Discrete Mathematics 55 (1985) 167–180
North-Holland

COVERING AND COLORING PROBLEMS FOR RELATIVES OF INTERVALS

A. GYÁRFÁS and J. LEHEL

Computer and Automation Institute, Hungarian Academcy of Sciences, Budapest, Hungary

Received 26 December 1983
Revised 15 May 1984

The following generalizations and relatives of interval families are studied in the paper: arcs of a circle, multiple intervals, chords of a circle, d-dimensional boxes and multiple boxes. For these families we survey results and problems concerning the dependence of the transversal number on the packing number and the dependence of the coloring number on the clique number.

The intersection graphs of the underlying set systems are circular arc graphs, multiple interval graphs, circle graphs (called also overlap graphs), box graphs and multiple box graphs. Thus most of our problems and results concern the relation between the clique-cover number and stability number (ϑ and α), or between the chromatic number and clique number (χ and ω) of these graphs.

1. Introduction

In this paper we study the following generalizations and relatives of interval families: arcs of a circle, multiple intervals, chords of a circle, d-dimensional boxes and multiple boxes. Applications and results on these families and their intersection graphs (circular arc graphs, multiple interval graphs, circle graphs, box graphs) can be found in [7, 8, 10, 15, 17, 18]. Here we focus our attention on results and problems concerning the dependence of the transversal number τ (or the clique cover number ϑ) on the packing number ν and the dependence of the coloring number q on the clique number ω.

An unpublished result of Gallai states that for any family of intervals the maximum number of pairwise disjoint intervals is equal to the minimum number of points meeting all intervals ($\nu = \tau$). Bielicki and Rado proved [3, 14] that the maximum number of pairwise intersecting intervals is equal to the minimum number of classes in a partition into sub-families containing pairwise disjoint intervals ($\omega = q$). These results can be summarized in the statement that interval hypergraphs are normal or, in terms of intersection graphs, interval graphs are perfect (cf. [2]).

The equality of τ and ν and the equality of q and ω is not true for the families which we are concerned with here, but it is reasonable to suspect that they are 'close' to each other or at least that τ is bounded by some function of ν, and q is bounded by some function of ω. The question of closeness and boundedness of

0012-365X/85/$3.30

these numbers was explicitly formulated for convex structures in [5], a hypergraph theoretical approach was presented in [11].

If F is a finite family of sets in the d-dimensional Euclidean space R^d, then the numbers in question are defined as follows.

The *packing number* $\nu(F)$ is the maximum number of pairwise disjoint sets in F; the *clique-cover number* $\vartheta(F)$ is the minimum number of classes in a partition of F into pairwise intersecting sets; the *transversal number* $\tau(F)$ is the minimum number of points meeting all sets of F; the *clique number* $\omega(F)$ is the maximum number of pairwise intersecting sets in F; the *coloring number* $q(F)$ is the minimum number of classes in a partition of F into pairwise disjoint sets.

These parameters are commonly used in hypergraph theory and all of them but τ correspond to well-known graph parameters. Let F be a family of sets. If G is the intersection graph of F (the vertices of G are the sets of F and the edges of G correspond to intersecting pairs of sets). Then

$\nu(F)=\alpha(G)$, the stability number of G;

$\vartheta(F)=\vartheta(G)$, the clique-cover number of G;

$\omega(F)=\omega(G)$, the clique number of G;

$q(F)=\chi(G)$, the chromatic number of G.

The order of magnitude of the functions expressing the interdependence of these numbers varies according to the different families we are concerned with. In some cases the function is linear, for instance, for circular arcs $\tau \leqslant \nu+1$ and $q \leqslant 2\omega-1$ hold. Sometimes only polynomial bounds are known (e.g., $\tau \leqslant \nu^d$ for d-dimensional boxes, or $q \leqslant 4\omega^2-3\omega$ for two-dimensional boxes) and there are cases when only exponential bounds are obtained (e.g., $q \leqslant 2^{\omega}\omega^2(\omega-1)$ for chords of a circle).

In certain cases no functional dependence connects these numbers; for instance, τ can be arbitrary large while $\nu=1$ if the underlying structure is the family of chords of a circle or the family of double boxes in the plane. A highly non-trivial example in this direction is a result due to Burling [4][1]: q cannot be bounded by any function of ω for three-dimensional boxes. Finally, there are cases when the existence of functional dependence between two numbers is an open question (see Problems 5.7, 5.8 and 5.9).

2. Arcs of a circle and multiple intervals

A well-known generalization of interval families is the family of *circular arcs*: a finite collection of closed arcs of a circle. The equality $\tau=\nu$ for interval families (due to Gallai) immediately implies $\tau(F) \leqslant \nu(F)+1$ if F is a family of circular arcs; the inequality is sharp. It is also straightforward that $q(F) \leqslant 2\omega(F)-1$, with equality for $\omega=2$.

