ON \((n,k)\)-COLORINGS OF COMPLETE GRAPHS

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ABSTRACT

An \((n,k)\)-coloring of a complete graph \(K\) means a coloring of the edges of \(K\) with \(k\) colors so that all monochromatic connected subgraphs have at most \(n\) vertices. We are interested in the maximum number of vertices of complete graphs with \((n,k)\)-colorings. We survey results concerning this problem and give some new results which lead to the complete solution for \(k \leq 5\).

0. Introduction.

Let \(f(n,k)\) denote the smallest integer \(m = m(n,k)\) with the following property: if the edges of \(K_m\) are colored with \(k\) colors then there exists a monochromatic connected subgraph of more than \(n\) vertices. The function \(f(n,k)\) has been introduced in [12] and \(f(n,3)\) was determined in [12] and [1]. The observation \(f(n,2) = n+1\) is equivalent with a remark of Erdős and Rado saying that for any graph \(G\), either \(G\) or its complement is connected. The second author has further results on \(f(n,k)\) in [13]. The problem of determining \(f(n,k)\) was rediscovered by the first author and Brandis in [3].

From the point of view of Ramsey theory, \(f(n,k)-1\) is a lower bound for the Ramsey number \(R(T_n,k)\), where \(T_n\) is any tree of \(n\) edges. Bounds on \(R(T_n,k)\) have been studied in [10].

We shall use the term \((n,k)\)-coloring introduced by Bierbauer and Brandis in [3]. An \((n,k)\)-coloring is a coloring of the edges of a complete graph with \(k\) colors so that all connected monochromatic subgraphs have at most \(n\) vertices.

\(^1\) Partly supported by the AKA Research Fund of the Hungarian Academy of Sciences

CONGRESSUS NUMERANTUM 58(1987), pp.123-139
n vertices. The function \( f(n,k)-1 \) clearly give the largest number of vertices of a complete graph which has an \((n,k)\)-coloring.

An \((n,k)\)-coloring can be viewed as \( k \) partitions of a ground set into sets of cardinality at most \( n \), so that all pairs of elements appear together is some of the sets. Thus resolvable block designs with \( \lambda = 1 \), \( k \) parallel classes and with blocksize \( n \) are natural examples of \((n,k)\)-colorings. However, \((n,k)\)-colorings are much more "relaxed" structures: the "blocks" may have any sizes up to \( n \) and the pairs of the ground set appear together in at least one block. In extremal \((n,k)\)-colorings, i.e. in \((n,k)\)-colorings of complete graphs of \( f(n,k)-1 \) vertices, the structure of connected monochromatic components is often close to the block structure of resolvable block designs.

In section 2 we review results on \( f(n,k) \) and present some new results. We give a new lower bound of \( f(n,k) \):

\[
f(n,k) > (k-1)^2 (p+1) - W_i(k-1)
\]

if \( n = (k-1)p+k-1-i \), \( 0 < i < k-1 \), and an affine plane of order \( k-1 \) exists (Theorem 1.5). Here \( W_i(q) \) denotes the minimum number of points of an affine plane \( A_q \) of order \( q \) which meet every line of \( A_q \) in at least \( i \) points. The minimum is taken over all affine planes of order \( q \). The bound is always sharp for \( 3 \leq k \leq 5 \) (see Theorems 1.16, 1.17, 1.18). If we compare this lower bound with the upper bound of \( n(k-1)+1 \) (Theorem 1.1), we see that for fixed \( k \) the function \( f(n,k)-n(k-1) \) is smaller than a function depending only on \( k \). It is unknown whether a similar statement holds if no affine plane of order \( k-1 \) exists (Problem 1.15). The main results of the paper are prepared in section 2, where a method is described to get an upper bound of \( f(n,k) \). The upper bound \( n(k-1)+1 \) (Theorem 1.1) comes from a joint result of J. Lehel and the second author: if the edges of a complete bipartite graph \( K_{m,n} \) are colored with \( s \) colors then there exists a monochromatic connected subgraph of at least \( \lceil (m+n)/s \rceil \) vertices (Corollary 2.2). Our main concern is to push this method to
its limit, i.e. to prove that $K_{m,n}$ contains a monochromatic connected subgraph of at least $\lceil m/s \rceil + \lceil n/s \rceil$ vertices in every $s$-coloring (Conjecture 2.4). We can prove this for $2 \leq s \leq 4$ (Corollary 2.3), which gives an essential part in determining $f(n,k)$ for $3 \leq k \leq 5$.

In section 2 the properties of $M$-extremal graphs play an important role. For fixed $m,n$, we call a bipartite graph $G$ $M$-extremal if the vertex classes of $G$ contain $m$ and $n$ vertices, the connected components of $G$ have at most $M$ vertices and $G$ has as many edges as possible under these conditions. The important properties of $M$-extremal graphs are summarized in Lemma 2.5. The main application of $M$-extremal graphs is the following Theorem (Theorem 2.1).

If $G$ is a bipartite graph with $m$ and $n$ vertices in its colour classes and $G$ has at least $\lceil mn/s \rceil$ edges, then $G$ contains a connected component of at least $\lceil (m+n)/s \rceil$ vertices. Moreover, if $2 \leq s \leq 4$ then $G$ contains a connected component of at least $\lceil m/s \rceil + \lceil n/s \rceil$ vertices.

