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Ramsey Numbers for Local Colorings

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Abstract. The concept of a local k-coloring of a graph G is introduced and the corresponding local k-Ramsey number $r_{\text{loc}}^k(G)$ is considered. A local k-coloring of G is a coloring of its edges in such a way that the edges incident to any vertex of G are colored with at most k colors. The number $r_{\text{loc}}^k(G)$ is the minimum m for which K_m contains a monochromatic copy of G for every local k-coloring of K_m. The number $r_{\text{loc}}^k(G)$ is a natural generalization of the usual Ramsey number $r^k(G)$ defined for usual k-colorings. The results reflect the relationship between $r^k(G)$ and $r_{\text{loc}}^k(G)$ for certain classes of graphs.

1. Introduction

The Ramsey number $r^k(G)$ is the smallest positive m such that there exists a monochromatic copy of G in any coloring of the edges of K_m (the complete graph on m vertices) by k colors. The purpose of this paper is to introduce Ramsey numbers for local colorings.

A *local k-coloring* of a graph H is a coloring of the edges of H in such a way that the edges incident to each vertex of H are colored with at most k different colors. The Ramsey number $r_{\text{loc}}^k(G)$ is defined as the smallest integer m such that K_m contains a monochromatic copy of G for every local k-coloring of K_m. Since a k-coloring is a special case of a local k-coloring, it is clear that $r_{\text{loc}}^k(G) \geq r^k(G)$. On the other hand, the number of colors in a local k-coloring generally does not depend on k.

The existence of $r_{\text{loc}}^k(G)$ will be established in Theorem 1.

It is easy to determine $r_{\text{loc}}^k(K_{1,n})$ (Proposition 2) where $K_{1,n}$ denotes the star with n edges. This local Ramsey number $r_{\text{loc}}^k(K_{1,n})$ differs by at most one from $r^k(K_{1,n})$ (cf. [3]). It might be a bit more surprising that $r_{\text{loc}}^k(P_4)$ is also very close to $r^k(P_4)$, the later of which was determined by Irving [11] (Theorem 3).

Generally, the ratio of local and usual Ramsey numbers can be arbitrary large. One can show that $r_{\text{loc}}^2(nK_t)/r^2(nK_t) \geq ct$ if n is sufficiently large with respect to t (see Proposition 10). For connected G it is shown $r_{\text{loc}}^2(G)/r^2(G) \leq 3/2$ (Propositon 9). We do not know, however, whether $r_{\text{loc}}^k(G)/r^k(G) \leq c_k$ for all connected G where

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c_k depends only on k (Problem 15). It is worth mentioning that $r^3_{\text{loc}}(K_3) = r^3(K_3) = 17$ (Proposition 5), but we do not know whether $r^k_{\text{loc}}(K_3) = r^k(K_3)$ for $k \geq 4$.

The value of $r^k_{\text{loc}}(G)$ is closely related to the concept of k-admissibility. Chung [4] defines a graph G as k-admissible if, for $m = r^k(G)$, K_m contains a monochromatic G whenever K_m is locally k-colored with $k + 1$ colors. Any graph G for which $r^k(G) = r^k_{\text{loc}}(G)$ is surely k-admissible. The results of this paper imply that K_3 is 3-admissible, K_m, C_m, P_{2m} are 2-admissible, and P_4 is k-admissible when $k \not\equiv 0 \pmod 3$.

Most of our results concern local 2-colorings. Admittedly, local 2-colorings of complete graphs are closely related to their usual 2-colorings, since all such local 2-colorings are partitionable into usual 2-colored complete graphs with well defined interconnections (Proposition 6). In spite of this, the relation between $r^2_{\text{loc}}(G)$ and $r^2(G)$ is not clear. In some cases these numbers are equal or nearly equal. We prove that $r^2_{\text{loc}}(K_n - K_m) = r^2(K_n - K_m)$ if $n \geq 2m - 1$, in particular, $r^2_{\text{loc}}(K_n) = r^2(K_n)$ (Theorem 11). It is also proved that $r^2_{\text{loc}}(C_n) = r^2(C_n)$ and $r^2_{\text{loc}}(P_n)$ is equal to either $r^2(P_n)$ or to $r^2(P_n) + 1$ depending upon the parity of n (Theorem 12). On the other hand, there are examples showing that $r^2_{\text{loc}}(G) - r^2(G)$ is not bounded: $r^2(nK_3) = 5n$ for $n \geq 2$ is proved in [2] and we show that $r^2_{\text{loc}}(nK_3) = 7n - 2$ for $n \geq 2$ (Theorem 14). Similar behavior can be observed for some connected graphs as well. It is easy to see that $r^2_{\text{loc}}(G) \geq \dfrac{3|V(G)|}{2}$ for each connected graph G (Proposition 8), but there are trees T for which $r^2(T) \leq \left\lfloor \dfrac{4|V(T)|}{3} - 1 \right\rfloor$ (see [1] and [6]). For such trees T, $r^2_{\text{loc}}(T) - r^2(T)$ is not bounded. In spite of these differences, $r^2_{\text{loc}}(G)/r^2(G) \leq \frac{3}{2}$ for all connected graphs G (Proposition 9).