[1] The authors are grateful to Prof. Branko Grünbaum for communicating this result.

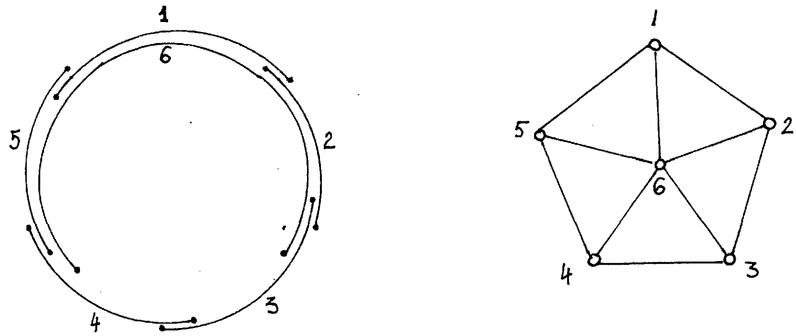

Fig. 1. Circular arcs and their intersection graphs; $\omega = 3$, $\chi = 4$.

Proposition 2.1. *If F is a family of arcs on a circle C satisfying $\omega(F)=3$, then $q(F)\leq 4$ and the bound is tight, as shown in Fig.* 1.

Proof. We may assume that there is an open arc $C' \subset C$ such that C' is contained in exactly two arcs $A, B \in F$ and no member of $F-\{A, B\}$ meets C'. If we cut C' from C then we get a family F' where A and B are replaced by two arcs A_1, A_2 and B_1, B_2, respectively (see Fig. 2). We may consider F' as a family of intervals on a line satisfying $\omega(F')\leq 3$. Therefore F' has a 3-coloring, denote by $c(I)\in\{1,2,3\}$ the color of $I\in F'$. We shall transform this 3-coloring of F' into a good 4-coloring of F.

Without restricting generality, we assume that $A_1 \subset B_1$.

Case 1. $c(A_1)=c(A_2)$. Color A with $c(A_1)$ and color B with 4. (If $c(B_1)=c(B_2)$ then similarily, define $c(B)=c(B_1)$ and $c(A)=4$.)

Case 2. $c(A_1)\neq c(A_2)=c(B_1)$. Then define $c(A)=c(A_2)$ and $c(B)=4$. (If $c(A_2)\neq c(A_1)=c(B_2)$ and $A_2\subset B_2$, then similarily, define $c(A)=c(A_1)$ and $c(B)=4$.)

Now the only remaining case we have to handle (apart from the permutations of colors) is the following (see Fig. 2):

Case 3. $A_2 \supset B_2$, $c(A_1)=c(B_2)=1$, $c(A_2)=2$ and $c(B_1)=3$. Denote by F_1' the arcs of $F'-\{A_1, A_2, B_1, B_2\}$ which meet A_1 and have color 2. Let F_2' denote the

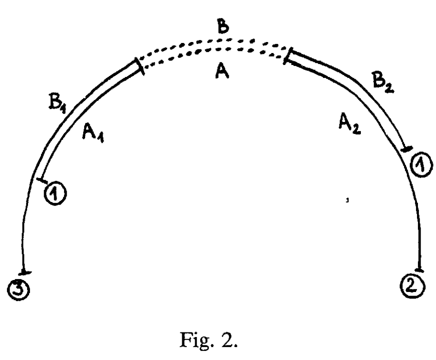

Fig. 2.

arcs of $F'-\{A_1, A_2, B_1, B_2\}$ which meet B_2 and have color 3. Now $F_1' \cup F_2'$ contains pairwise disjoint arcs, since $D \in F_1'$, $E \in F_2'$ and $D \cap E \neq \emptyset$ would imply that A, B, D and E are four pairwise intersecting arcs, contradicting $\omega(F)=3$. Now the arcs of $F_1' \cup F_2'$ can be colored with 4. Define $c(A)=2$ and $c(B)=3$. $\square$

Proposition 2.1 proves a special case of the following question of Tucker [18].

Problem 2.2. Is it true that $q(F) \leq \frac{3}{2}\omega(F)$ holds for any family F of circular arcs? (It is easy to give an F with $q = \lfloor \frac{3}{2}\omega \rfloor$.)

Another generalization of interval families is the family of *multiple intervals*: a finite collection of sets of the real line which can be written as the union of c closed intervals. If $c=2$ then we speak about *double intervals*. The name double and multiple intervals were introduced by Harary and Trotter in [17] and by Griggs and West in [8]. It is easy to see that the notion of double intervals extends the notion of circular arcs. The relation of τ and ν for multiple intervals has been studied by the authors in [10].

Proposition 2.3. *If F is a family of double intervals satisfying $\nu(F)=1$ then $\tau(F) \leq 3$ and the bound is tight as shown in Fig.* 3.

