ON (n,k)-COLORINGS OF COMPLETE GRAPHS

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ABSTRACT

An (n,k)-coloring of a complete graph K means a coloring of the edges of K with k colors so that all monochromatic connected subgraphs have at most n vertices. We are interested in the maximum number of vertices of complete graphs with (n,k)-colorings. We survey results concerning this problem and give some new results which lead to the complete solution for $k \leq 5$.

0. Introduction.

Let f(n,k) denote the smallest integer m=m(n,k) with the following property: if the edges of K_m are colored with k colors then there exists a monochromatic connected subgraph of more than n vertices. The function f(n,k) has been introduced in [12] and f(n,3) was determined in [12] and [1]. The observation f(n,2)=n+1 is equivalent with a remark of Erdos and Rado saying that for any graph G, either G or its complement is connected. The second author has further results on f(n,k) in [13]. The problem of determining f(n,k) was rediscovered by the first author and Brandis in [3].

From the point of view of Ramsey theory, f(n,k)-1 is a lower bound for the Ramsey number $R(T_n,k)$, where T_n is any tree of n edges. Bounds on $R(T_n,k)$ have been studied in [10].

We shall use the term (n,k)-coloring introduced by Bierbauer and Brandis in [3]. An (n,k)-coloring is a coloring of the edges of a complete graph with k colors so that all connected monochromatic subgraphs have at most

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n vertices. The function $f(n,k)-1$ clearly give the largest number of verices of a complete graph which has an (n,k)-coloring.

An (n,k)-coloring can be viewed as k partitions of a ground set into sets of cardinality at most n, so that all pairs of elements appear together is some of the sets. Thus resolvable block designs with $\lambda=1$, k parallel classes and with blocksize n are natural examples of (n,k)-colorings. However, (n,k)- colorings are much more "relaxed" structures: the "blocks" may have any sizes up to n and the pairs of the ground set appear together in at least one block. In extremal (n,k)-colorings, i.e. in (n,k)-colorings of complete graphs of $f(n,k)-1$ vertices, the structure of connected monochromatic components is often close to the block structure of resolvable block designs.

In section 2 we review results on $f(n,k)$ and present some new results.

We give a new lower bound of $f(n,k)$:

$$f(n,k)>(k-1)^2\ (p+1)-w_i(k-1)$$

if $n=(k-1)p+k-1-i$, $0<i\leq k-1$, and an affine plane of order $k-1$ exists (Theorem 1.5). Here $w_i(q)$ denotes the minimum number of points of an affine plane A_q of order q which meet every line of A_q in at least i points. The minimum is taken over all affine planes of order q. The bound is always sharp for $3\leq k\leq 5$ (see Theorems 1.16,1.17,1.18). If we compare this lower bound with the upper bound of $n(k-1)+1$ (Theorem 1.1), we see that for fixed k the function $f(n,k)-n(k-1)$ is smaller than a function depending only on k. It is unknown whether a similar statement holds if no affine plane of order $k-1$ exists (Problem 1.15). The main results of the paper are prepared in section 2, where a method is described to get an upper bound of $f(n,k)$. The upper bound $n(k-1)+1$ (Theorem 1.1) comes from a joint result of J. Lehel and the second author: if the edges of a complete bipartite graph $K_{m,n}$ are colored with s colours then there exists a monochromatic connected subgraph of at least $\lceil (m+n)/s \rceil$ vertices (Corollary 2.2). Our main concern is to push this method to

its limit, i.e. to prove that $K_{m,n}$ contains a monochromatic connected subgraph of at least $\lceil m/s\rceil+\lceil n/s\rceil$ vertices in every s-coloring (Conjecture 2.4). We can prove this for $2\leq s\leq 4$ (Corollary 2.3), which gives an essential part in determining $f(n,k)$ for $3\leq k\leq 5$.

In section 2 the properties of M-extremal graphs play an important role. For fixed m,n, we call a bipartite graph G M-extremal if the vertex classes of G contain m and n vertices, the connected components of G have at most M vertices and G has as many edges as possible under these conditions. The important properties of M-extremal graphs are summarized in Lemma 2.5. The main application of M-extremal graphs is the following Theorem (Theorem 2.1). If G is a bipartite graph with m and n vertices in its colour classes and G has at least $\lceil mn/s\rceil$ edges, then G contains a connected component of at least $\lceil (m+n)/s\rceil$ vertices. Moreover, if $2\leq s\leq 4$ then G contains a connected component of at least $\lceil m/s\rceil+\lceil n/s\rceil$ vertices.

