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ON THE SUM OF THE RECIPROCALS OF CYCLE LENGTHS IN SPARSE GRAPHS

A. GYÁRFÁS, H. J. PRÖMEL, E. SZEMERÉDI and B. VOIGT

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For a graph G let $\mathscr{L}(G)=\Sigma\left\{\frac{1}{k}\middle| G \text{ contains a cycle of length } k\right\}$. Erdős and Hajnal [1] introduced the real function $f(\alpha)=\inf\left\{\mathscr{L}(G)\middle|\frac{|E(G)|}{|V(G)|}\geqq\alpha\right\}$ and suggested to study its properties. Obviously $f(1)=0$. We prove $f\left(\frac{k+1}{k}\right)\geqq(300k\log k)^{-1}$ for all sufficiently large k, showing that sparse graphs of large girth must contain many cycles of different lengths.

1. Introduction

Let $G=(V,E)$ be a finite undirected graph. Let $\mathscr{L}(G)=\sum\left\{\frac{1}{k}\middle| \text{there exists a cycle of length } k \text{ in } G\right\}$. The number $\mathscr{L}(G)$ is the sum of the reciprocals of cycle lengths occuring in G. In a sense, $\mathscr{L}(G)$ measures how rich the graph G is with respect to cycles. E.g. $\mathscr{L}(K_n)=\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{n}\approx\left(\gamma-\frac{3}{2}\right)+\log n$, where $\gamma=.5772\ldots$ is the Mascheroni constant; $\mathscr{L}(K_{n,n})=\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\approx\frac{1}{2}(\gamma-1+\log n)$. Erdős and Hajnal [1] introduced the real function $f(\alpha)=\inf\left\{\mathscr{L}(G)\middle|\frac{|E(G)|}{|V(G)|}\geqq\alpha\right\}$. They asked about the behaviour of $f(\alpha)$ as α tends to infinity. The complete bipartite graph $K_{n,n}$ shows that $f(n)\leqq c\cdot\log n$ for $n\geqq2$, but originally it was unknown whether $f(\alpha)$ is bounded or not. Recently Gyárfás, Komlós and Szemerédi [2] showed that $f(\alpha)\geqq a\cdot\log\alpha$, provided that $\alpha>b$, where a and b are suitable contants. Obviously $f(\alpha)=0$ for $\alpha\leqq1$. The problem of determining the behaviour of $f(\alpha)$ for $\alpha\in(1, 1+\varepsilon)$ has been raised in [2]. It was not even clear whether $f(1+\varepsilon)>0$ for every $\varepsilon>0$.

In this paper we prove the following theorem, answering this question in the affirmative:

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Theorem 1. *There exists a positive integer* n_0 *such that* $f\left(\frac{k+1}{k}\right) \geqq \frac{1}{300k \log k}$ *for all integers* $k \geqq n_0$*; in other words: for every graph G with* $\frac{|E(G)|}{|V(G)|} \geqq \frac{k+1}{k}$ *it follows that* $\mathscr{L}(G) \geqq \frac{1}{300k \log k}$, *although the girth of G may be arbitrarily large.*

Basically our proof follows the pattern of [2]: Generalizing the notion of 3/2-tree [2], we construct a subtree of G. Since $\frac{|E(G)|}{|V(G)|} > 1$, eventually we will detect sufficiently many cycles of different lengths, so that $\mathscr{L}(G)$ can be estimated from below.

2. Proof of the theorem

Lemma. *Let k be a positive integer and let G be a graph such that* $\frac{|E(G)|}{|V(G)|} \geqq \frac{k+1}{k}$. *Then there exists a graph* G^* *such that*

(1) G^* *does not contain k vertices* $x_0, \ldots, x_{k-1}$, *each of which has degree* 2 *and which form a path of length k.*

(2) $\frac{|E(G^*)|}{|V(G^*)|} \geqq \frac{k+1}{k}$ *and* $\mathscr{L}(G^*) \leqq \mathscr{L}(G)$.