In section 3 we apply our methods to determine $f(n,k)$ for $3 \leq k \leq 5$:

$$f(n,3) = \begin{cases} 
4p+1 & \text{if } n=2p \\
4p+2 & \text{if } n=2p+1 \\
16p+1 & \text{if } n=4p 
\end{cases}$$

$$f(n,4) = \begin{cases} 
9p+1 & \text{if } n=3p \\
9p+2 & \text{if } n=3p+1 \\
9p+5 & \text{if } n=3p+2 
\end{cases}$$

$$f(n,5) = \begin{cases} 
16p+2 & \text{if } n=4p+1 \\
16p+7 & \text{if } n=4p+2 \\
16p+10 & \text{if } n=4p+3 
\end{cases}$$

The authors recently learned that $f(n,4)$ have been determined independently by Bialostocki and Dierker.

1. Results on $f(n,k)$.

In this section we survey results concerning $f(n,k)$ and present some new results. We start with upper bounds.

**Theorem 1.1.** ([13]) $f(n,k) \leq (k-1)n + 1$.

We note that Theorem 1.1 immediately follows from the following result of J. Lehel and the second author: if the edges of a complete bipartite graph $G$ are colored with $k-1$ colors then $G$ contains a monochromatic connected
subgraph of at least \( \lceil |V(G)|/(k-1) \rceil \) vertices. This result appears in section 2 as Corollary 2.2. Another proof of Theorem 1.1 is obtained if we consider the following hypergraph \( H \), determined by an \((n,k)\)-coloring of a complete graph \( K \). The vertices of \( H \) are the vertices of \( K \) and the edges of \( H \) are the vertex sets of the connected monochromatic components of \( K \). The dual hypergraph \( H^* \) of \( H \) is a \( k \)-partite intersecting hypergraph (every two edges of \( H^* \) have at least one common vertex). A result of Furedi ([11]) says that a \( k \)-uniform intersecting hypergraph \( H \) has a vertex of degree at least \( \lceil |E(H)|/(k-1) \rceil \), unless \( H \) is a projective plane of order \( k-1 \). Since a \( k \)-partite hypergraph is never a projective plane, \( H^* \) contains a vertex of degree at least \( \lceil |E(H^*)|/(k-1) \rceil \) which implies that \( H \) contains an edge with at least \( \lceil |V(H)|/(k-1) \rceil \) vertices and Theorem 1.1 follows.

The following upper bound is due to the first author and Brandis:

**Theorem 1.2 ([3]).** Assume \( k \equiv K \pmod{n} \), \( 2 \leq K < n \) and let \( \lambda = (k-K)/(k-1) \). If \( 4K > 3n+\lambda - \lceil \left(n(n+8-2\lambda) - \frac{2(8-\lambda)}{\lambda} \right) \rceil^{1/2} \), then
\[
f(n,k) \leq (k-1)n-(k-K)+1.
\]
Otherwise
\[
f(n,k) \leq k(n-1)+1/2-(n(k-\lambda-k(K-\lambda-1)-(\lambda-1/4)^{1/2})+1.
\]

The cases not covered by Theorem 1.2 are covered by the following two results.

**Theorem 1.3 ([3]).** If \( n \geq 2 \), \( 1 \leq k \equiv 1 \pmod{n} \), then
\[
f(n,k) \leq k(n-1)+2.
\]

Equality holds iff a resolvable block design exists with \( \lambda = 1 \), block size \( n \) and replication \( k \).

**Theorem 1.4 ([3]).** If \( k \equiv 0 \pmod{n} \), then
\[
f(n,k) \leq k(n-1)+1,
\]
and for \( n > 2 \), \( f(n,n) \leq n(n-1) \).

For comparison, it is easy to see that for \( n > k \geq 2 \), Theorem 1.1 is better than Theorem 1.2. If \( n \leq k \) then Theorems 1.2–1.4 are better than Theorem 1.1.

Concerning lower bounds of \( f(n,k) \), first we give a construction which uses the existence of an affine plane of order \( k-1 \). The lower bound is close to the upper bound of
Theorem 1.1 and for \( k=3,4,5 \) it gives the exact value of \( f(n,k) \). Let \( A_q \) denote an affine plane of order \( q \) and let \( X_1,\ldots,X_{q+1} \) be the ideal points of \( A_q \). Assume that we have a complete graph \( K \) whose vertex set is partitioned into \( q^2 \) parts, \( S_1,S_2,\ldots,S_{q^2} \). Consider a one-to-one mapping between the points of \( A_q \) and the sets \( S_1,\ldots,S_{q^2} \). We color an edge \( PQ \) of \( K \) with color \( k \) if \( P \in S_i, Q \in S_j, i \neq j \) and the points corresponding to \( S_i \) and \( S_j \) in \( A_q \) determine a line containing \( X_k \). The edges of \( K \) whose endpoints belong to the same set \( S_i \) may be colored arbitrarily. The colorings of complete graphs obtained by this method are called normal \( (q+1) \)-colorings. Note that normal \( (q+1) \)-colorings are defined only for those values of \( q \) for which an affine plane of order \( q \) exists.