In section 3 we apply our methods to determine $f(n,k)$ for $3\leq k\leq 5$:

$$f(n,3)=\begin{cases}4p+1 & \text{if } n=2p\\ 4p+2 & \text{if } n=2p+1\end{cases}\qquad f(n,4)=\begin{cases}9p+1 & \text{if } n=3p\\ 9p+2 & \text{if } n=3p+1\\ 9p+5 & \text{if } n=3p+2\end{cases}$$

$$f(n,5)=\begin{cases}16p+1 & \text{if } n=4p\\ 16p+2 & \text{if } n=4p+1\\ 16p+7 & \text{if } n=4p+2\\ 16p+10 & \text{if } n=4+3\end{cases}$$

The authors recently learned that $f(n,4)$ have been determined independently by Bialostocki and Dierker.

1. Results on $f(n,k)$.

In this section we survey results concerning $f(n,k)$ and present some new results. We start with upper bounds.

Theorem 1.1. ([13]) $f(n,k)\leq(k-1)n+1$.

We note that Theorem 1.1 immediately follows from the following result of J. Lehel and the second author: if the edges of a complete bipartite graph G are colored with $k-1$ colors then G contains a monochromatic connected subgraph of at least $\lceil |V(G)|/(k-1) \rceil$ vertices. This result appears in section 2 as Corollary 2.2. Another proof of Theorem 1.1 is obtained if we consider the following hypergraph H, determined by an (n,k)-coloring of a complete graph K. The vertices of H are the vertices of K and the edges of H are the vertex sets of the connected monochromatic components of K. The dual hypergraph H^* of H is a k-partite intersecting hypergraph (every two edges of H^* have at least one common vertex). A result of Furedi ([11]) says that a k-uniform intersecting hypergraph H has a vertex of degree at least $\lceil |E(H)|/(k-1) \rceil$, unless H is a projective plane of order $k-1$. Since a k-partite hypergraph is never a projective plane, H^* contains a vertex of degree at least $\lceil |E(H^*)|/(k-1) \rceil$ which implies that H contains an edge with at least $\lceil |V(H)|/(k-1) \rceil$ vertices and Theorem 1.1 follows.

The following upper bound is due to the first author and **Brandis:**

<u>Theorem 1.2</u> ([3]). Assume $k \equiv K \pmod{n}$, $2 \le K < n$ and let $\varsigma = (k-K)/(k-1)$. If $4K > 3n + \varsigma - (n(n+8-2\varsigma) - \varsigma(8-\varsigma))^{1/2}$ then

$$f(n,k) \le (k-1)n - (k-K) + 1. \quad \text{Otherwise}$$

$$f(n,k) \le \lfloor k(n-1) + 1/2 - (n(K-\varsigma) - K(K-\varsigma-1) - (\varsigma - 1/4))^{1/2} \rfloor + 1$$

The cases not covered by Theorem 1.2 are covered by the following two results.

<u>Theorem 1.3</u> ([3]). If $n \ge 2$, $1 < k \equiv 1 \pmod{n}$, then

$$f(n,k) \le k(n-1) + 2.$$

Equality holds iff a resolvable block design exists with $\lambda = 1$, block size n and replication k.

<u>Theorem 1.4</u> ([3]). If $k \equiv 0 \pmod{n}$, then

$$f(n,k) \le k(n-1) + 1,$$

and for $n > 2$, $f(n,n) \le n(n-1)$.

For comparison, it is easy to see that for $n > k \ge 2$, Theorem 1.1 is better than Theorem 1.2. If $n \le k$ then Theorems 1.2-1.4 are better than Theorem 1.1.

Concerning lower bounds of $f(n,k)$, first we give a construction which uses the existence of an affine plane of order $k-1$. The lower bound is close to the upper bound of

Theorem 1.1 and for $k=3,4,5$ it gives the exact value of $f(n,k)$. Let A_q denote an affine plane of order q and let $X_1,\dots,X_{q+1}$ be the ideal points of A_q. Assume that we have a complete graph K whose vertex set is partitioned into q^2 parts, $S_1,S_2,\dots,S_{q^2}$. Consider a one-to-one mapping between the points of A_q and the sets $S_1,\dots,S_{q^2}$. We color an edge PQ of K with color k if $P\in S_i, Q\in S_j, i\neq j$ and the points corresponding to S_i and S_j in A_q determine a line containing X_k. The edges of K whose endpoints belong to the same set S_i may be colored arbitrarily. The colorings of complete graphs obtained by this method are called <u>normal</u> $(q+1)$-colorings. Note that normal $(q+1)$-colorings are defined only for those values of q for which an affine plane of order q exists.

An i-transversal of an affine plane A_q is a set of points in A_q which meet every line of A_q in at least i points. Let $w_i(A_q)$ denote the minimum cardinality of an i-transversal of A_q and let $w_i(q)$ be min $w_i(A_q)$, where the minimum is taken over all affine planes of order q. The following Theorem gives a lower bound for $f(n,k)$ in terms of $w_i(k-1)$.