Proof. Let $p \in \bigcap_{A \in F} \operatorname{conv}(A)$, where $\operatorname{conv}(A)$ is the convex hull of A, i.e., the minimal closed interval containing the double interval A. It is enough to show that $\tau(F') \leq 2$ where $F' = \{A \in F\colon p \notin A\}$.

The interval components of the double interval $A \in F'$ lying on the left and right side of p are denoted by $L(A)$ and $R(A)$, respectively. If r is a right endpoint of $L(A)$ for some $A \in F'$ then we define the interval family I_r as follows:

$$I_r = \{R(A)\colon A \in F', L(A) \subset (-\infty, r)\}.$$

We may assume that I_r is non-empty for some r since otherwise $\tau(F') \leq 1$. We choose the point s to the extreme right with the property: the intervals of I_s have non-empty intersection. Now it is easy to check that the interval family $\{R(A)\colon A \in F', s \notin L(A)\}$ has non-empty intersection, thus $\tau(F') \leq 2$. $\square$

It is proved in [10] that any family F of c-intervals satisfies $\tau(F) \leq f(\nu(F))$ with a suitable function f depending only on c. The values of f are not known for $c \geq 2$ except the case $c=2$ and $\nu=1$ in Proposition 2.3. The smallest unsolved cases are considered in the next questions: $c=2$, $\nu=2$, and $c=3$, $\nu=1$.

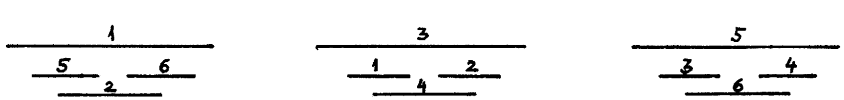

Fig. 3. Double intervals with $\nu=1$, $\tau=3$.

Problem 2.4. Let F be a family of double intervals satisfying $\nu(F)=2$. Find the smallest integer t such that $\tau(F)\leqslant t$.

Problem 2.5. Let F be a family of triple intervals satisfying $\nu(F)=1$. Find the smallest t such that $\tau(F)\leqslant t$.

Now we continue by the coloring problem of multiple intervals.

Proposition 2.6. *If F is a family of double intervals satisfying $\omega(F)=2$ then $q(F)\leqslant 4$ and the bound is the best possible as shown in Fig. 4.*

Proof. Let G be the intersection graph of F and direct an edge $xy\in E(G)$ as follows. If the corresponding double intervals are J_x and J_y then at least one of their interval components, say I_x and I_y, are intersecting; the edge xy is directed from x to y if the right endpoint of I_x is covered by I_y, otherwise, xy is directed from y to x. The condition $\omega(F)=2$ implies that each outdegree of the directed graph $\vec{G}$ is at most two. Moreover, it is easy to see that any set $S\subset V(\vec{G})$ induces a subgraph of $\vec{G}$ which contains a vertex of outdegree at most one. Thus a subgraph of $\vec{G}$ induced by S contains at most $2|S|-1$ edges. By this observation, one can index the vertices of G by $1,2,\ldots,|F|$ in such a way that x_i has indegree at most one in the subgraph $\vec{G}_i$ of $\vec{G}$ induced by $\{x_i, x_{i+1},\ldots,x_{|F|}\}$, for all i, $1\leqslant i\leqslant|F|$. Since all vertices of $\vec{G}$ have outdegree at most two, x_i is adjacent to at

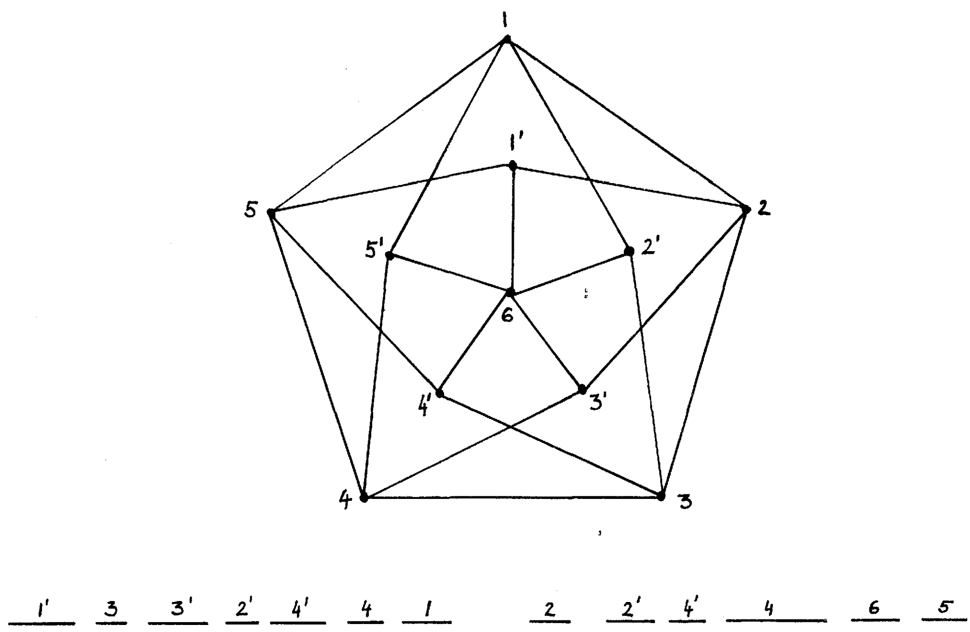

Fig. 4. Double intervals and their intersection graph; $\omega=2$, $\chi=4$.

