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Local k-Colorings of Graphs and Hypergraphs

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A local k-coloring of a graph is a coloring of its edges in such a way that each vertex is incident to edges of at most k different colors. We investigate the similarities and differences between usual and local k-colorings, and the results presented in the paper give a general insight to the nature of local colorings. We are mainly concerned with local variants of Ramsey-type problems, in particular, with Ramsey's theorem for hypergraphs, the existence of minimal Ramsey graphs and further questions from noncomplete Ramsey Theory. © 1987 Academic Press, Inc.

1. INTRODUCTION

Many results of graph and hypergraph theory can be formulated as edge coloring theorems. A *coloring* of a graph or hypergraph is an assignment of

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colors (or numbers) to the edges. The basic notion of this paper involves *local k-colorings.* A local k-coloring of a graph G is a coloring of its edges such that the edges incident to any vertex of G are colored with at most k different colors.

The notions of local k-colorings and local Ramsey numbers were introduced in [15]. In this paper we pursue the study of local k-colorings, investigating similarities and differences between usual and local k-colorings. Moreover we consider a possible extension of this concept for hypergraphs. Our paper is mostly devoted to local versions of some basic results of noncomplete Ramsey theory (Sects. 5, 6, 7).

In Sections 2, 3, and 4, we present three results giving general insight to the nature of local k-colorings. The first one is the local version of Ramsey's theorem for hypergraphs (Theorem 1) which seems to be a useful tool for handling certain graph problems for local k-colorings.

The second basic result is a density lemma. It says that every local k-coloring of a graph with average degree d^* has a monochromatic subgraph with average degree at least d^*/k (Theorem 2).

The third result is of a negative nature. It shows that for all n, there exists an n-chromatic graph with a local 2-coloring such that the edges of all color classes determine bipartite graphs (Theorem 5). This is a striking difference between usual and local colorings, since the chromatic number of the union of two bipartite graphs is clearly at most four.

Section 5 deals with minimal Ramsey graphs of forests. It turns out that in case of local k-colorings the family of minimal Ramsey graphs of a forest F is infinite unless F is an odd star or a two-star (Theorems 7 and 8).

In Section 6, we prove the induced Ramsey theorem for local colorings: for all graphs G and positive integers k there exists a graph H such that every local k-coloring of H contains a monochromatic *induced* copy of G (Theorem 11).

Further questions and results are mentioned in Section 7 concerning the generalization of the Ramsey number and the size Ramsey number for local k-colorings. We claim there that the linearity of the Ramsey number of graphs with bounded maximum degree (a result of Chvátal *et al.* [6]) and the linearity of the size Ramsey number of the path (a result of Beck in [2]) remain true in case of local k-colorings. The proofs of these results fall outside of the scope of the present paper.

2. Ramsey Theorem for Local k-Colorings of Hypergraphs

Here we prove the generalization of Ramsey's theorem for hypergraphs for local colorings. Let K_n^r denote the complete r-uniform hypergraph on n vertices (the edges are the r-element subsets of an n-element set). A local

k-coloring of K_n^r is a coloring of its edges such that the set of edges containing any $(r-1)$-element subset of vertices are colored with at most k different colors. A usual k-coloring is obviously a local k-coloring as well, and the definition is consistent with the definition given for graphs in Section 1. For convenience, we allow $r=1$ so that a local k-coloring is a vertex coloring with at most k colors.

THEOREM 1. *Let k, r, and n be positive integers, $r \leq n$. Then there exists $N = N(k, r, n)$ such that every local k-coloring of K_N^r contains a monochromatic K_n^r.*

Proof. We proceed by induction on r. The initial step, when $r=1$, is easy. Obviously $(n-1)k+1$ is a suitable choice for N, by the pigeonhole principle.

The inductive step is based on the following observation:

> If the edges of K_m^r are locally k-colored and $x \in V(K_m^r)$, then the edges incident to x in K_m^r induce a local k-coloring of the $(r-1)$-element subsets of $V(K_m^r) \backslash \{x\}$. $(*)$

Assume that $r \geq 2$ and choose $x_1 \in V(K_N^r)$. Using $(*)$ and the inductive hypothesis, if N is large, then there exists $Y_1 \subset V(K_N^r)$ such that $|Y_1|$ is large, $x_1 \notin Y_1$, and all $(r-1)$-subsets of Y_1 have the same color in the coloring induced by x_1. Let $x_2 \in Y_1$. Since $|Y_1|$ is large, $(*)$ and the inductive hypothesis allow one to find a subset $Y_2 \subset Y_1$ such that $|Y_2|$ is still large, $x_2 \notin Y_2$ and all $(r-1)$-subsets of Y_2 have the same color in the coloring induced by x_2. We continue until $x_1, ..., x_t$ are defined with $t = k(n-r)+1$ and let Y_t be such that $|Y_t| = r-1$, $x_t \notin Y_t$.