<u>Theorem 1.5.</u> Assume that an affine plane of order $k-1$ exists, let $n=(k-1)p+k-1-i$, where $0<i\leq k-1$. Then

$$f(n,k)>(k-1)^2(p+1)-w_i(k-1).$$

<u>Proof.</u> Let A_{k-1} be an affine plane of order $k-1$ possessing an i-transversal T of $w_i(k-1)$ elements. Let $m=(k-1)^2(p+1)-w_i(k-1)$ and consider a normal k-coloring of K_m, where we associate a set of p elements to the points of T and we associate a set of $p+1$ elements to the points outside T. By the definition of the normal coloring and the i-transversal, a monochromatic connected component of K_m has at most $pi+(k-1-i)(p+1)=n$ vertices. Thus we have an (n,k)-coloring on K_m and the theorem follows.

<u>Corollary 1.6.</u> If an affine plane of order $k-1$ exists, then

$$f((k-1)p,k)>(k-1)^2p \text{ for all } p\geq 1.$$

This follows as $w_{k-1}(k-1)=(k-1)^2$. Another easy application of Theorem 1.5 occurs if $i=k-2$. Now $w_{k-2}(k-1)=(k-1)^2-1$ is obvious which yields

Corollary 1.7. If an affine plane of order $k-1$ exists, then

$$f((k-1)p+1,k)>(k-1)^2p+1.$$

We observe that $w_1(k-1)\leq 2k-3$ since two intersecting lines of A_{k-1} give a 1-transversal. Thus we have

Corollary 1.8. If an affine plane of order $k-1$ exists, then

$$f((k-1)p+k-2,k)>(k-1)^2p+(k-2)^2.$$

A fundamental result of Jamison's ([7]) implies that 1-transversals (also called "affine blocking sets") in desarguesian A_q have at least $2q-1$ points. It is however possible to obtain 1-transversals of smaller size in other affine planes. Bruen and de Resmini ([8]) use the Hughes plane of order 9 to show $w_1(9)\leq 16$.

Corollary 1.9. $f(9p+8,10)>81p+65$.

We note that Corollary 1.6 is sharp for all k and corollary 1.7 is sharp for $k=3,4,5$.

A resolvable BIBD with blocks of cardinality n, with $\lambda=1$ and with k parallel classes is clearly suitable to define an (n,k)-coloring. In this case we have $k(n-1)+1$ points. If we substitute t points for all points of this design, an (nt,k)-coloring can be defined on $t(k(n-1)+1)$ vertices in analogy to normal colorings. Thus we have

Proposition 1.10. If a resolvable BIBD exists with blocks of cardinality n, with $\lambda=1$ and with k parallel classes then

$$f(nt,k)>t(k(n-1)+1)$$

An example for the application of Proposition 1.10 is the case $t=1,n=4,k=9$. Now $f(4,9)=29$ follows from Theorem 1.3 and Proposition 1.10. This example is taken from [3]. As there are resolvable BIBD with block size 4, with $\lambda=1$ and replication $4t+1$ $(t\geq 1)$ (see [17,18]), we get

$$f(4a,4t+1)>a(12t+4) \quad (a\geq 1,t\geq 1).$$

We obtain a general lower bound.

Corollary 1.11. $f(n,k)>(n-3)(3k-8)/4$.

The following generalizes a result of [3], which in turn relies upon a construction of Linstrom in [16]:

Proposition 1.12. If $f(ap,k_o)>kp$ and if there is a set of

$a-1$ mutually orthogonal latin squares of order k, then

$$f(ap,k_0+k)\geq apk.$$

As there is a set of five mutually orthogonal latin squares of order 12 ([5,9]) and $f(6p,3)\geq 12p$ (Theorem 1.16 below) we get $f(6p,15)\geq 72p$.

Applications of Proposition 1.12 in case $p=1$ are given in [3]. For instance $f(4,10+28\sum_{0}^{i}4^j)\geq 16\cdot 4^{i+1}$ $(i\geq 0)$ can be derived by repeated applications of Proposition 1.12 with $u=16\cdot 4^i$. Theorem 1.2 implies that the lower bound is sharp. The comparison of the upper bound of Theorem 1.1 and the lower bound of Theorem 1.5 shows that $n(k-1)$ is close to $f(n,k)$ for large n and fixed k.

Corollary 1.13. If an affine plane of order $k-1$ exists, then

$$f(n,k)-n(k-1)-1\leq w_i(k-1)-(k-1)i\leq (k-1)i,$$

for $n=(k-1)p+k-1-i$, $0<i\leq k-1$.

In particular, if $i=k-1$ then $w_i(k-1)=(k-1)^2$ and we get

Corollary 1.14. If an affine plane of order $k-1$ exists, then

$$f((k-1)p,k)=p(k-1)^2+1.$$

It would be interesting to get rid of the existence problem of affine planes of order $k-1$ in a lower bound close to $n(k-1)$ for large n. We have the following problem.

Problem 1.15. It is true that $f(n,k)-n(k-1)$ is less than a function depending only on k ?