An \( i \)-transversal of an affine plane \( A_q \) is a set of points in \( A_q \) which meet every line of \( A_q \) in at least \( i \) points. Let \( w_i(A_q) \) denote the minimum cardinality of an \( i \)-transversal of \( A_q \) and let \( w_i(q) \) be \( \min w_i(A_q) \), where the minimum is taken over all affine planes of order \( q \). The following Theorem gives a lower bound for \( f(n,k) \) in terms of \( w_i(k-1) \).

**Theorem 1.5.** Assume that an affine plane of order \( k-1 \) exists, let \( n=(k-1)p+k-1-i \), where \( 0<i \leq k-1 \). Then

\[
f(n,k) \geq (k-1)^2(p+1)-w_i(k-1).
\]

**Proof.** Let \( A_{k-1} \) be an affine plane of order \( k-1 \) possessing an \( i \)-transversal \( T \) of \( w_i(k-1) \) elements. Let \( m=(k-1)^2(p+1)-w_i(k-1) \) and consider a normal \( k \)-coloring of \( K_m \), where we associate a set of \( p \) elements to the points of \( T \) and we associate a set of \( p+1 \) elements to the points outside \( T \). By the definition of the normal coloring and the \( i \)-transversal, a monochromatic connected component of \( K_m \) has at most \( pi+(k-1-i)(p+1)=n \) vertices. Thus we have an \( (n,k) \)-coloring on \( K_m \) and the theorem follows.

**Corollary 1.6.** If an affine plane of order \( k-1 \) exists, then

\[
f((k-1)p,k) \geq (k-1)^2p \text{ for all } p \geq 1.
\]

This follows as \( w_{k-1}(k-1)=(k-1)^2 \). Another easy application of Theorem 1.5 occurs if \( i=k-2 \). Now \( w_{k-2}(k-1)=(k-1)^2-1 \) is obvious which yields
Corollary 1.7. If an affine plane of order $k-1$ exists, then
\[ f((k-1)p+1,k) > (k-1)^2 p + 1. \]

We observe that $w_1(k-1) \leq 2k-3$ since two intersecting lines of $A_{k-1}$ give a 1-transversal. Thus we have

Corollary 1.8. If an affine plane of order $k-1$ exists, then
\[ f((k-1)p+k-2,k) > (k-1)^2 p + (k-2)^2. \]

A fundamental result of Jamison's ([7]) implies that 1-transversals (also called "affine blocking sets") in desarguesian $A_9$ have at least $2q-1$ points. It is however possible to obtain 1-transversals of smaller size in other affine planes. Bruen and de Resmini ([8]) use the Hughes plane of order 9 to show $w_1(9) \leq 16$.

Corollary 1.9. $f(9p+8,10) > 81p+65$.

We note that Corollary 1.6 is sharp for all $k$ and corollary 1.7 is sharp for $k=3,4,5$.

A resolvable BIBD with blocks of cardinality $n$, with $\lambda=1$ and with $k$ parallel classes is clearly suitable to define an $(n,k)$-coloring. In this case we have $k(n-1)+1$ points. If we substitute $t$ points for all points of this design, an $(nt,k)$-coloring can be defined on $t(k(n-1)+1)$ vertices in analogy to normal colorings. Thus we have

Proposition 1.10. If a resolvable BIBD exists with blocks of cardinality $n$, with $\lambda=1$ and with $k$ parallel classes then
\[ f(nt,k) > t(k(n-1)+1) \]

An example for the application of Proposition 1.10 is the case $t=1$, $n=4$, $k=9$. Now $f(4,9)=29$ follows from Theorem 1.3 and Proposition 1.10. This example is taken from [3]. As there are resolvable BIBD with block size 4, with $\lambda=1$ and replication $4t+1$ ($t \geq 1$) (see [17,18]), we get
\[ f(4a,4t+1) > a(12t+4) \quad (a \geq 1, t \geq 1). \]

We obtain a general lower bound.

Corollary 1.11. $f(n,k) > (n-3)(3k-8)/4$.

The following generalizes a result of [3], which in turn relies upon a construction of Linstrom in [16]:

Proposition 1.12. If $f(ap,k_0) > kp$ and if there is a set of
If an affine plane of order \( k-1 \) exists, then

\[ f(n,k) \leq n(k-1) + (k-1) - (k-1)i \leq (k-1)i, \]

for \( n=(k-1)p+k-1-i, 0<i\leq k-1. \)

In particular, if \( i=k-1 \) then \( w_i(k-1) = (k-1)^2 \) and we get

Corollary 1.13. If an affine plane of order \( k-1 \) exists, then

\[ f((k-1)p,k) = p(k-1)^2 + 1. \]

It would be interesting to get rid of the existence problem of affine planes of order \( k-1 \) in a lower bound close to \( n(k-1) \) for large \( n \). We have the following problem.

Problem 1.15. It is true that \( f(n,k) - n(k-1) \) is less than a function depending only on \( k \) ?

Now we consider \( f(n,k) \) for small values of \( k \). An old remark of Erdos and Rado says that a graph or its complement is connected. Thus \( f(n,2) = n+1 \). The case \( k=3 \) has been settled in [12] and [11]. A new proof is given in section 3 based on results of section 2.