Since we have a local k-coloring on K_m^r, the edges $\{x_i\} \cup Y_t \in E(K_m^r)$, for $i = 1, ..., t$, are colored with at most k colors. By the choice of t, there exists a subset $I \subset \{1, ..., t\}$ such that $|I| = n-r+1$ and the edges $\{x_i\} \cup Y_t \in E(K_m^r)$ for $i \in I$ have the same color. Hence $(\bigcup_{i \in I} \{x_i\}) \cup Y_t$ gives a set on n vertices all of whose r-subsets have the same color.

Note that the condition "$|Y_i|$ is large" can be traced back by choosing $|Y_t| = r-1$, $|Y_{i-1}| = N(k, r-1, |Y_i|)+1$ for $i = t, t-1, ..., 1$, where $|Y_0| = N$. ∎

3. DENSITY THEOREM FOR LOCAL k-COLORINGS OF GRAPHS

Assume that a graph is locally k-colored. Denote by G_i the subgraph of G formed by the set of edges assigned color i. Let $d^*(G)$ denote the average degree of G, i.e.,

$$d^*(G) = \sum_{x \in V(G)} d(x)/|V(G)| = 2|E(G)|/|V(G)|.$$

The following theorem formulates an important property of local k-colorings.

THEOREM 2. *If G is locally k-colored, then for some monochromatic subgraph G_i, $d^*(G_i) \geqq d^*(G)/k$ holds.*

Proof. Denote by $d_j(x)$ the number of edges in color j incident to vertex x. Clearly,

$$d(x) = \sum_j d_j(x) \tag{1}$$

and

$$\sum_x d(x) = d^*(G)|V(G)|. \tag{2}$$

Using (1) and applying (2) with G and G_j, we obtain

$$d^*(G)|V(G)| = \sum_j \sum_x d_j(x) = \sum_j d^*(G_j)|V(G_j)| \leqq d^*(G_i) \sum_j |V(G_j)|, \tag{3}$$

where $d^*(G_i)$ is the maximum average degree of some monochromatic subgraph. Observing that in local k-colorings

$$\sum_j |V(G_j)| \leqq k|V(G)|,$$

we obtain by (3) that $d^*(G_i) \geqq d^*(G)/k$. ∎

The fact $|V(G_i)| > d^*(G_i)$ implies $|E(G_i)| > (d^*(G_i))^2/2$, so that Theorem 2 has the following immediate corollary.

COROLLARY 3. *If G is locally k-colored then $E(G_i) \geqq (d^*(G))^2/(2k^2)$ holds for some monochromatic subgraph G_i.*

Corollary 3 can be used to show when G has many edges then G_i has many edges as well, for some i. For example if the complete graph K_n is locally k-colored then at least $c_k n^2$ edges have the same color. (It is easy to prove that the largest value of c_2 is $\frac{1}{6}$.)

Another application of the density theorem shows there exists a monochromatic subgraph with $O(n^2)$ edges in any local k-coloring of the complete bipartite graph $K_{n,n}$ (with n vertices in both color classes). Therefore, by using density results, bipartite Ramsey theorems follow for local k-colorings (cf. [13, p. 95]).

Using the fact that a graph of average degree d^* contains a subgraph of minimum degree at least $d^*/2$, we get another corollary of Theorem 2.

COROLLARY 4. *If G is locally k-colored then it contains a monochromatic subgraph of minimum degree at least $d^*(G)/2k$.*

4. LOCAL k-COLORINGS OF LARGE CHROMATIC GRAPHS

Let $\chi(G)$ denote the chromatic number of the graph G. It is an elementary fact that $\chi(G) \geqq m^k + 1$ implies G contains a monochromatic subgraph of chromatic number at least $m+1$ in every coloring of G with k colors. On the other hand, it is easy to k-color any graph G satisfying $\chi(G) = m^k$ in such a way that the edges in the same color classes determine an m-chromatic graph. These statements belong to folklore (for $m = 2$ see, e.g., in [1, 16]).