most three vertices in $\vec{G}_i$. Now we can perform a greedy coloring of G with at most four colors by taking its vertices in the order $x_{|F|}, \ldots, x_2, x_1$. That coloring implies the 4-coloring of F. □

The proof of Proposition 2.6 can be easily generalized to show that $q(F) \leq 2c(\omega(F)-1)$ holds if F is a family of c-intervals (see [9]). This bound on $q(F)$, however, is probably not sharp.

Problem 2.7. Is it true that $q(F) \leq 2\omega(F)$ for any family F of double intervals? (This is true for $\omega = 2$ by Proposition 2.6.)

3. Chords of a circle

Well-known relatives of interval families are the families of *chords of a circle.* It is usual to assume that no chords share common endpoints. (Another way of simplification is to consider open chords.) The intersection graphs defined by families of chords are called *circle graphs* or *overlap graphs.* An overview of this topic can be found in [7].

In constrast with the case of circular arcs and multiple intervals, there are families of chords F_m, $m = 1, 2, \ldots$, such that $\nu(F_m) = 1$ and $\tau(F_m) \geq m$. However, the clique-cover number is bounded by a function of ν.

Proposition 3.1. *If F is a family of chords satisfying $\nu(F) = 2$ then $\vartheta(F) \leq 3$ and the circle graph C_5 (i.e. the cycle on five vertices) shows that the bound is sharp.*

Proof. Assume that $|F| = n$ and index the distinct endpoints of the chords of F by $1, 2, \ldots, 2n$, consecutively in clockwise direction. The starting point of a chord of F is the endpoint having the smaller index. Let $\vec{G}$ be the directed graph whose vertices correspond to the members of F, $(x, y) \in E(\vec{G})$ is an edge from x to y if and only if the corresponding chords C_x and C_y do not intersect and the starting point of C_x has smaller index than the index of the starting point of C_y. It is easy to check that $\vec{G}$ has not directed path of four vertices, thus $\chi(\vec{G}) \leq 3$ by a theorem of Gallai [6] and Roy [16].[2] Since the (undirected) complement of $\vec{G}$ is the intersection graph of F, $\vartheta(F) \leq 3$ follows. □

The argument of the proof of Proposition 3.1 easily gives $\vartheta(F) \leq \binom{\nu(F)+1}{2}$ for any family F of chords. The best bound of $\vartheta(F)$ is probably linear in $\nu(F)$.

Problem 3.2. Let F be a family of chords of a circle. Is there a constant c such that $\vartheta(F) \leq c\nu(F)$?

[2] If the longest directed path of $\vec{G}$ has k vertices then $\chi(\vec{G}) \leq k$.

Concerning the functional dependence of q on ω, we start with the very special case of all (open) chords generated by a given point set of a circle.

Proposition 3.3. *Let $p_1, p_2, \dots, p_n$ be n distinct points on a circle and $F = \{\mathrm{chord}(p_i, p_j)\colon 1 \leqslant i < j \leqslant n\}$, where $\mathrm{chord}(p_i, p_j)$ denotes the open line segment between p_i and p_j. Then $\omega(F) = \lfloor n/2 \rfloor$ and $q(F) = \lceil n/2 \rceil$, for every $n > 3$.*

Proof. If $k = \lfloor n/2 \rfloor$ then the intersection graph of the subfamily $\{\mathrm{chord}(p_i, p_{i+k})\colon 1 \leqslant i \leqslant k\}$ is a clique with k vertices. On the other hand, any chord meets at most $k-1$ chords having no common endpoints. Thus $\omega(F) = \lfloor n/2 \rfloor$.

If $n = 2k+1$ $(k > 1)$ then the intersection graph of the subfamily $\{\mathrm{chord}(p_i, p_{i+k})\colon 1 \leqslant i \leqslant 2k+1\}$ is the antihole $\bar{C}_{2k+1}$ (i.e., the complement of the cycle C_{2k+1}) which has chromatic number $k+1$. Thus $q(F) \geqslant \lceil n/2 \rceil$ follows for every $n \geqslant 2$.

Now let us consider our family F as a complete graph K_n on the vertex set $\{p_1, p_2, \dots, p_n\}$. If $n = 2k$ then the decomposition of K_n into k disjoint hamiltonian chains given in [2, p. 233] defines a good k-coloring of F, since the initial hamiltonian chain and its rotations contain no intersecting chords (see Fig. 5).