Theorem 1.16.

\[
  f(n,3) = \begin{cases} 
    4p+1 & \text{if } n=2p \\
    4p+2 & \text{if } n=2p+1.
  \end{cases}
\]

The following two Theorems are the main new results of this paper. The proofs are in section 3.
Theorem 1.17.
\[ f(n,4) = \begin{cases} 
9p+1 & \text{if } n=3p \\
9p+2 & \text{if } n=3p+1 \\
9p+5 & \text{if } n=3p+2 
\end{cases} \]

Theorem 1.18.
\[ f(n,5) = \begin{cases} 
16p+1 & \text{if } m=4p \\
16p+2 & \text{if } n=4p+1 \\
16p+7 & \text{if } n=4p+2 \\
16p+10 & \text{if } n=4p+3 
\end{cases} \]

Concerning the values of \( f(n,k) \) for small values of \( n \), the following observation is in [3].

Proposition 1.19.
\[ f(2,k) = \begin{cases} 
2k+2 & \text{if } k \text{ is odd} \\
2k+1 & \text{if } k \text{ is even} 
\end{cases} \]

The case \( n=3 \) is also completely solved. The following results is in [3].

Theorem 1.20.
\[ f(3,k) = \begin{cases} 
2k+2 & \text{if } k \equiv 1 \pmod{3} \\
2k & \text{if } k \equiv 2 \pmod{3} 
\end{cases} \]

We note that theorem 1.20 follows by combining the upper bound of Theorem 1.2 with the lower bound of Proposition 1.10 and using the existence theorem of D.K. Ray-Chaudhuri, R.M. Wilson on resolvable triple systems ([17]). For \( n=3 \) and \( k \equiv 0 \pmod{3} \), it is easy to see that \( f(3,k) \) is either \( 2k \) or \( 2k+1 \). It is easy to prove that \( f(3,3)=6 \) (see [3]). Zs. Tuza discovered a \((3,6)\) coloring of \( K_{12} \) (4 color classes are \( 4K_3 \), one color class is \( K_3 + 3K_1,2 \) and one color class is \( 2K_3 + 3K_2 \)). This construction shows that \( f(3,6)=13 \). The first author used a certain Steiner triple system on 19 points to show \( f(3,9)=19 \) ([4]). If \( k \equiv 0 \pmod{3} \) and \( k \geq 9 \), there exist \((3,k)\)-colorings of \( K_{2k} \) such that all but one color class are isomorphic to \( 2k/3 K_3 \) and the exceptional color class is isomorphic to \( k K_2 \). Such colorings are called Nearly Kirkman Triple Systems and their existence have been proved in a series of papers ([15],[2],[6],[19] in chronological order). The authors are grateful to
Professor Rosa for this information. Therefore the following theorem holds.

Theorem 1.21

\[
f(3,k) = \begin{cases} 
6 & \text{if } k = 3 \\
2k+1 & \text{if } k \geq 6 \text{ and } k \equiv 0 \pmod{3}.
\end{cases}
\]

2. \textbf{M-extremal bipartite graphs.}

Let \( B(m,n) \) denote the set of bipartite graphs with vertex classes of cardinality \( m \) and \( n \). We shall always assume that \( m,n \geq 1 \). The purpose of this section is to prove the following

Theorem 2.1. Assume \( G \in B(m,n) \) and \( G \) has at least \( \lfloor mn/s \rfloor \) edges for some positive integer \( s \). Then \( G \) contains a connected component of at least \( \lfloor (m+n)/s \rfloor \) vertices. Moreover, if \( 2 \leq s \leq 4 \) then \( G \) contains a connected component of at least \( \lfloor m/s \rfloor + \lfloor n/s \rfloor \) vertices.

The first part of Theorem 2.1 gives a joint result of the second author with J. Lehel:

Corollary 2.2 ([13]). If the edges of \( K_{m,n} \) are colored with \( s \) colors, then there exists a monochromatic connected subgraph of at least \( \lfloor (m+n)/s \rfloor \) vertices.

The second part of Theorem 2.1 implies

Corollary 2.3. If the edges of \( K_{m,n} \) are colored with \( s \) colors and \( 2 \leq s \leq 4 \), then there exists a monochromatic connected subgraph of at least \( \lfloor m/s \rfloor + \lfloor n/s \rfloor \) vertices. We conjecture, that Corollary 2.3 holds for every \( s \).

Conjecture 2.4. If the edges of \( K_{m,n} \) are colored with \( s \) colors, then there exists a monochromatic subgraph of at least \( \lfloor m/s \rfloor + \lfloor n/s \rfloor \) vertices.

It is worth noting, that conjecture 2.4 cannot be obtained from a density result since the second part of Theorem 2.1 is not true for \( s \geq 5 \). To see this for \( s = 5 \), let \( m = 5p+1, n = 20p+1 \). Now \( 2K_{p,4p+1} + 2K_{p+1,4p} + K_{p-1,4p-1} \) has \( \lfloor mn/5 \rfloor \) edges, but its components have at most \( 5p+1 = \lfloor m/5 \rfloor + \lfloor n/5 \rfloor - 1 \) vertices.