Rather surprisingly, a similar result does not hold for local k-colorings.

THEOREM 5. *There exist graphs with arbitrary large chromatic number and a local two-coloring such that all monochromatic subgraphs are bipartite.*

Proof. The graph G_m is defined by Erdös and Hajnal in [10] as follows: the vertices are the pairs (i, j) satisfying $1 \leqq i < j \leqq m$; two vertices (i, j) and (k, l) are adjacent if and only if either $j = k$ or $l = i$.

Let us color the edge between (i, j) and (j, k) in color j, for all pairs i and k with $1 \leqq i < j < k \leqq m$. Edges of color j clearly determine a complete bipartite graph for all j, $2 \leqq j \leqq m-1$. The edges incident to a vertex (k, l) are colored with colors k and l so that this is a local two-coloring for G_m.

In [10], it is proved that $\chi(G_m)$ tends to infinity with m, thus the graphs G_m have the required property. ∎

It is not clear which graphs G have the following property: G has a local two-coloring with bipartite color classes. An obvious necessary condition is that $K_5 \not\subset G$ and a sufficient condition is $\chi(G) \leqq 4$.

Certain triangle-free large chromatic graphs may have the property (see Theorem 5) but there are also triangle-free large chromatic graphs without the property, an example is Zykov's construction in [29]. It is also possible to show that there are graphs with arbitrary large girth and chromatic number with or without the property.

Hajnal pointed out that a result in [9] implies if a graph G satisfies $\chi(G) \geqq 4 \log|V(G)|$, then G cannot be locally two-colored with bipartite color classes.

Concerning local k-colorings of complete graphs, a sharp theorem can be stated.

THEOREM 6. *If the complete graph on $m^k + 1$ vertices is locally k-colored, then there exists a color class with chromatic number at least $m+1$.*

We remark that the case $m=2$ is a consequence of a result of Katona and Szemerédi in [17], and a short proof is given by Tarján in [26]. The method used in Tarján's proof can be generalized to derive Theorem 6. Details are not given here, since a more general extremal result holds for set systems (see [28]).

We finally note that a theorem of Gallai in [12] and Roy in [24] combined with Theorem 6 gives the next result.

COROLLARY. *If the arcs of a tournament on m^k+1 vertices are locally k-colored, then there exists a monochromatic directed path on $m+1$ vertices.*

Note that this corollary for usual k-colorings gives the diagonal case of path Ramsey numbers of tournaments. It is proved independently in [5 and 14].

5. Minimal Ramsey Graphs for Local k-Colorings

Let G be a graph and let $\mathscr{R}^k_{\text{loc}}(G)$ denote the set of graphs H without isolated points which satisfy the following property: every local k-coloring of H contains a monochromatic copy of G, but for any edge $e \in E(H)$ the graph $H-\{e\}$ can be locally k-colored without a monochromatic G. The elements of $\mathscr{R}^k_{\text{loc}}(G)$ are called *minimal Ramsey graphs* of G for local k-colorings.

It is clear from Theorem 1 that $\mathscr{R}^k_{\text{loc}}(G) \neq \varnothing$ for all G and k. If we change the requirement that the graph H be locally k-colored to require only a usual k-coloring, then we obtain the usual set of minimal Ramsey graphs for G. This set is denoted by $\mathscr{R}^k(G)$. A survey on minimal Ramsey graphs is given in [4].

The basic problem in the theory of minimal Ramsey graphs is to try to decide whether $\mathscr{R}^k(G)$ is finite or infinite for a given graph G (k is fixed). The case when G is acyclic is of particular interest and has not been completely solved (see [4]). However, if G contains a non-star component, then $\mathscr{R}^k(G)$ is infinite (see [19]).

We shall prove in Theorems 7 and 8 that $\mathscr{R}^k_{\text{loc}}(G)$ is infinite for all acyclic graphs G except when $G=S_2$ or $G=S_m$ with odd m, where S_m denotes the star with m edges.

As an example where $\mathscr{R}^k_{\text{loc}}(G)$ and $\mathscr{R}^k(G)$ differ, note when G is a matching $\mathscr{R}^k(G)$ is finite while $\mathscr{R}^k_{\text{loc}}(G)$ is infinite. On the other hand, $\mathscr{R}^k(S_2)$ is infinite (odd cycles are in $\mathscr{R}^2(S_2)$) but $\mathscr{R}^k_{\text{loc}}(S_2)=\{S_{k+1}\}$.

