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NOTE

AN UPPER BOUND ON THE RAMSEY NUMBER OF TREES

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In this note we give a Ramsey-type inequality and its consequence concerning the Ramsey number of trees.

Let $f(k, n)$ denote the smallest integer m with the following property. If the edges of K_m (the complete graph on m vertices) are colored with k colors, then there exists a monochromatic subgraph of minimum degree at least n. It is an obvious and well-known fact (see [1, p. xvii]) that a graph with m vertices and more than $(m-n)(n-1)+\binom{n}{2}$ edges contains a subgraph of minimum degree at least n. The straightforward upper bound for $f(k, n)$ is the minimal m satisfying the following inequality:

$$k\left((m-n)(n-1)+\binom{n}{2}\right)<\binom{m}{2}. \tag{1}$$

By solving (1), we get the following result.

Theorem. $f(k, n) \leqslant (n-1)(k+\sqrt{k(k-1)})+2.$

Let T_n denote a tree with n edges. The Ramsey number $r(T_n, k)$ is the smallest integer m such that every k-coloring of the edges of K_m contains a monochromatic T_n. Obviously, $T_n \subset G$ whenever G has minimum degree at least n, therefore $r(T_n, k) \leqslant f(k, n)$, and the above theorem implies

Corollary. $r(T_n, k) \leqslant (n-1)(k+\sqrt{k(k-1)})+2.$

Our upper bound on $r(T_n, k)$ slightly improves the estimate of $2kn+1$ given by Erdös and Graham [2]. The gain (asymptotically $\frac{1}{2}n$) is significant for small k only. For $k=2$, it gives $n(2+\sqrt{2})-2$ instead of $4n+1$.

The upper bound on $f(k, n)$ coming from (1) is probably sharp. For $k=2$, a nontrivial construction shows $f(2, n) \geqslant (n-1)(2+\sqrt{2}) - c$, where c is a constant. We note that $f(2, 2) = 5$, $f(2, 3) = 8$, $f(2, 4) = 11$, $f(2, 5) = 15$.

References

[1] B. Bollobás, Extremal Graph Theory (Academic Press, New York, 1978).
[2] P. Erdös and R.L. Graham, On partition theorems for finite graphs, in: A. Hajnal et al., eds., Infinite and Finite Sets (North-Holland, Amsterdam, 1975) 515–527.