It is convenient to introduce at this point the notion of \( M \)-extremal bipartite graphs. We shall always assume that \( M \) is an integer, \( M \geq 2 \). A graph \( G \in B(m,n) \) is called
If every connected component of $G$ has at most $M$ vertices and $G$ has the largest number of edges under this condition. It is clear, that an $M$-extremal bipartite graph is the union of disjoint complete bipartite graphs and possibly some isolated vertices in one of the vertex classes. If we accept $K_{a,t}$ and $K_{t,0}$ for $t \geq 1$ as degenerate complete bipartite graphs, then an $M$-extremal graph of $B(m,n)$ is the vertex-disjoint union of $K_{a_1,t_1}, K_{a_2,t_2}, \ldots, K_{a_r,t_r}$, where the numbers $a_i, b_i$ are non-negative integers, at most one of them can be zero, and they satisfy

1. $a_1 + a_2 + \ldots + a_r = m$, $b_1 + b_2 + \ldots + b_r = n$ and $a_i + b_i \leq M$ for all $i$ such that $1 \leq i \leq r$ and $a_i b_i \neq 0$.

Thus the description of $M$-extremal members of $B(m,n)$ is equivalent with finding values of $r$ and for the pairs $(a_i, b_i)$ such that (1) is satisfied and $E = \sum a_i b_i$ is maximum. Such a sequence is also called $M$-extremal.

$M$-extremal sequences (or $M$-extremal bipartite graphs) are not necessarily unique, for instance if $m = 4, n = 6, M = 4$, the the following sequence define $M$-extremal graphs:

$(2,2), (1,2), (1,2); (2,2), (2,2), (0,2); (2,2), (1,3), (1,1)$.  

Lemma 2.5. Let $m, n, M$ be fixed, $M > 2$, $r = \lceil (m+n)/M \rceil$, and let $\{(a_i, b_i) | i = 1, 2, \ldots, s\} \subseteq B(m,n)$ be an $M$-extremal sequence, $E = \sum a_i b_i$.

Then one of the following holds:

(i) $a = a_1 = a_2 = \ldots = a_r$, $a + 1 = a_{r-1} = \ldots = a_2$, $a = a_{r-1} = a_r$

$\begin{align*}
 b + 1 &= b_1 = b_2 = \ldots = b_r, \\
 m &= ra + r - 2 - t, \\
 n &= rb + t, \\
 m + n &= rM - 2, \\
 E &= rab + ta + b(r - 2 - t), \\
 r &= s.
\end{align*}$

(ii) $a = a_1 = a_2 = \ldots = a_r$, $a + 1 = a_{r-1} = \ldots = a_2$

$\begin{align*}
 b + 1 &= b_1 = b_2 = \ldots = b_r, \\
 m &= ra + r - t, \\
 n &= rb + t, \\
 m + n &= rM, \\
 E &= rab + ta + (r - t)b, \\
 r &= s.
\end{align*}$

(iii) $a = a_1 = a_2 = \ldots = a_r$, $a + 1 = a_{r-1} = \ldots = a_2$

$\begin{align*}
 b + 1 &= b_1 = b_2 = \ldots = b_r, \\
 m &= (r-1)a + r - 1 - t + a_r, \\
 n &= (r-1)b + t + b_r, \\
 m + n &= (r-1)M + a_r + b_r < rM, \\
 E &= (r-1)ab + ta + (r-1-t)b + a_r b_r, \\
 s &= r.
\end{align*}$
Here \( t, a, b \) are non-negative integers, \( a+b+1=M \).

We note that \( M \)-extremal sequences in forms (i), (ii) and (iv) may occur only for special choices of \( m, n, M \). It is easy to check, that \( M \) must be a divisor of \( m+n+2 \) if (i) occurs, \( M \) must divide \( m+n \) if (ii) occurs. Form (iv) appears if either \( n > (m+1)(M-1)+1 \) or \( m > (n+1)(M-1)+1 \) holds.

**Proof.** Let \( (a_1, b_1), \ldots, (a_m, b_m) \) be a \( M \)-extremal sequence. A pair \( (a_i, b_i) \) is called **saturated** if \( a_i + b_i = M \) and **unsaturated** if \( a_i + b_i < M \). A pair \( (a_i, b_i) \) is **exceptional** if \( a_i + b_i > M \). Clearly \( a_i \) or \( b_i \) is zero for an exceptional pair.

**Claim 1.** If \( (a_i, b_i) \) and \( (a_j, b_j) \) are unsaturated, then \( a_i = a_j \), \( b_i = b_j \) and \( a_i + b_i = M-1 \).

If \( a_i < a_j \) and \( b_i > b_j \) then \( b_i \neq 0 \) and then pairs \( (a_i, b_i), (a_j, b_j) \) can be changed into \( (a_i, b_i+1), (a_j, b_j+1) \) and the value of \( E \) increases by this change, contradiction. If \( b_i < b_j \), then either \( a_i \neq 0 \) or \( b_i \neq 0 \). Assume that \( a_i \neq 0 \). Now our pairs can be changed to \( (a_i-1, b_i), (a_i+1, b_j) \) to increase \( E \). The case \( b_i \neq 0 \) is symmetric. Thus \( a_i = a_j, b_i = b_j \). If \( a_i + b_i < M-1 \), then our pairs can be changed to \( (a_i-1, b_i-1), (a_j+1, b_j+1) \), increasing \( E \). The claim is proved.