When G is a star deciding whether $\mathscr{R}^k_{\text{loc}}(G)$ (and $\mathscr{R}^k(G)$) is infinite relates to a factorization problem. For example, $\mathscr{R}^2(S_4)$ consists of the graph S_7 together with the set of 6-regular grahs with an odd number of vertices (see Murty's result in [4]). The set $\mathscr{R}^2_{\text{loc}}(S_4)$ consists of the graph S_7 together with the set of 6-regular graphs with an odd number of vertices with the property that their edge set can not be partitioned into 3-regular subgraphs.

It is not trivial to see that the 6-regular graphs of the latter type exist. The smallest one we know has 37 vertices and is constructed as follows. Take three disjoint copies of $K_{6,6}$ with one additional vertex z. Delete from the ith bipartite graph the single edge $x_i y_i$, for $i = 1, 2, 3$, and add the six new edges zx_i and zy_i. The proof of Theorem 8 will show this 6-regular graph can not be partitioned into 3-regular subgraphs.

THEOREM 7. *Let F be a forest which is not a single star. Then $\mathscr{R}^k_{\mathrm{loc}}(F)$ is infinite for all $k \geqq 2$.*

Proof. Let F be a forest which is not a star and let $H_1 \in \mathscr{R}^k_{\mathrm{loc}}(F)$. We will show that there exists $H_2 \in \mathscr{R}^k_{\mathrm{loc}}(F)$ such that $|V(H_2)| > |V(H_1)|$ which will establish the result.

Let H be a graph of minimum degree $2|V(F)|\, k$ such that the shortest cycle in H is of length at least $|V(H_1)| + 1$. The existence of such graphs is well known (see [3, p. 104, Theorem 1.1]).

First, we show that every local k-coloring of H contains a monochromatic copy of F. Since the average degree of H is at least $2|V(F)|\, k$, we have, by Corollary 4, that H contains a monochromatic subgraph with minimum degree at least $|V(F)|$. This subgraph obviously contains a copy of F.

Next let H_2 be a minimal Ramsey graph for F contained in H. Clearly, H_2 must contain a cycle, since an acyclic graph can be locally two-colored such that each monochromatic subgraph is a single star. This implies, since H_2 has girth $|V(H_1)| + 1$, that $|V(H_2)| > |V(H_1)|$. ∎

THEOREM 8. *Let S_n be a star on n edges. Then $\mathscr{R}^k_{\mathrm{loc}}(S_n)$ is infinite for n even, $n \geqq 4$, and finite for $n = 2$ or n odd.*

In fact $\mathscr{R}^k_{\mathrm{loc}}(S_n) = \{S_{k(n-1)+1}\}$ when $n = 2$ or n is odd. The behavior of $\mathscr{R}^k(S_n)$ is similar to that of $\mathscr{R}^k_{\mathrm{loc}}(S_m)$ the odd cycle except when $n = 2$. In that case the odd cycle $C_{2m+1} \in \mathscr{R}^2(S_2)$ does not belong to $\mathscr{R}^2_{\mathrm{loc}}(S_2)$ for each $m \geqq 1$.

Proof of Theorem 8. We begin with some observations. If $H \in \mathscr{R}^k_{\mathrm{loc}}(S_n)$ and there is a vertex of H with degree at least $k(n-1)+1$, then $H = S_{k(n-1)+1}$. Thus all members of $\mathscr{R}^k_{\mathrm{loc}}(S_n) \setminus \{S_{k(n-1)+1}\}$ have maximum degree at most $k(n-1)$. We show when $n = 2$ or when n is odd that all regular graphs of degree $k(n-1)$ can be locally k-colored with no monochromatic S_n. This will complete the proof of the second part of the theorem (since each graph of maximum degree at most $k(n-1)$ is contained in some $k(n-1)$-regular graph). Thus consider a $k(n-1)$-regular graph H. If $n=2$ then simply color all edges of H with a different color. If n is odd, then by Petersen's theorem [23] H is factorable into $((n-1)/2)\,k$ 2-factors. Therefore H is the union of k $(n-1)$-factors and we can color each $(n-1)$-factor with a different color. The argument just given parallels the one given in[4] when considering the cardinality of $\mathscr{R}^k(S_n)$ for odd n. For n even, $n \geqq 4$, an entirely different approach is needed as we will now see.