Now we consider $f(n,k)$ for small values of k. An old remark of Erdos and Rado says that a graph or its complement is connected. Thus $f(n,2)=n+1$. The case $k=3$ has been settled in [12] and [1]. A new proof is given in section 3 based on results of section 2.

Theorem 1.16.

$$f(n,3)=\begin{cases}4p+1 & \text{if } n=2p\\ 4p+2 & \text{if } n=2p+1.\end{cases}$$

The following two Theorems are the main new results of this paper. The proofs are in section 3.

Theorem 1.17.

$$f(n,4) = \begin{cases} 9p+1 & \text{if } n=3p \\ 9p+2 & \text{if } n=3p+1 \\ 9p+5 & \text{if } n=3p+2 \end{cases}$$

Theorem 1.18.

$$f(n,5) = \begin{cases} 16p+1 & \text{if } m=4p \\ 16p+2 & \text{if } n=4p+1 \\ 16p+7 & \text{if } n=4p+2 \\ 16p+10 & \text{if } n=4p+3 \end{cases}$$

Concerning the values of $f(n,k)$ for small values of n, the following observation is in [3].

Proposition 1.19.

$$f(2,k) = \begin{cases} k+2 & \text{if } k \text{ is odd} \\ k+1 & \text{if } k \text{ is even} \end{cases}$$

The case $n=3$ is also completely solved. The following results is in [3].

Theorem 1.20.

$$f(3,k) = \begin{cases} 2k+2 & \text{if } k \equiv 1 \pmod 3 \\ 2k & \text{if } k \equiv 2 \pmod 3 \end{cases}$$

We note that theorem 1.20 follows by combining the upper bound of Theorem 1.2 with the lower bound of Proposition 1.10 and using the existence theorem of D.K. Ray-Chauohuri, R.M. Wilson on resolvable triple systems ([17]). For $n=3$ and $k \equiv 0 \pmod 3$, it is easy to see that $f(3,k)$ is either $2k$ or $2k+1$. It is easy to prove that $f(3,3)=6$ (see [3]). Zs. Tuza discovered a $(3,6)$ coloring of K_{12} (4 color classes are $4K_3$, one color class is $K_3 + 3K_{1,2}$ and one color class is $2K_3 + 3K_2$). This construction shows that $f(3,6)=13$. The first author used a certain Steiner triple system on 19 points to show $f(3,9)=19$ ([4]). If $k \equiv 0 \pmod 3$ and $k \geq 9$, there exist $(3,k)$-colorings of K_{2k} such that all but one color classes are isomorphic to $2k/3\ K_3$ and the exceptional color class is isomorphic to $k\ K_2$. Such colorings are called Nearly Kirkman Triple Systems and their existence have been proved in a series of papers ([15]),[2],[6],[19] in chronological order). The authors are grateful to

Professor Rosa for this information. Therefore the following theorem holds.

Theorem 1.21

$$f(3,k) = \begin{cases} 6 \text{ if } k=3 \\ 2k+1 \text{ if } k\geq 6 \text{ and } k\equiv 0 \pmod 3). \end{cases}$$

2. M-extremal bipartite graphs.

Let $B(m,n)$ denote the set of bipartite graphs with vertex classes of cardinality m and n. We shall always assume that $m,n\geq 1$. The purpose of ths section is to prove the following

Theorem 2.1. Assume $G\in B(m,n)$ and G has at least $\lceil mn/s\rceil$ edges for some positive integer s. Then G contains a connected component of at least $\lceil (m+n)/s\rceil$ vertices. Moreover, if $2\leq s\leq 4$ then G contains a connected component of at least $\lceil m/s\rceil + \lceil n/s\rceil$ vertices.

The first part of Theorem 2.1 gives a joint result of the second author with J. Lehel:

Corollary 2.2 ([13]). If the edges of $K_{m,n}$ are colored with s colors, then there exists a monochromatic connected subgraph of at least $\lceil (m+n)/s\rceil$ vertices.

The second part of Theorem 2.1 implies

Corollary 2.3 If the edges of $K_{m,n}$ are colored with s colors and $2\leq s\leq 4$, then there exists a monochromatic connected subgraph of at least $\lceil m/s\rceil + \lceil n/s\rceil$ vertices.

We conjecture, that Corollary 2.3 holds for every s.

Conjecture 2.4. If the edges of $K_{m,n}$ are colored with s colors, then there exists a monochromatic subgraph of at least $\lceil m/s\rceil + \lceil n/s\rceil$ vertices.

It is worth noting, that conjecture 2.4 cannot be obtained from a density result since the second part of Theorem 2.1 is not true for $s\geq 5$. To see this for $s=5$, let $m=5p+1$, $n=20p+1$. Now $2K_{p,4p+1}+2K_{p+1,4p}+K_{p-1,4p-1}$ has $\lceil mn/5\rceil$ edges, but its components have at most $5p+1=\lceil m/5\rceil + \lceil n/5\rceil -1$ vertices.