2. Local k-Colorings

The existence of $r^k_{\text{loc}}(G)$ follows from the first theorem.

Theorem 1. *For every* $n \geq 3$, $k \geq 2$, $r^k_{\text{loc}}(K_n) \leq \lceil (k^{k(n-2)+1})/(k-1) \rceil$

Proof. Let G be a locally k-colored complete graph with $\geq (k^{k(n-2)+1})/(k-1)$ vertices. Consider a spanning tree T of G, rooted at fixed vertex x_0, such that the following two conditions hold.

(a) Any vertex x of T has at most k successors $x_1, \ldots, x_s (s \leq k)$, and the colors assigned to edges xx_i, $1 \leq i \leq s$, are distinct.

(b) Edges xy and xz have the same color for any vertices x, y, z of T with $x < y < z$, where the ordering corresponds to the partial order defined by T (with x_0 as minimal element).

(The existence of such a T is clear. Moreover, every maximal T satisfying properties (a) and (b) is a spanning tree of G.)

Thus, by (a), T has an endvertex y such that the path from the root to y contains at least $k(n-2)+1$ edges. Some $n-1$ of these edges, say $x_1y_1, \ldots, x_{n-1}y_{n-1}$, have the same color, since by (b), this path should contain at most k distinct colors.

Assuming that $x_i < y_i$, $1 \le i \le n-1$, it again follows from property (b), that the vertices $x_1, x_2, \ldots, x_{n-1}, v_{n-1}$ induce a monochromatic copy of K_n in G. □

Proposition 2. *For all* $k, n \ge 1$, $r^k_{\text{loc}}(K_{1,n}) = k(n-1) + 2$.

Proof. Clearly $r^k_{\text{loc}}(K_{1,n}) \le k(n-1) + 2$, since $K_{k(n-1)+2}$ contains $K_{1,k(n-1)+1}$. To prove $r^k_{\text{loc}}(K_{1,n}) > k(n-1) + 1$ we give a local k-coloring of $K_{k(n-1)+1}$ without a monochromatic $K_{1,n}$. Let x be an arbitrary vertex of $K_{k(n-1)+1}$ and $A_1, A_2, \ldots, A_k$ a partition of $V(K_{k(n-1)+1}) - x$ such that $|A_i| = n-1$ for $i = 1, 2, \ldots, k$. The complete graph $\langle A_i \cup x\rangle$ is colored with color i for $i = 1, 2, \ldots, k$. The edges between different A_i's are colored with either k or $k-1$ additional colors as described below depending upon the parity of k.

It is well-known that the edge set of K_k has a 1-factorization into $k-1$ perfect matchings (when k is even) or can be decomposed into k maximal matchings (when k is odd). In both cases, assign a color to each matching of the decomposition. Identify in the natural way edges of K_k with pairs of sets A_i, $A_j (1 \le i < j \le k)$ and color all edges joining vertices of A_i with vertices of A_j the color of the corresponding matching. □

Theorem 3. *For every* $k \ge 1$,

$$r^k_{\text{loc}}(P_4) = \begin{cases} 2k+2 & \text{if } k \equiv 0 \text{ or } 1 \pmod 3 \\ 2k+1 & \text{if } k \equiv 2 \pmod 3 \end{cases}$$

We derive this theorem from the following somewhat stronger statement (which, at the same time, characterizes the structure of extremal P_4-free colorings whenever $k \equiv 0$ or $1 \pmod 3$).

Lemma 4. *The monochromatic connected subgraphs of* K_{2k+1} *in any* P_4*-free local* k*-coloring define an edge partition isomorphic to some Steiner triple system.*

Proof. Let $C_1, \ldots, C_m$ be the connected monochromatic subgraphs. Since $P_4 \not\subset C_i$, $1 \le i \le m$, every C_i is a triangle or a star;

set

$$T = \{C_i\colon 1 \le i \le m,\ C_i \text{ is a triangle}\}$$

and

$$S = \{C_i\colon 1 \le i \le m,\ C_i \text{ is a star}\}.$$

Denote by $c(i)$ and $E(i)$ the center vertex and the set of endvertices of $C_i \in S$, respectively. For each vertex x in C_i, $1 \le i \le m$, define the weight $w_i(x)$ as follows:

$$w_i(x) = \begin{cases} 0 & \text{if } C_i \in T \text{ or } x \notin V(C_i) \\ -1 & \text{if } x \in E(i) \\ |E(i)| - 2 & \text{if } x = c(i). \end{cases}$$

Clearly,

$$W = \sum_{i=1}^{m} \sum_{x} w_i(x) = -2|S|.$$

We show that $\sum_{i=1}^{m} w_i(x) \geq 0$ holds for any fixed vertex $x \in V(K_{2k+1})$.