**Claim 2.** If \( (a_i, b_i) \) and \( (a_j, b_j) \) are unsaturated, then all other pairs are saturated.

Assume there is an unsaturated pair \( (a_p, b_p) \). Using claim 1, \( a_i = a_p = a, b_i = b_p = b \) and \( A+B=M-1 \). Now the three \( (A,B) \)-pairs can be changed to \( (A+1,B), (A,B+1), (A-1,B-1) \) \((AB \neq 0)\) and \( E \) increases. An exceptional pair \( (c,0) \) can not occur as otherwise \( (c,0), (a_i, b_i) \) can be changed to \( (c-1,0), (a_i+1, b_i) \).

**Claim 3.** Let \( (a_i, b_i) \) and \( (a_j, b_j) \) be two pairs such that \( a_i + b_i \geq M-1 \), \( a_j + b_j \geq M-1 \), \( a_i b_i a_j b_j > 0 \). Then \( |a_i - a_j| \leq 1, |b_i - b_j| \leq 1 \).

Assume \( a_i - a_j \geq 2 \). Now \( b_j - b_i \geq 1 \) as otherwise \( a_i + b_i \geq M-1 + 2 = M+1 \) which contradicts (1). If we change our pairs to \( (a_i-1, b_i+1), (a_j+1, b_j-1) \), then \( E \) increases since \( (a_i-1) (b_i+1) + (a_j+1) (b_j-1) = a_i b_i + a_j b_j + a_i - a_j + b_j - b_i - 2 \geq a_i b_i + a_j b_j + 1 \). The claim is proved.
To continue with the proof of Lemma 2.5, assume that there are two unsaturated pairs in an M-extremal sequence. Claim 1 ensures, that both pairs are in the form \((a, b)\) where \(a+b=M-1\). If \((a_0, b_0)\) is any other pair then \(a_0+b_0=M\) by claim 2. Clearly \(ab \neq 0\) and claim 3 implies \(a_i=a\) or \(a_i=a+1\). Now we have our M-extremal sequence in form (i), where \(t\) denotes the number of indices \(i\) for which \(a_i=a\) and \((a_i, b_i)\) is saturated. Since \(m+n=sM-2\), \(r=s\) follows.

Assume that a M-extremal sequence contains exactly one unsaturated pair, \((a_\ast, b_\ast)\). All other pairs are saturated since an exceptional pair \((a_\ast, 0)\) would allow the changes \((a_\ast-1, 0), (a_\ast+1, b_\ast)\) contradicting the M-extremal property. It is obvious that \(r=s\). In order to see that our sequence is in form (iii), we have to prove \(a_\ast \geq a_\ast\) and \(b_\ast \geq b_\ast\) for all \(i\), \(1 \leq i \leq r-1\). Assume \(a_i < a_\ast\). Then \(b_i < b_\ast+2\) since \((a_i, b_i)\) is saturated and \((a_\ast, b_\ast)\) is unsaturated. We can change the pairs \((a_i, b_i), (a_\ast, b_\ast)\) to \((a_\ast-1, b_\ast+1), (a_\ast+1, b_\ast-1)(a_\ast>0, b_\ast \geq 2)\). We get a contradiction since \((a_i-1)(b_i+1)+(a_\ast+1)(b_\ast-1)=a_\ast b_\ast+a_\ast b_\ast+b_\ast-\ast b_\ast+\ast a_\ast-2+a_\ast b_\ast+a_\ast b_\ast+1\).

Finally, assume that there are no unsaturated pairs in an M-extremal sequence. If no exceptional pair is present, then all pairs are saturated, \(r=s\) and the sequence is in form (ii) by claim 3. Assume there is exactly one exceptional pair. By symmetry we can choose it as \((0, c)\). Since \(m \geq 1\), there exist other pairs. Let \((a_1, b_1)\) be any such pair. Obviously \((a_1, b_1)\) is saturated and \(a_1 > 0\). Now we change the pairs \((0, c), (a_1, b_1)\) to \((0, c-M+1), (1, M-1), (a_1-1, b_1)\). If \(b_1 < M-1\), the \(a_1 b_1 < 1(M-1)+(a_1-1)b_1\) and we reach a contradiction. Therefore \(b_1 = M-1\), i.e. the M-extremal sequence has form (iv).

Proof of Theorem 2.1 Let \(c(G)\) denote the maximal number of vertices in a connected component of \(G\). To prove the first part of the Theorem, we have to show:

If \(c(G) < \lceil(m+n)/s \rceil\), then \(|E(G)| < \lceil mn/s \rceil\), equivalently: if \(G\) is M-extremal, \(sM < m+n\), then \(s|E(G)| < mn\). Consider \(M\) as fixed. It suffices to prove \(s|E(G)| < mn\) for the maximal number \(s\) satisfying \(sM < m+n\), i.e. for \(s=r-1\).
In case (i) of Lemma 2.5 we have to show 
\[(r-1)ab+(r-1)ta+(r-1)(r-2-t)b < r^2ab+rat+rb(r-2-t)+t(r-2-t),\]
equivalently 
\[-a(t+rb) < (t+b)(r-2-t),\]
which is clearly true.