It is convenient to introduce at this point the notion of M-extremal bipartite graphs. We shall always assume that M is an integer, $M>2$. A graph $G\in B(m,n)$ is called

M-extremal if every connected component of G has at most M vertices and G has the largest number of edges under this condition. It is clear, that an M-extremal bipartite graph is the union of disjoint complete bipartite graphs and possibly some isolated vertices in one of the vertex classes. If we accept $K_{0,t}$ and $K_{t,0}$ for $t \geq 1$ as degenerate complete bipartite graphs, then an M-extremal graph of $B(m,n)$ is the vertex-disjoint union of K_{a_1,b_1}, K_{a_2,b_2}, ...K_{a_r,b_r}, where the numbers a_i, b_i are non-negative integers, at most one of them can be zero, and they satisfy

(1) $a_1+a_2+...+a_r=m$, $b_1+b_2+...b_r=n$ and $a_i+b_i \leq M$ for all i such that $1 \leq i \leq r$ and $a_i b_i \neq 0$.

Thus the description of M-extremal members of $B(m,n)$ is equivalent with finding values of r and for the pairs (a_i, b_i) such that (1) is satisfied and $E = \sum_1^r a_i b_i$ is maximum. Such a sequence is also called M-extremal.

M-extremal sequences (or M-extremal bripartite graphs) are not necessarily unique, for instance if $m=4, n=6, M=4$, the the following sequence define M-extremal graphs: (2,2),(1,2),(1,2); (2,2),(2,2),(0,2); (2,2),(1,3),(1,1).

Lemma 2.5. Let m,n,M be fixed, $M>2$, $r=\lceil (m+n)/M \rceil$, and let $\{(a_i,b_i) | i=1,2,..,s\}$ - be an M-extremal sequence, $E=\sum_1^s a_i b_i$.

Then one of the following holds:

(i) $a=a_1=a_2=...=a_t$, $a+1=a_{t+1}=...=a_{r-2}$, $a=a_{r-1}=a_r$
$b+1=b_1=b_2=...=b_t$, $b=b_{t+1}=...=b_{r-2}$, $b=b_{r-1}=b_r$,
$m=ra+r-2-t$, $n=rb+t$, $m+n=rM-2$, $E=rab+ta+b(r-2-t)$, $r=s$.

(ii) $a=a_1=a_2=...=a_t$, $a+1=a_{t+1}=...=a_r$
$b+1=b_1=b_2=...=b_t$, $b=b_{t+1}=...=b_r$
$m=ra+r-t$, $n=rb+t$, $m+n=rM$, $E=rab+ta+(r-t)b$, $r=s$.

(iii) $a=a_1=a_2=...=a_t$, $a+1=a_{t+1}=...=a_{r-1}$.
$b+1=b_1=b_2=...=b_t$, $b=b_{t+1}=...=b_{r-1}$,
where $a_r+b_r < M$, $a_i \geq a_r$, $b_i \geq b_r$ for all i, $1 \leq i \leq r-1$,
$m=(r-1)a+r-1-t+a_r$, $n=(r-1)b+t+b_r$, $m+n=(r-1)M+a_r+b_r < rM$,
$E=(r-1)ab+ta+(r-1-t)b+a_r b_r$, $s=r$.

(iv) $1=a_1=a_2=...=a_m$, $a_{m+1}=0$, $M-1=b_1=b_2...=b_m$, $b_{m+1} > M$ or
$1=b_1=b_2=...=b_n$, $b_{n+1}=0$, $M-1=a_1=a_2=...=a_n$, $a_{n+1} > M$,
$E=(M-1)\mathrm{Min}(m,n)$.

Here t,a,b are non-negative integers, $a+b+1=M$.

We note that M-extremal sequences in forms (i),(ii) and (iv) may occur only for special choices of m,n,M. It is easy to check, that M must be a divisor of $m+n+2$ if (i) occurs, M must divide $m+n$ if (ii) occurs. Form (iv) appears if either $n > (m+1)(M-1)+1$ or $m > (n+1)(M-1) + 1$ holds.

Proof. Let $(a_1,b_1),...,(a_m,b_m)$ be a M-extremal sequence. A pair (a_i,b_i) is called saturated if $a_i+b_i = M$ and unsaturated if $a_i+b_i < M$. A pair (a_i,b_i) is exceptional if $a_i+b_i > M$. Clearly a_i or b_i is zero for an exceptional pair.

Claim 1. If (a_i,b_i) and (a_j,b_j) are unsaturated, then $a_i=a_j$, $b_i=b_j$ and $a_i+b_i=M-1$.

If $a_i < a_j$ and $b_i \geq b_j$ then $b_i \neq 0$ and then pairs (a_i,b_i), (a_j,b_j) can be changed into $(a_i,b_i-1),(a_j,b_j+1)$ and the value of E increases by this change, contradiction. If $b_i < b_j$, then either $a_i \neq 0$ or $b_i \neq 0$. Assume that $a_i \neq 0$. Now our pairs can be changed to $(a_i-1,b_i),(a_j+1,b_j)$ to increase E. The case $b_i \neq 0$ is symmetric. Thus $a_i=a_j, b_i=b_j$. If $a_i+b_i < M-1$, then our pairs can be changed to (a_i-1,b_i-1), (a_j+1,b_j+1), increasing E. The claim is proved.