By taking an inventory of the $2k$ edges covered by at most k components incident to x, we obtain

$$\begin{aligned} 2k &= 2|\{i\colon x \in C_i\}| + \sum_{c(i)=x} (|E(i)| - 2) - |\{i\colon x \in E(i)\}| \\ &\leq 2k + \sum_{i=1}^{m} w_i(x). \end{aligned}$$

Consequently, $\sum_x \sum_{i=1}^{m} w_i(x) = W = -2|S| \geq 0$, implying $S = \varnothing$, which proves the lemma. □

Proof of Theorem 3. First we give a coloring which shows that $r_{\text{loc}}^k(P_4)$ is at least as large as stated. For $k = 3l$ or $k = 3l + 1$ take a Steiner triple system on $2k + 1$ points and associate each triangle with a different color. ($2k + 1 = 6l + 1$ or $6l + 3$ so that such Steiner triple systems exist, see [12]). For $k = 3l + 2$ locally $(k - 1)$-color a K_{2k-1} according to a Steiner triple system ($2k - 1 = 6l + 3$) and complete it with a star in a new color.

Consider an arbitrary P_4-free local k-coloring of K_{2k+1}. Then Lemma 4 implies that every vertex x is contained in k triangles of an appropriate Steiner triple system, i.e., k distinct colors occur at x. Consequently, such a coloring cannot be extended to a P_4-free coloring of K_{2k+2}. On the other hand, Steiner triple systems do not exist on $2k + 1$ vertices when $k \equiv 2 \pmod 3$. □

Proposition 5. $r_{\text{loc}}^3(K_3) = 17$.

Proof. Assume that we have a local 3-coloring of K_{17}. We prove that a monochromatic K_3 is present. This local 3-coloring gives a monochromatic star on 7 vertices. The 6 endvertices of that star determine a locally 2-colored K_6 which contains a monochromatic K_3. On the other hand, $r_{\text{loc}}^3(K_3) \geq r^3(K_3) = 17$ (see [10]). □

The above proof shows that the well-known recursive upper bound $r^k(K_3) \leq kr^{k-1}(K_3) - k + 2$ is valid for $r_{\text{loc}}^k(K_3)$ as well. To give a lower bound for $r_{\text{loc}}^k(K_3)$ necessitates constructions of local k-colorings without monochromatic triangles.

3. Local 2-Colorings

Assume that the edges of K_n are locally 2-colored with colors $1, 2, \ldots, m$. We can define a partition $\mathscr{P}(K_n)$ on the vertices of K_n in a natural way as follows. Let A_{ij} denote the set of vertices in K_n incident to edges of color i and color j. The vertices incident to edges of only one color (say color i) can be distributed arbitrarily in the sets A_{ij}. Every partition class A_{ij} induces a 2-colored complete graph in K_n. Moreover the graph G^* with vertex set $\{1, 2, \ldots, m\}$ and edge set $\{(i,j)\colon A_{ij} \neq \varnothing\}$

has pairwise adjacent edges. Thus G^* is a triangle or a star. This observation is summarized in the next proposition.

Proposition 6. *Let K_n be locally 2-colored with colors $1, 2, \ldots, m$. Then either $m = 3$ and*

$$(1) \qquad \mathscr{P}(K_n) = \{A_{12}, A_{13}, A_{23}\}$$

or there exists a color, say color 1, such that

$$(2) \qquad \mathscr{P}(K_n) = \{A_{12}, A_{13}, \ldots, A_{1m}\}. \qquad \square$$

Observe further that in case of *connected* forbidden graphs, partitions of type (2) can be viewed as usual 2-colorings. Indeed, if we have a G-free coloring of K_n with partition classes $A_{12}, \ldots, A_{1m}$, then the 2-coloring obtained by identifying colors $2, \ldots, m$ is also G-free (by the connectivity of G). Thus, we have the following proposition.

Proposition 7. *If G is an arbitrary connected graph, and K_n has a G-free local 2-coloring with $n \geq r^2(G)$, then $\mathscr{P}(K_n) = \{A_{12}, A_{13}, A_{23}\}$ and no A_{ij} is empty $(1 \leq i < j \leq 3)$.* $\square$

If we have a local 2-coloring on K_{3m} such that $\mathscr{P}(K_{3m}) = \{A_{12}, A_{13}, A_{23}\}$ and $|A_{12}| = |A_{13}| = |A_{23}| = m$ then the largest connected monochromatic subgraph has $2m$ vertices. Thus we obtain the following result.