In case (ii) we have to show 
\[(r-t)b < r^2 ab+rat+rb(r-t)+t(r-t),\]
equivalently 
\[-a(t+rb) < (t+b)(r-t),\]
true like before.

In case (iii) we have to show 
\[(r-t)a_b+a(r-t)(r-t)b < a(t+rb)(r-t),\]
equivalently 
\[-a(t+rb) < (t+b)(r-t),\]
true like before.

The inequality is true as either \(a \leq a\) or \(b \leq b\).

In case (iv) by symmetry we have to consider only the first case.

As \(m+n > (m+1)M\), we can choose \(s \geq m+1\). We have to prove, that \(s(M-1) < n\) if \(sM < m+n\). This is true as \(m < s\).

Let us proceed to the proof of the second part of Theorem 2.1. Let \(s \in \{2,3,4\}\), \(G\) an \(M\)-extremal graph, \(M < [m/s] + [n/s]\). We have to show \(s|E(G)| < mn\). Because of the first part of the theorem, it suffices to consider the case \(m=sx+1, n= sy+1, M=x+y+1\). Then \(m+n=sM+2-s\). As \(2 \leq s \leq 4\) we get \(s=r\). We inspect the cases of Lemma 2.5.

(i) As \(m+n=sM-2=sM+2-s\), it follows \(s=4\). Further \(m=4a+2-t=4x+1, n=4b+1=4y+1\), hence \(a=x, b=y, t=1\). Thus \(4|E(G)| = 16xy+4x+4y=mn-1\).

(ii) \(m+n=sM\), thus \(s=2\). The equations for \(m\) and \(n\) yield \(a=x, b=y, t=1\), thus \(2|E(G)| = 4xy+2x+2y=mn-1\).

(iii) \(m+n=(s-1)M+a_r+b_r=sM+2-s < sM\), thus \(M+2-s=a_r+b_r < M, s \in \{3,4\}\). We have \(n=(s-1)b+tb_r=sb+t+b_r-b \leq sb+t+1 \leq s(b+1)\).

It follows \(b \geq y\). The equations for \(m\) yield \(a \geq x\). As \(M=a+b+1=x+y+1\), we get \(a=x, b=y\). The equations for \(m\) and \(n\) now (*) \(b_r=y+t+1-a, a_r=x+t+2-s\).

We have to show the validity of the following inequality: \(s|E(G)| = s(s-1)xy+st+x+(s-1-t)y+sa_r+b_r < s^2 xy+sx+sy+1\), after simplification \(-sxy+s(t-1)x+s(s-2-t)y+sa_r+b_r < 1\).

This is equivalent to \(-xy+t+1+(s-2-t)y+a_r+b_r < 0\).

We substitute (*) for \(a_r, b_r\). It remains to show \((t-1)(s-2-t) \leq 0\).

This is true as either \(t \leq 1\) or \(t+2 \geq 4\).

(iv) We have to consider only the first case. The
equations for \( m+n \) yield \( s \leq m \). We have to show \( s \mid E(G) \mid = sm(M-1) < mn \), equivalently \( s(M-1) < n = m(M-1) + b_{m-1} \), which is obviously true.

3. Values of \( f(n,k) \) for \( 3 \leq k \leq 5 \).

In this section we prove Theorem 1.16-1.18. The notation \([A,B]\) is used for the complete bipartite graph with vertex classes \( A \) and \( B \). In the cases \( n = (k-1)p \) our Corollary 1.14 does the job. In order to prove \( f(4p + 2, 5) > 16p + 6 \) we invoke Theorem 1.5 and use \( w_2(4) \leq 10 \). Indeed, the affine plane of order has a 2-transversal of the following type:

The remaining lower bounds follow from Corollaries 1.7, 1.8. We have to prove the upper bounds for \( f((k-1)p + j, k) \) \((k=3,4,5; j=1, \ldots, k-2)\). A \(((k-1)p + j, k)\)-coloring of the appropriate complete graph \( K \) has to be considered. We want to derive a contradiction. Let us proceed inductively, starting from small values of \( k \) and \( j \). As \( f((k-1)p + j-1, k) \) is small enough by induction, we can assume that there exists a red connected subgraph \( R \) of \( K \) on \((k-1)p + j\) vertices. By definition of an \((n,k)\)-coloring, there are no red edges in \([R,K-R]\). Thus \([R,K-R]\) is colored with \( k-1 \) colors. If \((k,j) \in \{(3,1), (4,1), (5,1), (5,2)\}\) we get a contradiction by Corollary 2.3. Only two cases remain. Consider the case \( k=4, j=2 \). We have \(|R| = 3p + 2, |K-R| = 6p + 3\). Let \( H \) be a \((3p+2)\)-extremal bipartite graph with \( m = 3p + 2, n = 6p + 3 \). An easy inspection shows that only case \((iii)\) of Lemma 2.5 occurs. The unique extremal sequence is \((p+1, 2p+1), (p+1, 2p+1), (p, 2p+1)\). Further \(|E(H)| = (2p+1)(3p+2) = |E([R,K-R])|/3\).

