$ is $A = \{M_1, M_2, \ldots, M_\ell\}$ and $B = \{e_1, e_2, \ldots, e_m\}$. Let a vertex $M_i \epsilon A$ be adjacent in G' to the vertex $e_j \epsilon B$ if e_j is not incident in G to any vertex of M_i. Since e_j can be incident in G to at most two elements of A, it follows from $\ell \geq m+2$ that each element of B is adjacent in G' to at least m elements of A. Thus each t-element subset of B has at least t adjacencies in A. By the theorem of P. Hall [3] the elements of B can be matched in G' to m elements of A. Assume without loss of generality the e_i is matched with M_i for $i = 1, 2, \ldots, m$. For each i, select any fixed edge $e'_i \epsilon M_i$, $i = 1, 2, \ldots, m$. Letting $M'_i = (M_i - \{e'_i\}) \cup \{e_i\}$ for $i = 1, 2, \ldots, m$, $M'_i = M_i$ for $i = m+1, \ldots, \ell$, and $M'_{\ell+1} = \{e'_1, e'_2, \ldots, e'_\ell\}$ gives the desired partition.

THEOREM 3. *Let G be a graph of maximum degree d on $\binom{n+1}{2}$ edges. If $n \geq 4d^2 + 6d + 3$, then G has an ASD into graphs $G_1, G_2, \ldots, G_n$ with each G_i a matching (on i edges).*

PROOF: By Vizing's Theorem [4] G has edge chromatic number at most $d+1$. Hence the edges of G can be decomposed into sets (subgraphs) $\mathcal{M}_1, \mathcal{M}_2, \ldots, \mathcal{M}_r$ ($r \leq d+1$) with each $\mathcal{M}_i$ a matching in G. We wish to partition the edge set into graphs $G_1, G_2, \ldots, G_n$ such that each G_i is a matching with i edges. To do this we start by splitting each $\mathcal{M}_i$ into graphs (sets) such that $\bigcup_{i=1}^{r} \mathcal{M}_i$ contains the graphs $G_n, G_{n-1}, \ldots, G_{n-k}$, with each of these graphs G_i contained entirely in some $\mathcal{M}_j$, and such that k has the largest possible value. If this can be done such that $k = n-1$, then the proof is complete.

Assume $k < n-1$, set $s = n-k-1$, and let $R_j = \mathcal{M}_j - (\bigcup_{i=0}^{k} G_{n-i})$, $j = 1, 2, \ldots, r$. We have found all the desired graphs except for $G_1, G_2 \ldots, G_s$. Also $s(s+1)/2 = |E(\bigcup_{j=1}^{r} R_j)| \leq (s-1)(d+1)$ where the last inequality follows from the choice of s. This gives $s < 2d+1$.

The idea of the proof is repeated use of the lemma in the following way. For each m, $1 \leq m \leq s$, we find $m+2$ graphs $G_{j_1}, G_{j_2}, \ldots, G_{j_{m+2}}$ such that all are contained in some $\mathcal{M}_j$. Specifically for each m ($1 \leq m \leq s$) select a set of m (unused) edges $e_1, e_2, \ldots, e_m$ in $\bigcup_{j=1}^{r} R_j$. Letting $G_{j_1}, G_{j_2}, \ldots, G_{j_{m+2}}$ correspond to the matchings $M_1, M_2, \ldots, M_{m+2}$ of the lemma we obtain, by the lemma, a new graph G_m (corresponding to $M'_{\ell+1}$), new graphs isomorphic to $G_{j_1}, G_{j_2}, \ldots, G_{j_m}$ (corresponding to $M'_1, M'_2, \ldots, M'_m$), and retain the graphs $G_{j_{m+1}}, G_{j_{m+2}}$ (corresponding to M'_{m+1}, M'_{m+2}). This means that disjoint collections of m elments are needed for $m = 1, 2, \ldots, s$ plus 2 additional elements to always insure the existence of the collection $\{G_{j_1}, G_{j_2}, \ldots, G_{j_{m+2}}\}$ for each m. But the largest of these collections has $s+2$ elements, so that at most $s+1$ of the graphs G_{n-i} ($i = 0, 1, 2, \ldots, k$) which appear in any $\mathcal{M}_j$ may not be usable in finding the s disjoint collections. This means that the proof is complete if the number of graphs in the list $G_n, G_{n-1}, \ldots, G_{n-k}$ is as large as the sum $2 + \sum_{i=1}^{s} i$ plus the nonusable part in each $\mathcal{M}_j$, i.e. if $k+1 \geq 2 + \sum_{i=1}^{s} i + (s+1)r$. But

$$2+\sum_{i=1}^{s} i+(s+1)r = 2+s(s+1)/2+(s+1)r \leq 2+(2d)(2d+1)/2+(2d+1)(d+1) = 4d^2+4d+3.$$

Since $k+1 = n-s$, the proof is complete if $n \geq 4d^2+6d+3$. But this is a condition of the theorem, completing the proof.