Proposition 8. *Let G be a connected graph. Then*

$$r^2_{\text{loc}}(G) \geq \begin{cases} \dfrac{3|V(G)|}{2} - 1 & \textit{if } |V(G)| \textit{ is even} \\[2ex] \dfrac{3(|V(G)| - 1)}{2} + 1 & \textit{if } |V(G)| \textit{ is odd.} \end{cases} \qquad \square$$

It is known that there are trees T for which $r^2(T) \leq \left\lceil \dfrac{4|V(T)|}{3} - \right\rceil$ (see [1] and [6]). For such trees T, $r^2_{\text{loc}}(T) - r^2(T)$ can be arbitrary large. However, $r^2_{\text{loc}}(T)/r^2(T)$ is small as shown by the next proposition.

Propositon 9. $r^2_{\text{loc}}(G) \leq \lfloor \frac{3}{2} r^2(G) - \frac{1}{2} \rfloor$ *for all connected graphs G.*

Proof. Suppose to the contrary, that we have a G-free local 2-coloring of K_m, $m = \lfloor \frac{3}{2} r^2(G) - \frac{1}{2} \rfloor$. By Proposition 7 $\mathscr{P}(K_m) = \{A_{12}, A_{13}, A_{23}\}$. Clearly, the smallest partition class, say A_{23}, satisfies $|A_{23}| \leq \lfloor \frac{1}{2} r^2(G) - \frac{1}{2} \rfloor$, and consequently $|A_{12} \cup A_{13}| \geq r^2(G)$. This contradicts Proposition 7. $\square$

It is worth mentioning that $r^2_{\text{loc}}(G)/r^2(G)$ is not in general bounded. To see this let $G = nK_t$. It was proved in [2] that for n large with respect to t, $r^2(nK_t) \leq (2t - 1)n + c_t$. On the other hand, let $X = \bigcup_{i=1}^{t} X_i$ where $|X_i| = tn - 1$ for $i = 1, 2, \ldots, t - 1$, $|X_t| = n - 1$, $X_i \cap X_j = \varnothing$ for $i \neq j$. Define a local 2-coloring on

$K_{(t-1)(tn-1)+n-1}$ by coloring the edges inside X_i with color i for $i = 1, \ldots, t$ and coloring all edges connecting distinct X_i's with color t. Thus we obtain the following result.

Proposition 10. *For every* $n \geq 1$, $t \geq 2$, $r^2_{\text{loc}}(nK_t) \geq n(t^2 - t + 1) - t + 1$ *and*

$$r^2_{\text{loc}}(nK_t)/r^2(nK_t) \geq t/2 - 1/4 + o(1)$$

when t is fixed and n is large. □

Theorem 11. *For* $m \geq 2n - 1$, $(m, n) \neq (3, 2)$, $r^2_{\text{loc}}(K_m - K_n) = r^2(K_m - K_n)$.

Proof. Trivially, it is sufficient to prove that $r^2_{\text{loc}}(K_m - K_n) \leq r^2(K_m - K_n)$.

Let $r^2_{\text{loc}}(K_m - K_n) = t + 1$ and locally 2-color $K = K_t$ such that it contains no monochromatic $K_m - K_n$. By Proposition 7, we may assume $\mathscr{P}(K) = \{A_{12}, A_{13}, A_{23}\}$. We will show that this locally 2-colored graph K can be recolored with the two colors such that it contains no monochromatic $K_m - K_n$.

Our first objective is to recolor K with two colors such that each of $\langle A_{12} \cup A_{13} \rangle$, $\langle A_{12} \cup A_{23} \rangle$, and $\langle A_{13} \cup A_{23} \rangle$ contain no monochromatic $K_m - K_n$. For each i and j, $1 \leq i \leq j \leq 3$, let $u(i, ij)$ and $l(i, ij)$ be integers such that A_{ij} contains a $K_{u(i,ij)} - K_{l(i,ij)}$ in color i. Since we have a $(K_m - K_n)$-free coloring of K, it follows that each of the following inequalities hold for all possible values of $u(i, ij)$ and $l(i, ij)$.

(1) $$u(1, 12) + u(1, 13) - \max\{0, l(1, 12) + l(1, 13) - n\} \leq m - 1$$

(2) $$u(3, 13) + u(3, 23) - \max\{0, l(3, 13) + l(3, 23) - n\} \leq m - 1$$

(3) $$u(2, 12) + u(2, 23) - \max\{0, l(2, 12) + l(2, 23) - n\} \leq m - 1.$$

We suppose for the moment that K cannot be recolored in two colors such that each of $\langle A_{12} \cup A_{13} \rangle$, $\langle A_{12} \cup A_{23} \rangle$, and $\langle A_{13} \cup A_{23} \rangle$ contain no monochromatic $K_m - K_n$. Since we cannot eliminate color 1, if we attempt to do so by changing color 1 to 3 in A_{12}, color 1 to 2 in A_{13}, and all edges between A_{12} and A_{13} to either 3 or to 2, we obtain for some set of $u(i, ij)$ and $l(i, ij)$