This shows, that \( G \) has blue connected components of cardinalities \( 3p+2, 3p+2, 3p+1 \). Let \( B \) be the blue component of cardinality \( 3p+1 \). Then Corollary 2.3 applied to \([B,K-B]\) yields a contradiction.
Finally consider the case $k=5, j=3$.

We have $|R| = 4p+3$, $|K-R| = 12p+7$. A $(4p+3)$-extremal bipartite graph with $m=4p+3, n=12p+7$ has $12p^2 + 16p + 6 = (|E([R,K-R])| + 3)/4$ edges. As $[R,K-R]$ is colored with four colors, one of them, say green leads to a $(4p+3)$-extremal green graph on $[R,K-R]$. Only type (iii) of Lemma 2.5 occurs and the green subgraph of $[R,K-R]$ is defined by the sequence three times $(p+1, 3p+2)$, once $(p, 3p+1)$.

Thus the green subgraph of $K$ has four components $G_1, \ldots, G_4$ of the following cardinalities: $|G_1| = 4p+3$ $(i=1,2,3)$, $|G_4| = 4p+1$. Let $S= G_1 \cup G_2, T=G_3 \cup G_4$. Then $[S,T]$ is $4$-colored. This time Lemma 2.5 yields equality, i.e. a $(4p+3)$-extremal subgraph has $2(2p+1)(4p+3)=|E([S,T])|/4$ edges. Thus all non-green monochromatic subgraphs of $[S,T]$ are $(4p+3)$-extremal. Type (i) of Lemma 2.5 does not occur as this would yield a color with components $C_1, \ldots, C_4$ of sizes $|C_1|=|C_2|=4p+3$, $|C_3|=|C_4|=4p+2$, and Corollary 2.3 would produce monochromatic connected subgraphs on at least $4p+4$ vertices of $[C_1 \cup C_3, C_2 \cup C_4]$, contradiction. Hence only type (iii) occurs, and every non-green monochromatic component of $[S,T]$ is given by one of the sequences:

$(\omega) \quad 3x(2p+2, 2p+1) \quad$ or $\quad (\beta) \quad 1x(2p+1, 2p+2)$

$1x(2p, 2p+1) \quad 2x(2p+2, 2p+1)$

$1x(2p+1, 2p)$

Thus every complete monochromatic subgraph of $K$ has four components, three of size $4p+3$, one of size $4p+1$.

Let $c$ be a non-green color of type $(\omega)$. Then the number of $c$-colored edges of $[G_1, G_2]$ is $\leq 3(p+1)^2 + p^2 = 4p^2 + 3p + 6p + 3$ of $[G_3, G_4]$ is $\leq 4p(p+1)=4p^2 + 4p$.

If $c$ has type $(\beta)$, then the number of $c$-colored edges of $[G_1, G_2]$ is $\leq 2(p+1)^2 + 2p(p+1)=4p^2 + 6p + 2$ of $[G_3, G_4]$ is $\leq (p+1)^2 + 2p(p+1)+p^2=4p^2 + 4p + 1$.

As $|E([G_1, G_2])| = 16p^2 + 24p + 9$, $|E([G_3, G_4])| = 16p^2 + 16p + 3$, we get the following properties:

(i) Type $(\beta)$ occurs three times, type $(\omega)$ occurs once.

(ii) If $P \in G_1$, $Q \in G_2$ (or $P \in G_3$, $Q \in G_4$), and if $P$ and $Q$ are in the same $c$-component, then the edge $PQ$ is colored $c$. 

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(iii) If \( G \) is a green and \( H \) is a \( c \)-component, then 
\[ |G \cap H| \in \{p, p+1\}. \]
Let us fix notation: the colours are 1, 2, 3, 4, 5. Write \( c(PQ) = i \) if \( PQ \) is colored \( i \), write \( P \cong Q \) if \( P \) and \( Q \) are in the same \( i \)-component. As we could have started from any color instead of green and from any pairing of its components, and as we could have compared with any color instead of \( c \), we get

\[ (*) \text{ If } P \not\cong Q, P \cong Q, \text{ then } c(PQ) = i, \text{ equivalently:} \]
For any pair \( P, Q \) of distinct vertices, \( c(PQ) = j \), one of the following holds: either \( P \not\cong Q \) for every color \( i \), or \( P \cong Q \) only for the color \( i = j \).

Thus we get an equivalence relation \( \sim \) on \( K \) defined by \( P \sim Q \) if and only if \( P \not\cong Q \) for every color \( i \).
This relation has 16 equivalence-classes. By (ii) the coloring of \( K \) induces a coloring of \( K/\sim \). Let \( H \) be the hypergraph with \( K/\sim \) as vertex set and the monochromatic components of \( K/\sim \) as edges. Clearly \( H \) is the affine plane of order 4. By (iii) every equivalence class has \( p \) or \( p+1 \) elements of \( K \). Let \( B = \{ R | R \in K/\sim, |B| = p \} \). It is obvious, that \( |B| = 6 \) and that \( B \) is a 1-transversal (an affine blocking set) in \( H \). This contradicts [5].

References