For $n = 2k+1$ the previous hamiltonian chain decomposition plus the chords at p_{2k+1} define a $(k+1)$-coloring of F. In both cases $q(F) \leqslant \lceil n/2 \rceil$ follows. □

The existence of a function f such that $q(F) \leqslant f(\omega(F))$ holds for any family F of chords of a circle has been proved in [9]. The function obtained is exponential, namely $2^{\omega}\omega^2(\omega - 1)$. If $\omega = 2$ then $q \leqslant 5$ is known [13]; Fig. 6 shows a construction with $q(F) = 4$ (the proof is left to the reader).

Problem 3.4. Let F be a family of chords of a circle satisfying $\omega(F) = 2$. Is it true that $q(F) \leqslant 4$?

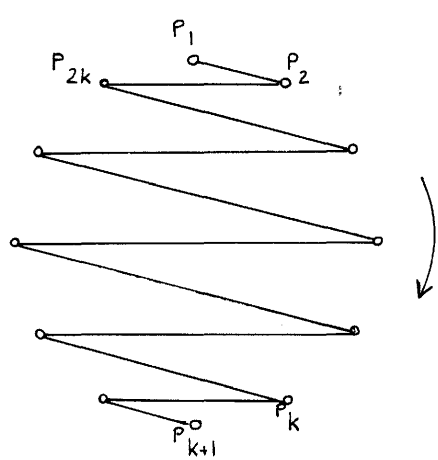

Fig. 5.

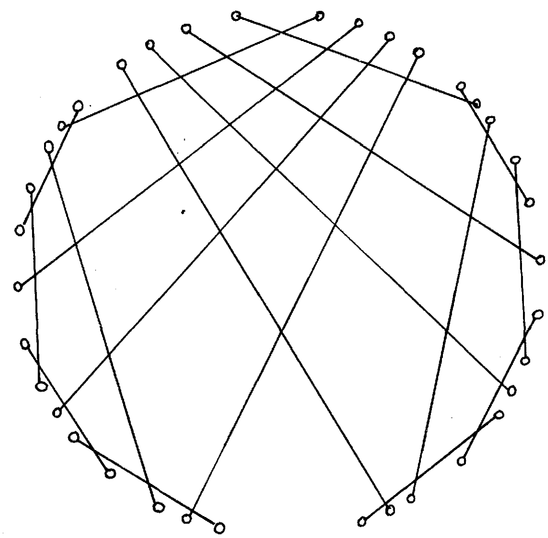

Fig. 6. Chords of a circle with $\omega = 2$, $q = 4$.

Problem 3.5. Give reasonable upper and lower bounds for $q(F)$ in terms of $\omega(F)$, where F is a family of chords of a circle. (The best known lower bound is linear, the best known upper bound is exponential.)

4. Boxes

The parallelopipeds whose faces are parallel to the coordinate axes of R^d are called *d-dimensional boxes*. The intersection graphs of d-dimensional boxes were introduced in [15].

Proposition 4.1. *If F is a family of two-dimensional boxes satisfying $\nu(F) = 2$ then $\tau(F) \leq 3$ and the bound is sharp, since C_5 is an intersection graph of two-dimensional boxes.*

Proof. Denote by F_x and F_y the interval families defined by the projections of the boxes of F into the x and y axes, respectively. Clearly, $\nu(F_x) \leq 2$ and $\nu(F_y) \leq 2$; therefore F_x and F_y possess two-element transversals. Their direct product gives a transversal set $\{p, q, r, s\}$ of F such that p, q, r and s are corners of a box in clockwise order. Assume that p is the upper right corner.

Now consider the box family

$$F_p = \{B \in F: B \cap \{s, p, q\} = \{p\}\}.$$

If $F_p = \emptyset$ then $T_0 = \{q, r, s\}$ is a transversal of F. Assume that $F_p \neq \emptyset$ and let B_1 be the box of F_p with the uppermost bottom side. The intersection of the bottom line of B_1 and the segment pq is denoted by x_1 (see Fig. 7).

Consider the family

$$F_q=\{B\in F\colon B\cap\{x_1, q, r\}=\{q\}\}.$$

If $F_q=\emptyset$ then $T_1=\{x_1, r, s\}$ is a transversal of F. Assume that $F_q\neq\emptyset$ and let B_2 be the box of F_q with the rightmost left side meeting the segment rq in x_2.

Define the family

$$F_r=\{B\in F\colon B\cap\{x_2, r, s\}=\{r\}\}.$$

If $F_r=\emptyset$ then $T_2=\{x_1, x_2, s\}$ is a transversal of F. Assume that $F_r\neq\emptyset$ and let B_3 be the box of F_r with the lowest top side meeting the segment rs in x_3.