Claim 2. If (a_i,b_i) and (a_j,b_j) are unsaturated, then all other pairs are saturated.

Assume there is an unsaturated pair (a_p,b_p). Using claim 1, $a_i=a_j=a_p=A$, $b_i=b_j=b_p=B$ and $A+B=M-1$. Now the three (A,B)-pairs can be changed to $(A+1,B),(A,B+1),(A-1,B-1)$ $(AB\neq 0)$ and E increases. An exceptional pair $(c,0)$ can not occur as otherwise $(c,0)$, (a_i,b_i) can be changed to $(c-1,0),(a_i+1,b_i)$.

Claim 3. Let (a_i,b_i) and (a_j,b_j) be two pairs such that $a_i+b_i \geq M-1$, $a_j+b_j \geq M-1$, $a_ib_ia_jb_j \neq 0$. Then $|a_i-a_j| \leq 1$ $|b_i-b_j| \leq 1$.

Assume $a_i-a_j \geq 2$. Now $b_j-b_i \geq 1$ as otherwise $a_i+b_i \geq M-1+2=M+1$ which contradicts (1). If we change our pairs to $(a_i-1,b_i+1),(a_j+1,b_j-1)$, then E increases since $(a_i-1)(b_i+1)+(a_j+1)(b_j-1) = a_ib_i+a_jb_j+a_i-a_j+b_j-b_i-2 \geq a_ib_i+a_jb_j+1$. The claim is proved.

To continue with the proof of Lemma 2.5, assume that there are two unsaturated pairs in an M-extremal sequence. Claim 1 ensures, that both pairs are in the form (a,b) where a+b=M-1. If (a_i,b_i) is any other pair then $a_i+b_i=M$ by claim 2. Clearly $ab\neq 0$ and claim 3 implies $a_i=a$ or $a_i=a+1$. Now we have our M-extremal sequence in form (i), where t denotes the number of indices i for which $a_i=a$ and (a_i,b_i) is saturated. Since m+n=sM-2, r=s follows.

Assume that a M-extremal sequence contains exactly one unsaturated pair, (a_r,b_r). All other pairs are saturated since an exceptional pair $(a_i,0)$ would allow the changes $(a_i-1,0),(a_r+1,b_r)$ contradicting the M-extremal property. It is obvious that r=s. In order to see that our sequence is in form (iii), we have to prove $a_i\geq a_r$ and $b_i\geq b_r$ for all i, $1\leq i\leq r-1$. Assume $a_i < a_r$. Then $b_i\geq b_r+2$ since (a_i,b_i) is saturated and (a_r,b_r) is unsaturated. We can change the pairs $(a_i,b_i),(a_r,b_r)$ to (a_r-1,b_r+1), (a_i+1,b_i-1) $(a_r>0, b_i\geq 2)$. We get a contradiction since $(a_r-1)(b_r+1)+(a_i+1)(b_i-1) = a_ib_i+a_rb_r+b_i-b_r+a_r-a_i-2 \geq a_ib_i+a_rb_r+1$.

Finally, assume that there are no unsaturated pairs in an M-extremal sequence. If no exceptional pair is present, then all pairs are saturated, r=s and the sequence is in form (ii) by claim 3. Assume there is exactly one exceptional pair. By symmetry we can choose it as (0,c). Since $m\geq 1$, there exist other pairs. Let (a_i,b_i) be any such pair. Obviously (a_i,b_i) is saturated and $a_i>0$. Now we change the pairs $(0,c),(a_i,b_i)$ to $(0,c-M+1)$, $(1,M-1)$, (a_i-1,b_i). If $b_i < M-1$, the $a_ib_i < 1(M-1)+(a_i-1)b_i$ and we reach a contradiction. Therefore $b_i= M-1$, i.e. the M-extremal sequence has form (iv).

Proof of Theorem 2.1 Let c(G) denote the maximal number of vertices in a connected component of G. To prove the first part of the Theorem, we have to show:

If $c(G) < \lceil (m+n)/s\rceil$, then $|E(G)| < \lceil mn/s\rceil$, equivalently: if G is M-extremal, $sM < m+n$, then $s|E(G)| < mn$. Consider M as fixed. It suffices to prove $s|E(G)| < mn$ for the maximal number s satisfying $sM < m+n$, i.e. for s=r-1.

In case (i) of Lemma 2.5 we have to show $(r-1)ab+(r-1)ta+(r-1)(r-2-t)b < r^2ab+rat+rb(r-2-t)+t(r-2-t)$, equivalently $-a(t+rb) < (t+b)(r-2-t)$, which is clearly true.