Proof. We proceed by induction on the number of k-element subsets $\{x_0, \ldots, x_{k-1}\}$ of vertices in G such that each x_i has degree 2 and $x_0, \ldots, x_{k-1}$ form a path of length k. Pick any such subset $\{x_0, \ldots, x_{k-1}\}$ and let $\hat{G}$ be the graph which is obtained from G by deleting $\{x_0, \ldots, x_{k-1}\}$ and deleting all edges which are incident with these vertices. Obviously then $\mathscr{L}(\hat{G}) \leqq \mathscr{L}(G)$, $|E(\hat{G})| = |E(G)| - k - 1$ and $|V(\hat{G})| = |V(G)| - k$. The assumption $\frac{|E(G)|}{|V(G)|} \geqq \frac{k+1}{k}$ implies that also $\frac{|E(\hat{G})|}{|V(\hat{G})|} \geqq \frac{k+1}{k}$. Hence, by induction, the assertion follows. ∎

Now let $G = (V, E)$ be a fixed finite graph such that

$$|E|/|V| \geqq (k+1)/k,$$

for some positive integer k.

According to the lemma we may assume that G is connected and satisfies

(1) *G does not contain k vertices* $x_0, \ldots, x_{k-1}$, *each of which has degree* 2 *and which form a path in G.*

We keep on establishing observations about the interior structure of G. These observations are based on certain reals α, β, γ and g. Ultimately we find out: if these numbers satisfy certain equations (viz. 7.1 and 9.1), then $\mathscr{L}(G)$ is sufficiently large (cf. 17). Since g is directly related to G (cf. 3), it remains to choose α, β and γ accordingly.

The final result is formulated under (20).

(2) Notation. A *rooted* $(\leqq 2)$*-tree* in G is a triple $\mathcal{T}=(T, r, F)$ such that
$T \subseteqq V$ is a set of vertices in G,
$F \subseteqq E$ is a set of edges in G,
$r \in T$ is a distinguished vertex, called the *root* of $\mathcal{T}$,
(T, F) is a tree, i.e. a connected graph without cycles, such that every vertex $x \in T$ has degree at most three in (T, F).

For vertices x and y in T let dist (x, y) be the number of edges in the (uniquely determined) path in (T, F) joining x and y. For nonnegative integers n let

$$\mathcal{T}(n) = \{x \in T \mid \operatorname{dist}(r, x) = n\}$$

be the n-th level of $\mathcal{T}$. The *height of* $\mathcal{T}$ is the maximal integer n such that $\mathcal{T}(n) \neq \emptyset$.

(3) Notation. Let g be the maximal positive integer such that G does not contain any cycle of length at most g, i.e. $g = \operatorname{girth}(G) - 1$. For a vertex $x \in V$ let $\deg(x)$ denote the number of adjacent vertices, i.e. the degree of x.

(4) Observation. *Let $0 < \alpha < 1$ be a real number. There exists a nonnegative integer N and there exists a rooted $(\leqq 2)$-tree $\mathcal{T}=(T, r, F)$ of height at most $N+1$ satisfying the following properties:*

(4.1) $$|\mathcal{T}(l+k)| \geqq \alpha^k \frac{2}{2-\alpha^{k-1}} |T(l)| \quad \textit{for every} \quad 0 \leqq l \leqq N-k.$$

Additionally, there exists a set $\mathrm{GOOD} \subseteqq \mathcal{T}(N)$ *and a mapping* $s\colon \mathrm{GOOD} \to T$ *such that*

(4.2) $|\mathrm{GOOD}| > (1-\alpha)\,|\mathcal{T}(N)|$,

(4.3) $\{x, s(x)\} \in E \setminus F$ *for every* $x \in \mathrm{GOOD}$, *i.e. the vertices x and $s(x)$ are joined by an edge in G not belonging to the rooted tree $\mathcal{T}$.*

Proof. Such a rooted tree $\mathcal{T}$ can be obtained by a straightforward recursive construction. Pick a vertex $r \in V$ arbitrarily and put $T_0 = \{r\}$ and $F_0 = \emptyset$. Assume that for some nonnegative integer m the rooted $(\leqq 2)$-tree $\mathcal{T}_m = (T_m, r, F_m)$ has been constructed such that (4.1) is satisfied for all $0 \leqq l \leqq m-k$.

Let

$$A_m = \{x \in \mathcal{T}_m(m) \mid \deg(x) = 2\}$$

be the set of degree 2 vertices of G belonging to the m-th level of $\mathcal{T}_m$.