Since $B_1\cap B_2=\emptyset$ and $B_2\cap B_3=\emptyset$, it follows from the condition $\nu(F)=2$ that $B_3\cap B_1\neq\emptyset$. Thus x_3 is placed higher than x_1 (or they are at the same height). Now we claim that $T_3=\{x_1, x_2, x_3\}$ is a transversal of F. Indeed, the existence of a box $B\in F$ such that $B\cap T_3=\emptyset$ would imply

$$B\cap B_3=B_2\cap B_3=B\cap B_2=\emptyset,$$

contradicting the condition $\nu(F)=2$. Thus one of the sets T_i $(i=0, 1, 2, 3)$ is a three-element transversal of F. $\square$

It is easy to see that $\tau(F)\leqslant\nu^d(F)$ holds for any family F of d-dimensional boxes (with equality for $\nu(F)=1$, since F has the Helly property). However, very few reasonable estimates are known. We formulate the next two problems.

Problem 4.2. Let F be a family of 3-dimensional boxes satisfying $\nu(F)=2$. What is the smallest integer t such that $\tau(F)\leqslant t$? ($t\leqslant 6$ comes easily from Proposition 4.1.)

Problem 4.3. Let F be a family of two-dimensional boxes. Is there a constant c such that $\tau(F)\leqslant c\nu(F)$?

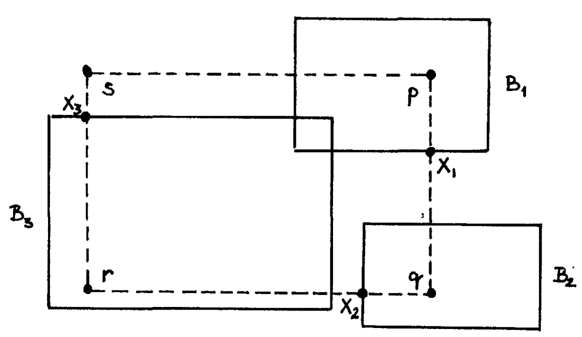

Fig. 7.

Concerning the functional dependence of q on ω, the following results are known. Asplund and Grünbaum prove in [1] that if F is a family of two-dimensional boxes such that $\omega(F)=2$ then $q(F)\leq 6$ and this bound is sharp. They give also the upper bound $q\leq 4\omega^2-3\omega$, for any family of two-dimensional boxes. The situation changes radically in the space R^3. Burling shows in [4] that for every $m=1,2,\ldots,$ there exists a family F_m of three-dimensional boxes such that $\omega(F_m)=2$ and $q(F_m)=m$.

5. Multiple boxes

The notion of multiple boxes is a common generalization of boxes and multiple interval structures. A set in R^d which is the union of c closed d-dimensional boxes is called a *d-dimensional c-box.* The study of families of multiple boxes, with given parameters d and c, leads to various interesting questions with a geometrical flavor. When $c=2$, we speak about double boxes.

Proposition 5.1. *A double box in the plane is called an L-gon if it is the union of two boxes having a common corner and common interior points. If F is a family of L-gons such that every three of them have a common point, then $\tau(F)\leq 2$ and this bound is tight as shown in Fig.* 8.

Proof. Let F_b be the subfamily of L-gons of F having box components with a common bottom corner; similarly, let $F_t\subset F$ be the subfamily containing all L-gons with box components having a common top corner. Clearly, $F=F_b\cup F_t$. Let $L_0\in F_b$ be the L-gon with uppermost bottom line Q. Then every two L-gons of $F_b-\{L_0\}$ have a common point in L_0, and by the speciality of F_b, any two of its members have a common point on the line Q. Thus $\tau(F_b)=1$ follows by Helly's theorem. By a similar argument one obtains $\tau(F_t)=1$. Therefore,

$$\tau(F)=\tau(F_b\cup F_t)\leq\tau(F_b)+\tau(F_t)=2. \quad \square$$

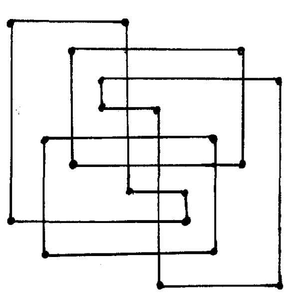

Fig. 8.

Proposition 5.1 is also sharp in the sense that there are families of pairwise intersecting L-gons with arbitrary transversal number (e.g. see Fig. 9).

To express in a more comprehensive manner the tightness of the results like Proposition 5.1, it is convenient to introduce the notions of Gallai-index and Gallai-numbers.

The kth *Gallai-number* ($k \geq 2$) of the d-dimensional c-boxes, $g(k, c, d)$, is the minimal integer t with the following property: if F is a family of d-dimensional c-boxes such that every k members of F have a nonempty intersection then $\tau(F) \leq t$. The smallest k for which $g(k, c, d) < \infty$ is called the *Gallai-index* of the d-dimensional c-boxes.