(4) $$u(1, 12) + u(3, 13) - \max\{0, l(1, 12) + l(3, 13) - n\} \geq m$$

(5) $$u(2, 12) + u(1, 13) - \max\{0, l(2, 12) + l(1, 13) - n\} \geq m.$$

Similarly if we attempt to eliminate color 2, we obtain

(6) $$u(1, 12) + u(2, 23) - \max\{0, l(1, 12) + l(2, 23) - n\} \geq m.$$

(7) $$u(2, 12) + u(3, 23) - \max\{0, l(2, 12) + l(3, 23) - n\} \geq m;$$

and if we attempt to eliminate color 3, we obtain

(8) $$u(1, 13) + u(3, 23) - \max\{0, l(1, 13) + l(3, 23) - n\} \geq m$$

(9) $$u(3, 13) + u(2, 23) - \max\{0, l(3, 13) + l(3, 23) - n\} \geq m.$$

It should be emphasized that under the supposition all inequalities (1)–(9) hold for some set of $u(i, ij)$ and $l(i, ij)$. (e.g., in which the $u(i, ij)$'s are maximal and the $l(i, ij)$'s are as small as possible).

We show that inequalities (1)–(9) are incompatible, which will imply the existence of a particular recoloring. First consider the case when a pair of three expressions

$$\max\{0, l(1,12) + l(1,13) - n\},$$
$$\max\{0, l(3,13) + l(3,23) - n\}, \quad \text{and} \quad \max\{0, l(2,12) + l(2,23) - n\},$$

appearing in (1)–(3), is zero. For example, suppose the latter two are zero. Then the sum of the left-hand sides of equations (2) and (3) is less than the sum of the left-hand sides of equations (7) and (9) which is impossible. Clearly, a similar incompatibility of inequalities occurs when some other pair is zero. Thus we may assume at least one pair of the expressions, say the first pair, is nonzero. Then it follows that

$$\begin{aligned} &l(1,12) + l(1,13) - n + l(3,13) + l(3,23) - n \\ &\quad \le \max\{0, l(1,12) + l(3,13) - n\} + \max\{0, l(1,13) + l(3,23) - n\} \end{aligned}$$

so that inequalities (1), (2), (4) and (8) are inconsistent.

Hence, the given supposition is false and K can be recolored in two colors such that each of $\langle A_{12} \cup A_{13} \rangle$, $\langle A_{12} \cup A_{23} \rangle$, and $\langle A_{13} \cup A_{23} \rangle$ contain no monochromatic $K_m - K_n$.

By symmetry we may assume that we can recolor K changing color 3 to 2 in A_{13}, color 3 to 1 in A_{23}, and all edges between A_{13} and A_{23} to color 2. Denote the resulting graph by K' and the changed A_{ij} by A'_{13} and A'_{23}.

It is clear that if the 2-colored graph K' contains a monochromatic $K_m - K_n$, then it must be in color 2 (since A'_{23} and $\langle A_{12} \cup A'_{13} \rangle$ are $(K_m - K_n)$-free), and must contain vertices from each of A_{12}, A'_{13}, and A'_{23}. Suppose this is the case with a_1, a_2, and a_3, the number of vertices taken from A_{12}, A'_{13}, and A'_{23}, respectively.

All edges between A_{12} and A'_{13} are in color 1, so that $a_1 + a_2 \le n$ and these $a_1 + a_2$ vertices must be part of the K_n deleted from K_m. This clearly implies $|A_{12}| \le n - 1$ and $|A'_{13}| \le n - 1$, since no $K_m - K_n$ of color 2 appears in $\langle A_{12} \cup A'_{23} \rangle$ or $\langle A'_{13} \cup A'_{23} \rangle$.

Now we define a final recoloring of K. Change the color 2 edges between A'_{13} and A'_{23} to color 1 and interchange colors 1 and 2 in the graph A'_{13}. The resulting graph will be denoted by K'' and the changed A'_{13} by A''_{13}.