The reader should observe that in both Theorems 1 and 3 the graphs G_i which appear in the decomposition are all of the same type. In Theorem 1 each member is a star and in Theorem 3 each member is a matching. In the next theorem we consider the case where G is a star forest and show it has an ASD. Surely such a star forest could contain too few stars to contain an n-matching and no star with n or more edges so that an ASD would not need to have each of its members of the same type.

THEOREM 4. *Let G be a star forest with $\binom{n+1}{2}$ edges. Then G has an ASD.*

PROOF: To describe the desired decompositon we will use the convention that each of the edges of graph G_i will be assigned the label i. Thus an ASD is an assignment of labels $1, 2, \ldots, n$ to the edges of G such that each subgraph G_i (generated by those edges with label i) is isomorphic to the subgraph G_{i+1} (generated by edges with label $i+1$).

The proof will be by induction on n and is trivial for small values. We assume throughout that all star forests with $\binom{t}{2}$ edges, $t \leq n$, have an ASD.

Let G be a star forest wtih $\binom{n+1}{2}$ edges. We consider three separate cases.

CASE 1: The stars of G with at most n edges have collectively at least n edges.

Let $H_1, H_2, \ldots, H_\ell$ be the stars of G with at most n edges. Assume $|H_1| \leq |H_2| \leq \cdots \leq |H_\ell|$. Delete exactly n edges from these ℓ stars, starting with all edges from the largest star H_ℓ. Thus assume that all the edges of $H_\ell, H_{\ell-1}, \cdots, H_{\ell-m}$ have been removed and possibly some (but not all) of the edges of $H_{\ell-m-1}$ have been removed in this deletion. We will eventually assign the labels $1, 2, \cdots, n$ to these n edges.

Let G' be the graph which results from the deletion of the n edges. By assumption G' has an ASD with members $G'_2, G'_3, \ldots, G'_n$ such that each G'_i has $i-1$ edges with each of its edges assigned label i. Further G'_i is isomorphic to a subgraph of G'_{i+1} for $i = 2, 3, \ldots, n-1$.

It is clear that the number of edges which remain on the star $H_{\ell-m-1}$ after the deletion is less than the total number of edges deleted from the stars $H_\ell, H_{\ell-1}, \ldots, H_{\ell-m}$. Thus the labels assigned to the n deleted edges may be done so that if $H_{\ell-m-1}$ and G'_i have an edge in common, then some edge deleted from one of the other stars receives label i. This gives the desired decomposition, i.e. if $L_1, L_2, \ldots, L_n$ represent the single edge graphs with labels $1, 2, \ldots, n$ respectively, then $G_1 = L_1$ and $G_i = G'_i \cup L_i$ for $i = 2, 3, \ldots, n$ is an ASD of G.

CASE 2: The graph G contains two stars $K_{1,\ell}$ and $K_{1,m}$ such that $n \leq \ell \leq m$.

Define the function f such that for a positive integer $k < n$, $f(k) = 1+2+\cdots+k+(k+$

1)$(n-k)$. Observe that $f(k+1)-f(k)=n-k-1$. Select the smallest positive k such that $m \leq f(k) \leq m+\ell$. This choice of k is possible since $n \leq \ell$ and $f(k+1)-f(k)=n-k-1$.

Our objective is to delete f(k) edges from the two stars, invoke induction on the resulting graph, and then join the $f(k)$ edges appropriately to the ASD found by induction. The $f(k)$ edges we delete include all the edges of the large star $K_{1,m}$ and an appropriate number from the small star $k_{1,\ell}$. We shall split the set of deleted edges into n stars $L_1, L_2, \ldots, L_n$ such that star L_i receives label i and $L_1 = K_{1,1}, L_2 = K_{1,2}, \ldots, L_k = K_{1,k}$ and $L_i = K_{1,k+1}$ for $i = k+1, k+2, \ldots, n$. It is obvious that the L_i's can be defined as described but we need further restrictions to apply the induction.