The definitions given here only for multiple boxes can be obviously interpreted on arbitrary structures. As an example, our observations concerning L-gons can be summarized in the following way: The Gallai-index of L-gons is three and the third Gallai-number of L-gons is two.

It was proved recently in [12] that the Gallai-index of the d-dimensional c-boxes is $\min\{c, d\}+1$, for every $c, d \geq 1$. There are very few results however on the kth Gallai-numbers for $k \geq \min\{c, d\}+1$.

Problem 5.2. Determine $g(3, 2, 2)$, i.e., what is the best upper bound on the transversal number of a family of double boxes in the plane such that every three members of the family have a common point. (We known only that $g(3, 2, 2) \geq 2$ by Proposition 5.1.)

The third Gallai-number is not known even in case of very special box families, such as crosses or T-gons.

Problem 5.3. Let F be a family of T-shaped double boxes in the plane such that every three of them have a nonempty intersection. Are there two points meeting every member of F?

We use $g(k, c)$ to denote the kth Gallai-number of multiple intervals (i.e., $g(k, c) = g(k, c, 1)$). The lower bound $\lceil ck/(k-1) \rceil - 1 \leq g(k, c)$ given in [12], together with the next proposition, yields the Gallai-number $g(c+1, c) = c$.

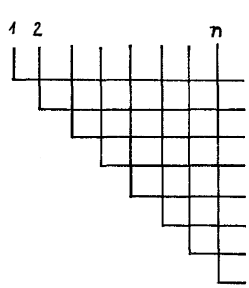

Fig. 9. L-shapes with $\nu = 1, \tau = \lceil n/2 \rceil$.

Proposition 5.4. *If F is a family of c-intervals such that every $c+1$ of them have a common point, then $\tau(F) \leqslant c$.*

Proof. We use induction on c. Assume that $c \geqslant 2$ and the claim is true for $c-1$. (The case $c=1$ is Helly's theorem.) Let p_0 be the left endpoint of the interval $\bigcap_{A \in F} \operatorname{conv}(A)$. Let p_0 be the left endpoint of $\operatorname{conv}(A_0)$ for some $A_0 \in F$, and denote by H the half line at p_0 containing A_0. Now clearly

$$F' = \{A \cap H \colon p_0 \notin A,\ A \in F\}$$

is a family of $(c-1)$-intervals such that every c of them have a common point in A_0. Thus by induction, for some points $p_1, p_2, \ldots, p_{c-1}$,

$$\{p_1, \ldots, p_{c-1}\} \cap A' \neq \emptyset \quad \text{for every } A' \in F'.$$

Therefore, the set $\{p_0, p_1, \ldots, p_{c-1}\}$ is a c-element transversal of F. $\square$

The value $g(c, c) = c+1$ can be obtained in the same way; thus $g(k, c)$ remains to be determined in the range $2 \leqslant k < c$. The smallest unknown value is the second Gallai-number of 3-intervals (see Problem 2.5). To close Gallai-type problems, we propose a question on the order of magnitude of the second Gallai-numbers of multiple intervals.

* **Problem 5.5.** Is $g(2, c)$ a linear function of c, perhaps $g(2, c) = 2c-1$?

Now we propose further questions on multiple boxes in the plane. If the parameter k is less than the Gallai-index then, by definition, the transversal number is unbounded. The question arises, what is the behavior of the clique-cover number ϑ in this case. Our first example concerns particular families of double boxes.

Assume that four real numbers x, x', y, y' are given and $x \leqslant x'$, $y \leqslant y'$. An *L-shape* $L(x, x', y, y')$ is the union of the line segments AB and AC, where $A = (x, y)$, $B = (x, y')$ and $C = (x', y)$. The segments AB and AC are called the vertical and horizontal sides of the L-shape.

Proposition 5.6. *If F is a family of L-shapes satisfying $\nu(F) = 2$ then $\vartheta(F) \leqslant 4$ and the bound is sharp as shown in Fig.* 10.

Proof. In order to avoid the discussion of some special cases, we assume that all sides of the L-shapes of F lie on the distinct lines. Choose $L', L'' \in F$ in such a way that L' has the uppermost horizontal side and L'' has the rightmost vertical side. Let F_1 and F_2 be the subfamilies of F whose members do not intersect L' and L'', respectively. Clearly F_1 and F_2 contain pairwise intersecting L-shapes. The members of $F - (F_1 \cup F_2)$ intersect the horizontal side of L' and the vertical side of L''. This fact implies that the intersection graph of the L-shapes in $F - (F_1 \cup F_2)$ is a