One can see that each pair of A_{12}, A'_{23}, and A''_{13} induce a $(K_m - K_n)$-free subgraph: $\langle A_{12} \cup A'_{23} \rangle$ has not been changed in the last step, $|A_{12} \cup A''_{13}| \le 2n - 2 < m$, and all color 1 edges of $\langle A''_{13} \cup A'_{23} \rangle$ were color 3 in (a $(K_m - K_n)$-free coloring of) K. If K'' has a $(K_m - K_n)$-free 2-coloring, the theorem is proved. But if a $K_m - K_n$ occurs in color 1 then $|A_{23}| \le n - 1$ follows in the same way as above for $|A_{12}|$ and $|A_{13}|$. Thus, $t \le 3n - 3 \le r^2(K_m - K_n) - 1$, because the complete 3-partite graph with color classes of cardinality $n - 1$ defines a $(K_m - K_n)$-free 2-coloring of K_{3n-3} when $(m, n) \ne (3, 2)$. □

Theorem 12. *Let P_n and C_n denote the path and the cycle on n vertices. Then*

(1) $$r^2_{\text{loc}}(C_3) = r^2_{\text{loc}}(C_4) = 6;$$

(2) $$r^2_{\text{loc}}(C_{2m}) = 3m - 1 \quad \text{if } m \ge 3;$$

(3) $$r^2_{\mathrm{loc}}(C_{2m+1}) = 4m + 1 \quad \text{if } m \geq 2;$$

(4) $$r^2_{\mathrm{loc}}(P_{2m}) = 3m - 1 \quad \text{if } m \geq 1;$$

(5) $$r^2_{\mathrm{loc}}(P_{2m+1}) = 3m + 1 \quad \text{if } m \geq 1.$$

Sketch of proof. All local Ramsey numbers but those given in (5) are identical with their usual Ramsey number values as proved in [9], [13] and [8]. (The value $r^2(P_{2m+1}) = 3m$ is established in [9].) Thus the local Ramsey numbers are at least as large as the values given in (1)–(4). Also the inequality $r^2_{\mathrm{loc}}(P_{2m+1}) \geq 3m + 1$ follows from Proposition 9.

The proof is completed by showing that the values given in (1)–(5) are upper bounds for $r^2_{\mathrm{loc}}(G)$ where G is the appropriate path or cycle of interest. We show this by induction, from P_t or C_t to P_{t+2} or C_{t+2}. Using Proposition 7, we may always assume that $\mathscr{P}(K_n) = \{A_{12}, A_{13}, A_{23}\}$ with $|A_{12}| \geq |A_{13}| \geq |A_{23}| > 0$, where n is the claimed Ramsey number in (1)–(5). Then, in most cases (whenever $|A_{13}| \geq 2$) the induction step is done by deleting three vertices, $x \in A_{12}$, $y \in A_{13}$, and $z \in A_{23}$ from K_n. Let G' be the monochromatic path or cycle found in $K_n - \{x, y, z\}$ and assume that G' is in color 1. If G' has an edge pq between $A_{12} - x$ and $A_{13} - y$ then G' is extended by changing the edge pq to the path $pyxq$. If G' lies completely in $\langle A_{12} - x \rangle$ then any vertex of G' can be replaced by a vertex of $\langle A_{13} - y \rangle$ so that the previous extension works.

The induction is anchored by checking the appropriate formula when the graph is C_3, C_4, C_6, P_2, or P_3. Technical details and the case $|A_{13}| = 1$ are left to the reader. □

It follows from a result of Cockayne and Lorimer [5] that $r^2(nK_2) = 3n - 1$. That result remains true for local 2-colorings as well.

Theorem 13. $r^2_{\mathrm{loc}}(nK_2) = 3n - 1$.

Proof. First assume that we have a local 2-coloring of K_{3n-1} such that $\mathscr{P}(K_{3n-1}) = \{A_{12}, A_{13}, \ldots, A_{1m}\}$, $|A_{12}| \geq \cdots \geq |A_{1m}|$. If $|A_{12}| \leq 2n - 1$ then, clearly, K_{3n-1} contains an nK_2 in color 1. Assume $|A_{12}| = 2n - 1 + k$ and suppose that A_{12} does not contain an nK_2 in color 2. Then since $r^2(nK_2) = 3n - 1$ implies $r(kK_2, nK_2) \leq 2n - 1 + k$, we have that A_{12} contains a kK_2 in color 1. But this kK_2 can be extended to a monochromatic nK_2, since $3n - |A_{12}| = n - k$ and $|A_{12}| - 2k \geq n - k$.

Next assume the local 2-coloring of K_{3n-1} is such that $\mathscr{P}(K_{3n-1}) = \{A_{12}, A_{13}, A_{23}\}$. When this occurs we use induction on n. Pick $x \in A_{12}$, $y \in A_{13}$, $z \in A_{23}$, giving a triangle whose edges are all colored differently. The (3-colored) complete graph $K_{3n-1} - \{x, y, z\}$ contains a monochromatic $(n-1)K_2$. This together with an appropriate edge of the triangle gives a monochromatic nK_2. □

Theorem 14. $r^2_{\mathrm{loc}}(nK_3) = 7n - 2$ *if* $n \geq 2$.

Proof. Proposition 10 implies $r^2_{\mathrm{loc}}(nK_3) \geq 7n - 2$. We note that there is more than one construction which shows that $r^2_{\mathrm{loc}}(nK_3) > 7n - 3$. Two of them are shown in Fig. 1.