Let n be even, $n \geqq 4$, $k \geqq 2$. We will construct a graph $G(m)$ for all $m=3, 4, \ldots$ with the following properties:

(1) the girth of $G(m)$ is at least m;

(2) the maximum degree of $G(m)$ is at most $k(n-1)$;

(3) if the edges of $G(m)$ are locally k-colored then it contains a monochromatic copy of S_n.

For the moment assume $G(m)$ has been constructed. For $i=0, 1, 2, \ldots$, we define H_0, H_1, $H_2, \ldots$, such that $|V(H_i)| < |V(H_{i+1})|$ with each $H_i \in \mathscr{R}^k_{\text{loc}}(S_n)$. Let $H_0 = S_{k(n-1)+1}$ so that $H_0 \in \mathscr{R}^k_{\text{loc}}(S_n)$. Assume that H_0, H_1, $H_2, \ldots, H_i$ have been defined for some $i \geqq 0$. Set $m = |V(H_i)| + 1$ and consider the graph $G(m)$. By property (3) we can choose a subgraph $H_{i+1} \subset G(m)$ such that $H_{i+1} \in \mathscr{R}^k_{\text{loc}}(S_n)$. The graph H_{i+1} is not acyclic, since a forest of maximum degree at most $k(n-1)$ can be easily locally k-colored without monochromatic S_n (see property (2)). Therefore H_{i+1} contains a cycle C which implies $|V(H_{i+1})| \geqq |C| \geqq m = |V(H_i)| + 1$ by property (1). Thus $\mathscr{R}^k_{\text{loc}}(S_n)$ is infinite for n even, $n \geqq 4$, when the infinite family of graphs $G(m)$, $m \geqq 3$, exists.

The construction of $G(m)$ utilizes a result of Neumann-Lara [22]. This result is the following: for fixed $m \geqq 3$ and $d \geqq 2$, there exists a d-regular bipartite graph of girth at least m. Let $B(m)$ be a $k(n-1)$-regular bipartite graph of girth at least m. Let $B_1, \ldots, B_t$ be disjoint copies of $B(m)$, with $t = ((n-2)\,k+2)/2$. We remove an edge (x_i, y_i) from B_i for each i, $1 \leqq i \leqq t$, add a new vertex z, and join the new vertex to both x_i and y_i for all i. We denote the resulting graph by $G(m)$.

Obviously the girth of $G(m)$ is not less than the girth of $B(m)$. Moreover all vertices of $G(m)$ are of degree $k(n-1)$ except z whose degree is $2t \leqq (n-2)\,k+2 \leqq (n-1)\,k$ $(k \geqq 2)$. Therefore $G(m)$ satisfies properties (1) and (2).

The graph $G(m)$ also satisfies property (3). To see this, assume the contrary, that there is a local k-coloring of $G(m)$ which does not contain a monochromatic copy of S_n. Since $G(m)$ is "almost" $k(n-1)$-regular, the edges belonging to any color class in this coloring determine an "almost" $(n-1)$-regular subgraph. The only vertex which can have degree less than $n-1$ is z. However, for some color, say red, the red subgraph is $(n-1)$-

regular, since the degree of z is larger than $k(n-2)$. But $(n-1)$ is odd, so there exists an i, $1 \leqq i \leqq t$, such that exactly one of the edges zx_i, zy_i is red. Assume that zx_i is red and zy_i is not. The red edges in $E(B_i) - \{x_i\, y_i\}$ determine a bipartite graph $B^* = (X, Y)$ with one vertex (x_i) of degree $n-2$ and all the other vertices of degree $n-1$. Counting the edges of B^* from X to Y and from Y to X we obtain $(|X|-1)(n-1)+n-2=|Y|(n-1)$, i.e., $(n-2)$ is divisible by $(n-1)$ which is impossible when $n \neq 2$. This contradiction proves property (3). ∎

Remark. For $k=2$, the construction just given for $G(m)$ can be simplified. In place of each $B(m)$ one can take any $2(n-1)$-regular graph. The resulting graphs $G(m)$ are easily shown to belong to $\mathscr{R}^2_{\text{loc}}(S_n)$. By the proof just given any such graph, when locally 2-colored, contains a monochromatic copy of S_n. Furthermore, in this case the vertex z has degree $2(n-2)+2=2(n-1)$ so that $G(m)$ is $2(n-1)$-regular. Thus when an edge, say zw, is deleted from $G(m)$, its edges can be alternately colored by two colors along an Eulerian path from x to y, giving a locally 2-colored graph without a monochromatic S_n.