In case (ii) we have to show $(r-1)rab+(r-1)ta+(r-1)(r-t)b < r^2 ab+rat+rb(r-t)+t(r-t)$, equivalently $-a(t+rb) < (t+b)(r-t)$, true like before.

In case (iii) we have to show $(r-1)^2ab+(r-1)at+(r-1)(r-1-t)b+(r-1) a_r b_r < (r-1)^2 ab+(r-1)a(t+b_r)+(r-1)b(r-1-t+a_r)+(t+b_r)(r-1-t+a_r)$, equivalently $(r-1)a_r b_r < (r-1)ab_r+(r-1)ba_r+(t+b_r)(r-1-t+a_r)$.

The inequality is true as either $a_r \leq a$ or $b_r \leq b$.

In case (iv) by symmetry we have to consider only the first case.

As $m+n > (m+1)M$, we can choose $s \geq m+1$. We have to prove, that $s(M-1) < n$ if $sM < m+n$. This is true as $m<s$.

Let us proceed to the proof of the second part of Theorem 2.1. Let $s \in \{2,3,4\}$, G an M-extremal graph, $M < \lceil m/s \rceil + \lceil n/s \rceil$. We have to show $s|E(G)| < mn$. Because of the first part of the theorem, it suffices to consider the case $m=sx+1$, $n=sy+1$, $M=x+y+1$. Then $m+n=sM+2-s$. As $2 \leq s \leq 4$ we get $s=r$. We inspect the cases of Lemma 2.5.

(i) As $m+n=sM-2=sM+2-s$, it follows $s=4$. Further $m=4a+2-t=4x+1$, $n=4b+t=4y+1$, hence $a=x$, $b=y$, $t=1$. Thus $4|E(G)|=16xy+4x+4y=mn-1$.

(ii) $m+n=sM$, thus $s=2$. The equations for m and n yield $a=x$, $b=y$, $t=1$, thus $2|E(G)| = 4xy+2x+2y=mn-1$.

(iii) $m+n=(s-1)M+a_r+b_r=sM+2-s < sM$, thus $M+2-s=a_r+b_r < M$, $s \in \{3,4\}$. We have $n=(s-1)b+t+b_r=sb+t+b_r-b \leq sb+t+1 \leq s(b+1)$. It follows $b \geq y$. The equations for m yield $a \geq x$. As $M=a+b+1=x+y+1$, we get $a=x$, $b=y$. The equations for m and n now (*) $b_r=y+1-t$, $a_r=x+t+2-s$.

We have to show the validity of the following inequality: $s|E(G)| = s(s-1)xy+stx+s(s-1-t)y+sa_r b_r < s^2 xy+sx+sy+1$, after simplification $-sxy+s(t-1)x+s(s-2-t)y+sa_r b_r < 1$. This is equivalent to $-xy+t-1+(s-2-t)y+a_r b_r < 0$. We substitute (*) for a_r, b_r. It remains to show

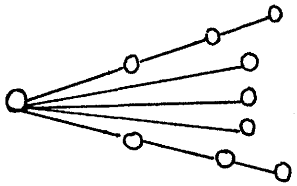
$$(t-1)(s-2-t) \leq 0.$$

This is true as either $t \leq 1$ or $t+2 \geq 4 \geq s$.

(iv) We have to consider only the first case. The equations for m+n yield $s \leq m$. We have to show $s|E(G)| = sm(M-1) < mn$, equivalently $s(M-1) < n = m(M-1)+b_{m+1}$, which is obviously true.

3. Values of f(n,k) for $3 \leq k \leq 5$.

In this section we prove Theorem 1.16-1.18. The notation [A,B] is used for the complete bipartite graph with vertex classes A and B. In the cases $n=(k-1)p$ our Corollary 1.14 does the job. In order to prove $f(4p+2,5) > 16p+6$ we invoke Theorem 1.5 and use $w_2(4) \leq 10$. Indeed, the affine plane of order has a 2-transversal of the following type:



The remaining lower bounds follow from Corollaries 1.7, 1.8. We have to prove the upper bounds for $f((k-1)p+j,k)$ $(k=3,4,5; j=1,...,k-2)$. A $((k-1)p+j,k)$-coloring of the appropriate complete graph K has to be considered. We want to derive a contradiction. Let us proceed inductively, starting from small values of k and j. As $f((k-1)p+j-1,k)$ is small enough by induction, we can assume that there exists a red connected subgraph R of K on $(k-1)p+j$ vertices. By definition of an (n,k)-coloring, there are no red edges in [R,K-R]. Thus [R,K-R] is colored with k-1 colors. If $(k,j) \in \{(3,1),(4,1),(5,1),(5,2)\}$ we get a condradiction by Corollary 2.3. Only two cases remain. Consider the case k=4, j=2. We have $|R| = 3p+2$, $|K-R| = 6p+3$. Let H be a (3p+2)-extremal bipartite graph with m=3p+2, n=6p+3. An easy inspection shows that only case (iii) of Lemma 2.5 occurs. The unique extremal sequence is (p+1,2p+1), (p+1,2p+1), (p,2p+1). Further $|E(H)| = (2p+1)(3p+2) = |E([R,K-R])|/3$.