Fig. 1

It remains to prove $r^2_{\text{loc}}(nK_3) \leq 7n - 2$. We first assume that K_{7n-2} is locally 2-colored such that

$$\mathscr{P}(K_{7n-2}) = \{A_{12}, A_{13}, \ldots, A_{1m}\}.$$

Note that $m \geq 3$, otherwise we have a 2-colored complete graph and $r^2(nK_3) = 5n \leq 7n - 2$. Let l be the maximum number of disjoint triangles in color 1 such that no three vertices of a triangle lie in A_{1j} for some j, $2 \leq j \leq m$. Let B denote the union of vertices of a monochromatic lK_3 in color 1. Clearly, from the maximality of l, $V(K_{7n-2}) - B$ contains vertices from at most two classes of the partition $\{A_{12}, \ldots, A_{1m}\}$. Assume that $V(K_{7n-2}) - B \subset A_{12} \cup A_{13}$ and $|A_{12} - B| \geq |A_{13} - B| \geq 2$. If a triangle xyz of B has two vertices, say x and y in $B - (A_{12} \cup A_{13})$ then for $p, q \in A_{12} - B$, $r, s \in A_{13} - B$, $B - \{x, y, z\} \cup prx \cup qsy$ defines a $(l + 1)K_3$ in color 1, contradicting the choice of l. If a triangle xyz of B is such that $x \in B - (A_{12} \cup A_{13})$ and y, z are both in A_{12} (or in A_{13}) then for $p \in A_{12} - B$, $r, s \in A_{13} - B$, $B - \{x, y, z\} \cup prx \cup yzs$ defines a $(l + 1)K_3$ in color 1, again contradicting the choice of l. Thus all triangles of B have at least one vertex in A_{12}. We claim that every triangle of B has a vertex in A_{12} such that it is connected to all but at most one vertex of $A_{12} - B$ in color 2. If xyz is a triangle of B such that $A_{12} \cap \{x, y, z\} = x$ and xp is an edge of color 1 for some $p \in A_{12} - B$, then choose $q \in A_{12} - B$ with $p \neq q$ and $r \in A_{13} - B$. Thus $B - \{x, y, z\} \cup xpr \cup qyz$ gives a $(l + 1)K_3$ in color 1, a contradiction. Next assume that xyz is a triangle of B such that $A_{12} \cap \{x, y, z\} = \{x, y\}$. If our claim is not true then we can choose $p, q \in A_{12} - B$ such that $p \neq q$, px and qy are edges of color 1. By choosing $r, s \in A_{13} - B$ with $r \neq s$, $B - \{x, y, z\} \cup pxr \cup qys$ is a $(l + 1)K_3$ in color 1, a contradiction. This establishes the claim.

Let C be a subset of $A_{12} \cap B$ such that $|C| = l$ and each vertex of C is connected to all but at most one vertex of $A_{12} - B$ in color 2. Now $|A_{12} - B| + |A_{13} - B| \geq 7n - 2 - 3l$ so that $|A_{12} - B| + |C| \geq \lceil (7n - l - 2)/2 \rceil$. But if $l \leq n - 1$, then $\lceil (7n - l - 2)/2 \rceil \geq 3n$ and $(A_{12} - B) \cup C$ contains an nK_3 in color 2.

Next we drop the condition that $|A_{13} - B| \geq 2$ and assume that $|A_{13} - B| \leq 1$. Then $|A_{12} - B| \geq 7n - 2 - 1 - 3(l - 1) = 7n - 3l$. In [2], it is shown that a 2-colored complete graph on $3n + 2(n - l)$ vertices contains either nK_3 in color 2 or $(n - l)K_3$ in color 1. Since $7n - 3l \geq 3n + 2(n - l)$ reduces to $2n \geq l$, we have a monochromatic nK_3. The only case left to consider is when $\mathscr{P}(K_{7n-2}) = \{A_{12} \cup A_{13} \cup A_{23}\}$. We do this case by induction on n leaving $n = 2$ to be checked by the interested reader. We wish to select at most seven vertices from the colored K_{7n-2} which induces monochromatic triangles in all the three colors. This will clearly complete the proof by induction. It is easy to see that (apart from obvious symmetries) there are only two cases when we can not find seven such vertices. The first case is when A_{12} and A_{13} are complete in color 1 and A_{23} is complete in color 2. In this case $|A_{12} \cup A_{13}|$ or $|A_{23}|$ is at least $3n$ and the existence of a monochromatic nK_3 is clear.