Concerning the behavior of $\mathscr{R}^2(G)$ for graphs other than forests, it follows from constructions of Nešetřil and Rödl [20] that $\mathscr{R}^2(G)$ is infinite if $\chi(G) \geqq 3$ and if G is 3-connected. Their method can be used for local k-colorings to show that $\mathscr{R}^2_{\text{loc}}(K_m)$ is infinite for $m \geqq 3$.

6. Induced Ramsey Theorems for Local k-Colorings

For usual k-colorings the following basic result is well known

THEOREM A (*Rödl* [25], *Deuber* [7], *Erdös, Hajnal, and Pósa* [11]). *For all graphs G and for all k there exists a graph H such that when the edges of H are k-colored, then it contains a monochromatic copy of G as an induced subgraph of H.*

In this section we generalize Theorem A for local k-colorings. We follow the idea of [18] and prove

THEOREM 9. *For all bipartite graphs B and for all k there exists a bipartite graph B' such that when B' is locally k-colored, then it contains a monochromatic copy of B as an induced subgraph of B'.*

Proof. To prove Theorem 9, we need a lemma involving special bipartite graphs. The bipartite graph $B(\binom{p}{q})$ is defined with color classes X, Y as follows: $|X| = p$, $|Y| = \binom{p}{q}$ and each vertex of Y is connected to a different q-element subset of X. It is easy to see that $B(\binom{p}{q})$ is a universal bipartite graph in the sense that all bipartite graphs appear as an induced subgraph of some $B(\binom{p}{q})$. Therefore, to prove Theorem 9, it is enough to prove

LEMMA 10. *For all positive integers p, q and k $(p \geqq q)$ there exists $M = M(p, q, k)$ and $N = N(p, q, k)$ with the following property: if $B(\binom{N}{M})$ is locally k-colored then it contains a monochromatic copy of $B(\binom{p}{q})$ as an induced subgraph of $B(\binom{N}{M})$.*

Proof. We choose $M = k(q-1)+1$ and locally k-color the graph $B(\binom{N}{M})$ for some $N \geqq M$. Let H be the complete M-uniform hypergraph on X, i.e., H has N vertices and the edges of H are the M-element subsets of X. We can naturally define a coloring on the edges of H as follows. If $e \in E(H)$ then consider the vertex $y(e) \in Y$ adjacent to the vertices of e in $B(\binom{N}{M})$. Among the edges of $B(\binom{N}{M})$ incident to $y(e)$, at least q have the same color since $|e| = M = k(q-1)+1$ and $B(\binom{N}{M})$ is locally k-colored. We can therefore fix a q-element subset $f = f(e) \subset e$ such that all edges of $B(\binom{N}{M})$ from $y(e)$ to f have the same color, say color i. We say that f is the *core* of e and we assign color i to f and to e. This gives a coloring on the edges of H. It is easy to see that our coloring is a local k'-coloring for H (in the sense defined in Sect. 2) if $k' = k\binom{k(q-1)}{q-1}$. (We are very generous in the choice of k'.)

Now we refine our coloring on the edges of H according to the types of cores. A type is a q-element subset of $\{1, 2, \ldots, M\}$. If we imagine X as an ordered set, then each $e \in H$ has a type determined by the various positions of the core $f(e)$ in e. Clearly, the number of types assigned to the edges of H is at most $\binom{k(q-1)+1}{q}$. Combining types and colors, the edges of H are locally k''-colored, where $k'' = k'\binom{k(q-1)+1}{q}$. We define

$$n = (p+1)(q-1)\,k + p.$$

Invoking Theorem 1, there exists $N = N(k'', M, n)$ such that K_N^M contains a monochromatic K_n^M under all local k''-colorings of K_N^M. Obviously, N depends only on k, p, q by the definition of k'', M, n. We prove that for this choice of N, $B(\binom{N}{M})$ contains an induced monochromatic $B(\binom{p}{q})$.