This shows, that G has blue connected components of cardinalities 3p+2,3p+2,3p+1. Let B be the blue component of cardinality 3p+1. Then Corollary 2.3 applied to [B,K-B] yields a condradiction.

Finally consider the case $k=5, j=3$.

We have $|R| = 4p+3$, $|K-R| = 12p+7$. A $(4p+3)$-extremal bipartite graph with $m=4p+3, n=12p+7$ has $12p^2+16p+6 = (|E([R,K-R])|+3)/4$ edges. As $[R,K-R]$ is colored with four colors, one of them, say green leads to a $(4p+3)$- extremal green graph on $[R,K-R]$. Only type (iii) of Lemma 2.5 occurs and the green subgraph of $[R,K-R]$ is defined by the sequence

three times $(p+1,3p+2)$, once $(p,3p+1)$.

Thus the green subgraph of K has four components $G_1,\dots,G_4$ of the following cardinalities: $|G_i| = 4p+3$ $(i=1,2,3)$, $|G_4|=4p+1$. Let $S=G_1\cup G_2, T=G_3\cup G_4$. Then $[S,T]$ is 4-colored. This time Lemma 2.5 yields equality, i.e. a $(4p+3)$-extremal subgraph has $2(2p+1)(4p+3)=|E([S,T])|/4$ edges. Thus all non-green monochromatic subgraphs of $[S,T]$ are $(4p+3)$-extremal. Type (i) of Lemma 2.5 does not occur as this would yield a color with components $C_1,\dots,C_4$ of sizes $|C_1|=|C_2| = 4p+3$, $|C_3| = |C_4| = 4p+2$, and Corollary 2.3 would produce monochromatic connected subgraphs on at least $4p+4$ vertices of $[C_1\cup C_3, C_2\cup C_4]$, condradiction. Hence only type (iii) occurs, and every non-green monochromatic component of $[S,T]$ is given by one of the sequences:

(α)	$3\times(2p+2,2p+1)$	or	(β)	$1\times(2p+1,2p+2)$
	$1\times(2p,2p+1)$			$2\times(2p+2,2p+1)$
				$1\times(2p+1,2p)$

Thus every complete monochromatic subgraph of K has four components, three of size $4p+3$, one of size $4p+1$.

Let c be a non-green color of type (α). Then the number of c-colored edges of $[G_1,G_2]$ is $\le 3(p+1)^2+p^2 = 4p^2+6p+3$

of $[G_3,G_4]$ is $\le 4p(p+1)=4p^2+4p$.

If c has type (β), then the number of c-colored edges

of $[G_1,G_2]$ is $\le 2(p+1)^2+2p(p+1)=4p^2+6p+2$

of $[G_3,G_4]$ is $\le (p+1)^2+2p(p+1)+p^2=4p^2+4p+1$.

As $|E([G_1,G_2])| = 16p^2+24p+9$, $|E([G_3,G_4])| = 16p^2+16p+3$, we get the following properties:

(i) Type (β) occurs three times, type (α) occurs once.

(ii) If $P\in G_1$, $Q\in G_2$ (or $P\in G_3$, $Q\in G_4$), and if P and Q are in the same c-component, then the edge PQ is colored c.

(iii) If G is a green and H is a c-component, then $|G \cap H| \in \{p, p+1\}$.

Let us fix notation: the colours are 1,2,3,4,5. Write $c(PQ)=i$ if PQ is colored i, write $P \underset{i}{\sim} Q$ if P and Q are in the same i-component. As we could have started from any color instead of green and from any pairing of its components, and as we could have compared with any color instead of c, we get

(*) If $P \underset{G}{\not\sim} Q$, $P \underset{i}{\sim} Q$, then $c(PQ)=i$, equivalently:

For any pair P,Q of distinct vertices, $c(PQ)=j$, one of the following holds: either $P \underset{i}{\sim} Q$ for every color i, or $P \underset{i}{\sim} Q$ only for the color $i=j$.

Thus we get an equivalence relation $\sim$ on K defined by $P \sim Q$ if and only if $P \underset{i}{\sim} Q$ for every color i.

This relation has 16 equivalence-classes. By (ii) the coloring of K induces a coloring of $K/\sim$. Let H be the hypergraph with $K/\sim$ as vertex set and the monochromatic components of $K/\sim$ as edges. Clearly H is the affine plane of order 4. By (iii) every equivalence class has p or $p+1$ elements of K. Let $B=\{R \mid R \in K/\sim, |B|=p\}$. It is obvious, that $|B|=6$ and that B is a 1-transversal (an affine blocking set) in H. This contradicts [5].

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