The second case is when A_{13} is complete in color 1 and A_{23} is complete in color 2. If $|A_{13}|$ (or $|A_{23}|$) is less than $2n-1$ then $|A_{12} \cup A_{23}|$ (or $|A_{12} \cup A_{13}|$) is at least $5n$ and the existence of a monochromatic nK_3 follows from $r^2(nK_3) = 5n$. Clearly we can assume $|A_{13}|$ and $|A_{23}|$ are both less than $3n$ so that $|A_{12}| \geq n$. If $|A_{13}|$ or $|A_{23}| \geq 2n$ then we find a nK_3 in color 1 in $A_{12} \cup A_{13}$ or in color 2 in $A_{12} \cup A_{23}$. Thus we can assume $|A_{13}| \leq |A_{23}| \leq 2n-1$ which implies $|A_{12}| \geq 3n$. In this case we find a nK_2 in color 1 or color 2 in A_{12}, since $r^2(nK_2) = 3n-1$, and we can complete it to a nK_3 by using vertices of A_{13} or of A_{23}. □

4. On the Ratio $r^k_{\text{loc}}(G)/r^k(G)$

Concerning the ratio $r^k_{\text{loc}}(G)/r^k(G)$ we have the following problem.

*Problem 15.** Is it true for connected graphs G that $r^k_{\text{loc}}(G)/r^k(G) \leq c_k$, where c_k is a constant depending only on k?

We note that the connectivity of G is essential (cf. Proposition 10) and for $k = 2$ the answer is affirmative (see Proposition 9). For $k = 3$ we can prove only the following weaker result.

Theorem 16. *Let G be a connected graph of order n. Then $r^3_{\text{loc}}(G) < c \cdot \log n \cdot r^3(G)$ for some constant c.*

Proof. Let $r = r^3_{\text{loc}}(G)$ and assume that a K_r is locally 3-colored with colors $1, \ldots, m$ in such a way that no monochromatic copy of G occurs. Let $A(i,j,k)$ denote the set of vertices in K_r incident to edges of color i, j and k $(1 \leq i < j < k \leq m)$. Without loss of generality one can assume that there are exactly three distinct colors at any vertex. For $A(i,j,k)$ non-empty, (i,j,k) will be called an *active triple*.

Obviously, any two active triples contain a common color. Hence, as one can easily show, either the number of active triples remains small (actually no more than 10) or there exists a set of at most two colors, say $\{1,2\}$, which has a common color with any active triple. (This claim also follows by a more general hypergraph result of Erdös and Lovász [7].)

Since each $A(i,j,k)$ induces a 3-colored complete graph K_r, $r < 10 \cdot r^3(G)$ in the first case (since there are at most ten active triples). Hence we assume that any active triple contains color 1 or 2, and that there are $r_1 \geq r/2$ vertices in K_r which are incident with edges colored 1. Consider the locally 3-colored K_{r_1} induced in K_r by this set of vertices and define the graph H with vertex set $\{2, \ldots, m\}$ and edge set $\{ij\colon A(1,i,j) \neq \varnothing\}$.

* This problem was solved in an article appearing earlier in this journal (see pp. 67–73, Vol. 3 No. 1. 1987, M. Truszczynski and Z. Tuza)

First observe that if ij is not an edge of H, then colors i and j can be identified in the coloring of K_{r_1} so that no monochromatic copy of the connected graph G is obtained.

By a series of possible recolorings with identification of vertices in H, the graph H reduces to a complete graph. Thus without loss of generality one can assume that H is a complete graph on $m-1$ vertices, i.e., $A(1,i,j) \neq \varnothing$ for every i and j, $1 \leq i < j \leq m$. If ij and kl are independent edges of H, then any edge of K_{r_1} between the vertex classes $A(1,i,j)$ and $A(1,k,l)$ is colored 1. Therefore H has at most $2n-1$ vertices, otherwise K_{r_1} would contain a complete graph on n vertices and thus a copy of G in color 1.

Clearly, the edges of H can be covered by at most $\log_2(2n-1)+1$ bipartite graphs $B_1, \ldots, B_p$. Define the vertex sets $V_k = \bigcup \{A(1,i,j)\colon ij \text{ is an edge of } B_k\}$ $(1 \leq k \leq p)$. Now $V_1 \cup \cdots \cup V_p$ is equal to the vertex set of K_{r_1} and thus $r_1 \leq \sum_{k=1}^{p} |V_k|$. We show now that $|V_k| < r^3(G)$ holds for every $k = 1, \ldots, p$.

Observe that for fixed k the locally 3-colored complete graph induced by V_k in K_{r_1} can be recolored by identifying colors i and j whenever ij is not an edge of B_k. This means that each color belonging to the same vertex class of the bipartite graph B_k can be identified. In this way we obtain a 3-coloring for the complete graph induced by V_k so that no monochromatic copy of G is obtained. Consequently, $|V_k| < r^3(G)$ $(1 \leq k \leq p)$, and the proposition follows, since

$$r^3_{\mathrm{loc}}(G) = r \leq 2r_1 \leq 2 \sum_{r=1}^{p} |V_k| < c \cdot \log n \cdot r^3(G). \qquad \square$$

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