Let G' be the graph with $\binom{n-k}{2}$ edges obtained when the $f(k)$ edges are deleted from G. By the induction assumption G' has an ASD with members $G'_{k+2}, G'_{k+3}, \ldots, G'_n$ where each G'_i has $i-k-1$ edges, each assigned label i.

Let H denote the part of the star $K_{1,\ell}$ which remains after the deletion of the $f(k)$ edges. Further let $S_i = H \cap G'_i$ for $i = k+2, k+3, \ldots, n$. We wish to form the ASD for G with members $G_1, G_2, \ldots, G_n$ by setting $G_i = L_i$ for $i = 1, 2, \ldots, k+1$ and $G_i = L_i \cup G'_i$ for $i = k+2, k+3, \ldots, n$. To insure that this gives our ASD of G we need to make certain that L_i and G_i are vertex disjoint. This may require exchanging some of the stars S_i $(k+1 \leq i \leq n)$ with substars of $K_{1,m}$ (part of the deleted set of edges).

If $\ell - |E(\bigcup_{i=k+2}^{n} S_i)| \leq \binom{k+1}{2}$, then no exchange is necessary. Simply select a subcollection of $L_1, L_2, \ldots, L_{k+1}$ whose total edge set has cardinality $\ell - |E(\bigcup_{i=k+2}^{n} S_i)|$. Members of this subcollection are obtained from the edges of $K_{1,\ell}$ that were deleted and the remaining L_i are all obtained from the large star $K_{1,m}$.

Therefore consider the remaining case when $\ell - |E(\bigcup_{i=k+2}^{n} S_i)| > \binom{k+1}{2}$. Since $f(k) \geq m \geq \ell$, there exists a subsequence $S_{i_1}, S_{i_2}, \ldots, S_{i_k}$ of $S_{k+2}, S_{k+3}, \ldots, S_n$ (each S_{i_j} with fewer than $k+1$ edges) such that

$$0 < \ell - [(k+1)r - |E(\bigcup_{j=1}^{n} S_{i_j})|] - |E(\bigcup_{i=k+2}^{n} S_i)| \leq \binom{k+1}{2}.$$

Let r be as small as possible such this inequality holds. Then, exchange $S_{i_1}, S_{i_2}, \ldots, S_{i_r}$ with disjoint substars of $K_{1,m}$, changing the set of deleted edges. In this case let $L_{i_1}, L_{i_2}, \ldots, L_{i_r}$ come from the star $K_{1,\ell}$ (part of the new set of deleted edges). In addition select a subcollection of $L_1, L_2, \ldots, L_{k+1}$ whose total edge set has cardinality

$$\ell - [(k+1)r - |E(\bigcup_{j=1}^{r} S_{i_j})|] - |E(\bigcup_{i=k+2}^{n} S_i)|.$$

Members of this subcollection also come from the star $K_{1,\ell}$ and are part of the new deleted edge set. All remaining L_i come from unusued edges of $K_{1,m}$.

Thus in each case an ASD for G with members $G_1, G_2, \ldots, G_n$ is obtained by setting $G_i = L_i$ for $i = 1, 2, \ldots, k+1$ and $G_i = L_i \cup G'_i$ for $i = k+2, k+3, \ldots, n$. This completes the proof of case 2.

CASE 3: . Cases 1 and 2 fail to hold.

In this case G consists of one star with more than n edges and the remaining stars contain collectively at most $n-1$ edges. For this particular case we can construct the members of an ASD directly. Let t be the total number of edges in the "small" stars, those which have fewer than n edges. Split these t edges into single edge graphs $L_{n+1-t}, L_{n+2-t}, \ldots, L_n$. The large star has precisely $\binom{n+1}{2} - t$ edges. Decompose it into the following stars: $G'_i = K_{1,i}$ for $i = 1, 2, \ldots, n-t$ and $G'_i = K_{1,i-1}$ for $i = n-t+1, n-t+2, \ldots, n$. Then letting $G_i = G'_i$ for $i = 1, 2, \ldots, n-t$ and $G_i = L_i \cup G'_i$ for $i = n-t+1, \ldots, n$ gives an ASD for G.

This completes the proof of this case and the proof of the theorem.

III. Conclusion.

At least two interesting questions are suggested by the results of this paper. The first of these was suggested by P. Erdös when he learned of Theorem 4.

QUESTION 1: Let G be a star forest with $\binom{n+1}{2}$ edges such that each star of the forest has more than n edges. Does G have an ASD in which each member is a star?

QUESTION 2: Let G be a graph with $\binom{n+1}{2}$ edges. Does G have an ASD such that each member is a star forest?

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