* no, Matousek showed $g(2,c) \geq \text{const}\,\dfrac{c^2}{\log c}$

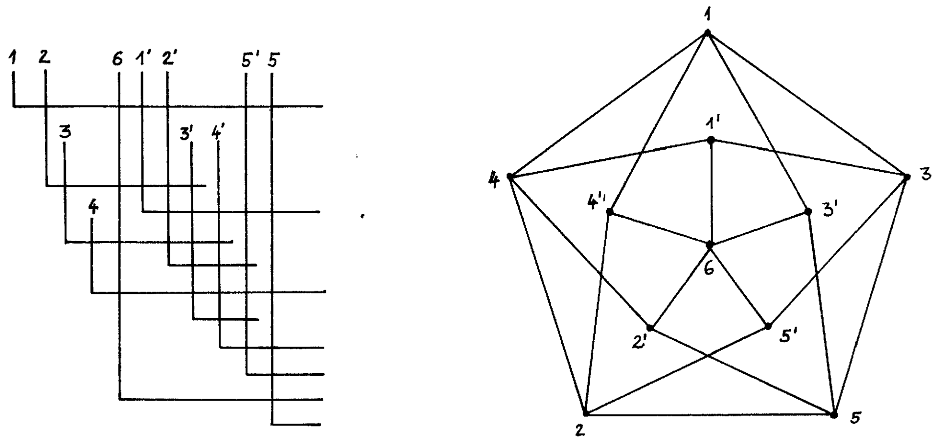

Fig. 10. L-shapes and their 'disjointness graph' showing $\nu=2$, $\vartheta=4$.

permutation graph (cf. [7]). Permutation graphs are perfect and then the equality $\vartheta=\alpha$ implies

$$\vartheta(F-(F_1\cup F_2))=\nu(F-(F_1\cup F_2))\leqslant 2,$$

that is, $F-(F_1\cup F_2)$ has a partition into families F_3 and F_4 containing pairwise intersecting L-shapes. Thus $\vartheta(F)\leqslant 4$. $\square$

Let us remark that by using the same method as in the proof of Proposition 5.6, one can obtain an exponential bound on ϑ as a function of ν, for the families of L-shapes. In case of arbitrary c-boxes in the plane there are no known existence results for $c\geqslant 2$.

Problem 5.7. Is there a constant t such that if F is a family of c-boxes in the plane satisfying $\nu(F)=2$ then $\vartheta(F)\leqslant t$?

The coloring problem on the same structure is also open.

Problem 5.8. Is there a constant k depending only on c such that $q(F)\leqslant k$ holds for any family F of c-boxes in the plane with $\omega(F)=2$?

The answer to Problem 5.8 is not known even if F consists of L-shapes.

Problem 5.9. Let F be a family of L-shapes satisfying $\omega(F)=2$. Is there a constant k such that $q(F)\leqslant k$?

Acknowledgment

The authors are grateful to one of the referees whose comments improved the presentation.

References

[1] E. Asplund and B. Grünbaum, On a coloring problem, Math. Scand. 8 (1960) 181–188.
[2] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).
[3] A. Bielicki, Problem 56, Colloq. Math. 1 (1948) 333.
[4] J.P. Burling, On coloring problems of families of prototypes, Ph.D. Thesis, University of Colorado, 1965.
[5] L. Danzer, B. Grünbaum and V. Klee, Helly's theorem and its relatives, Proc. Symp. Pure Math., Vol. VII, Amer. Math. Soc., Providence, RI (1963) 101–180.
[6] T. Gallai, On directed paths and circuits, in: P. Erdös and G. Katona, eds., Theory of Graphs (Academic Press, New York, 1968) 115–118.
[7] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs (Academic Press, New York, 1980).
[8] J.R. Griggs and D.B. West, Extremal values of the interval number of a graph, SIAM J. Algebraic Discrete Maths. 1 (1980) 1–7.
[9] A. Gyárfás, On the chromatic number of multiple interval graphs and overlap graphs, Discrete Math. 55 (1985) 161–166.
[10] A. Gyárfás and J. Lehel, A Helly-type problem in trees, in: P. Erdös, A. Rényi and V.T. Sós, eds., Combinatorial Theory and its Applications (North-Holland, Amsterdam, 1970) 571–584.
[11] A. Gyárfás and J. Lehel, Hypergraph families with bounded edge cover or transversal number, Combinatorica 3 (1983) 351–358.
[12] J. Lehel, Gallai-type results for multiple boxes and forests, to appear.
[13] J. Lehel, A 5-coloring algorithm for triangle-free overlap graphs, manuscript.
[14] R. Rado, Covering theorems for ordered sets, Proc. London Math. Soc. 50 (1948) 509–535.
[15] F.S. Roberts, On the boxicity and cubicity of a graph, in: W.T. Tutte, ed., Recent Progress in Combinatorics (Academic Press, New York, 1969) 301–310.
[16] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, RAIRO Automat. 1 (1967) 127–132.
[17] W.T. Trotter, Jr. and F. Harary, On double and multiple interval graphs, J. Graph Theory 3 (1979) 205–211.
[18] A. Tucker, Coloring a family of circular arcs, SIAM J. Appl. Math. 3 (1975) 493–502.