The definition of N is such that we can choose an n-element subset A of $V(H)$ with the following property: for all $e \in E(H)$ satisfying $e \subset A$, the color and the type of e are the same, say red of type T. Consider the elements of A in the order given on X. Let x_j be the $jk(q-1)+1$-th element of A for $j = 1, 2, \ldots, p$. The choice of n implies that between x_j and x_{j+1} (moreover before x_1 and after x_p) there is a "gap" consisting of $k(q-1)$ elements of A. Let X' be an arbitrary q-element subset of $x_1, x_2, \ldots, x_p$. Due to the large gaps, we can add $M-q$ elements of $A - \{x_1, x_2, \ldots, x_p\}$ to X' in such a way that we get $e = e(X') \in E(H)$ and X' is of type T in e. Therefore, by definition, the edges of $B(\binom{N}{M})$ between $y(e)$

and X' are red. Moreover, $y(e)$ is not adjacent to vertices of $\{x_1, x_2, \ldots, x_p\} - X'$ in $B(\binom{N}{M})$ from the definition of e. We conclude that $\{x_1, x_2, \ldots, x_p\}$ and $\{y(e(X')): |X'| = q, X' \subset \{x_1, \ldots, x_p\}\}$ induce a red $B(\binom{p}{q})$ in $B(\binom{N}{M})$. ∎

Once Theorem 9 is established, the extension of Theorem A for local k-colorings follows. The argument of [21] (presented as Theorem 1 in [13, p. 103]) can be paralleled to give the following stronger result.

THEOREM 11. *For all graphs G and for all k there exists an H having the same maximum clique size as G such that if H is locally k-colored, then it contains a monochromatic copy of G as an induced subgraph of H.*

Without going into details, we note that some proofs in noncomplete Ramsey theory can be similarly paralleled for local k-colorings. In this way a couple of results related to Theorem 11 can be obtained (e.g., concerning excluded short cycles form G and H).

7. RAMSEY NUMBERS AND SIZE RAMSEY NUMBERS

We denote by $r^k(G)$ the Ramsey number of a graph G, i.e., the minimum m for which K_m contains a monochromatic copy of G in every k-coloring of the edges of K_m. In [15], $r^k_{\mathrm{loc}}(G)$ was introduced as the minimum m for which K_m contains a monochromatic copy of G in every *local* k-coloring of K_m. In the spirit of [6], a result of Chvátal *et al.* can be extended for these local Ramsey numbers as follows.

THEOREM 12. *Let G be a graph on n vertices and of maximum degree d. There is a function $c = c(k, d)$ such that $r^k_{\mathrm{loc}}(G) \leqslant cn$.*

Our proof is similar to that in [6] but it is outside the scope of this paper. On the other hand, answering a conjecture raised in [15], quite recently Truszczyński and Tuza [27] proved the existence of a constant $c = c(k)$ such that $r^k_{\mathrm{loc}}(G) \leqslant cr^k(G)$ for all connected graphs G. This theorem implies a straightforward possibility to extend some upper bounds on $r^k(G)$ for local colorings. For example, Theorem 12 can be deduced from the result of [6] cited above since every graph G of maximum degree $d > 0$ is a subgraph of a connected graph G' of maximum degree at most $d+1$, $|V(G')| = |V(G)|$.

As another consequence of the existence of $c(k)$, for every k there exists an integer k' such that $r^k_{\mathrm{loc}}(G) \leqslant r^{k'}(G)$ for all connected graphs G. We note that, though the upper bound for k' in [27] is exponential, it can be shown that $k' = k^2$ is a suitable choice for k' if we consider the restricted family of connected graphs containing a triangle.

The size Ramsey number $\hat{r}^k(G)$ of a graph G was introduced in [8]. Using the notion of minimal Ramsey graphs (see Section 5), $\hat{r}^k(G)=\min\{|E(H)|: H\in\mathscr{R}^k(G)\}$. The size Ramsey number for local k-colorings, $\hat{r}^k_{\rm loc}(G)$, can be defined analogously as $\min\{|E(H)|: H\in\mathscr{R}^k_{\rm loc}(G)\}$. We may ask about the relation of these numbers. It is possible that $\hat{r}^k_{\rm loc}(G)\leqq c(k)\,\hat{r}^k(G)$, at least for connected graphs G. A supporting evidence is that Beck's theorem [2], which says $\hat{r}^2(P_n)\leqq 900n$, remains true for local 2-colorings. (To see that $\hat{r}^2_{\rm loc}(P_n)\leqq 900n$, it is enough to apply Theorem 2 for the locally 2-colored graph of $900n$ edges defined in [2].) The proof method of Beck also shows $\hat{r}^k(P_n)\leqq c(k)\,n$ which is again extendable (by using Theorem 2) for local k-colorings. The authors are grateful to J. Beck for the discussion on this remark.

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