Let

$$A_m^+ = \{y \in V \setminus T_m \mid \{x, y\} \in E \quad \text{for some } x \in A_m\}$$

be the set of neighbours of A_m not already belonging to $\mathcal{T}_m$. Let $B_m \subseteqq \mathcal{T}_m(m) \setminus A_m$ be a set of maximal cardinality such that there exist two mappings $b_i\colon B_m \to V \setminus (T_m \cup A_m^+)$, $i \in \{0, 1\}$, satisfying

(4.4) $\{x, b_i(x)\} \in E$ for every $x \in B_m$ and $i \in \{0, 1\}$,

(4.5) $b_i(x) \neq b_j(y)$ for every $x, y \in B_m$ and $i, j \in \{0, 1\}$ with $(x, i) \neq (y, j)$,

i.e. to every $x \in B_m$ there are associated two adjacent vertices $b_0(x)$ and $b_1(x)$ not already belonging to $T_m \cup A_m^+$ such that $\{b_0(x), b_1(x)\} \cap \{b_0(y), b_1(y)\} = \emptyset$ for any two different x and y in B_m.

Let

$$B_m^+ = \{b_0(x) | x \in B_m\} \cup \{b_1(x) | x \in B_m\}.$$

Put

$$T_{m+1} = T_m \cup A_m^+ \cup B_m^+$$

and

$$F_{m+1} = F_m \cup \{\{x, y\} \in E | x \in A_m \text{ and } y \in A_m^+\} \cup \{\{x, b_0(x)\} | x \in B_m\} \cup \{\{x, b_1(x)\} | x \in B_m\}.$$

Repeat this construction as long as possible, viz. let N be the minimal integer such that

$$|A_N| + |B_N| < \alpha |\mathcal{T}_N(N)|, \tag{4.6}$$

and put $\mathcal{T} = (T, r, F) = (T_{N+1}, r, F_{N+1})$.

Recall that $\mathcal{T}$ has been defined in such a way that

$$|A_m| + |B_m| \geqq \alpha |\mathcal{T}(m)| \quad \text{for every} \quad 0 \leqq m < N \tag{4.7}$$

and

$$|\mathcal{T}(m)| = |A_{m-1}| + 2|B_{m-1}| \quad \text{for every} \quad 0 < m \leqq N. \tag{4.8}$$

We show that $\mathcal{T}$ satisfies assertion (4.1): Let $0 \leqq l \leqq N-k$. By (4.7) then

$$\alpha^{k-1}|A_l| + \alpha^{k-1}|B_l| \geqq \alpha^k |T(l)|. \tag{4.9}$$

According to (4.7) and (4.8) it follows that

$$\alpha^{k-1-i}|A_{l+i}| + \alpha^{k-1-i}|B_{l+i}| - \alpha^{k-i}|A_{l-1+i}| - 2\alpha^{k-i}|B_{l-1+t}| \geqq 0 \tag{4.10}$$

for every $1 \leqq i < k$.

Summing up the inequalities (4.9) and (4.10) yields

$$|A_{l+k-1}| + |B_{l+k-1}| - \sum_{i=0}^{k-2} \alpha^{k-1-i}|B_{l+i}| \geqq \alpha^k |\mathcal{T}(l)|. \tag{4.11}$$

As the graph satisfies condition (1), it follows that

$$|A_{l+k-1}| \leqq \sum_{i=0}^{k-2} 2|B_{l+i}|, \tag{4.12}$$

and hence

$$-\frac{\alpha^{k-1}}{2} |A_{l+k-1}| + \sum_{i=0}^{k-2} \alpha^{k-1}|B_{l+i}| \geqq 0. \tag{4.13}$$

Summing up (4.11) and (4.13) yields

$$\frac{2-\alpha^{k-1}}{2} |A_{l+k-1}| + |B_{l+k-1}| + \sum_{i=1}^{k-2} (\alpha^{k-1} - \alpha^{k-1-i}) |B_{l+i}| \geqq \alpha^k |\mathcal{T}(l)|, \tag{4.14}$$

thus

(4.15)

$$|A_{l+k-1}| \geqq \frac{2\alpha^k}{2-\alpha^{k-1}}|\mathcal{T}(l)| + \frac{2}{\alpha^{k-1}-2}\Big(|B_{l+k-1}| + \sum_{i=1}^{k-2}(\alpha^{k-1}-\alpha^{k-1-i})|B_{l+i}|\Big)$$

$$\geqq \frac{2\alpha^k}{2-\alpha^{k-1}}|\mathcal{T}(l)| + \frac{2}{\alpha^{k-1}-2}|B_{l+k-1}|,$$

as $0<\alpha<1$.

From (4.8) and (4.15) then finally it follows that

(4.16) $$|\mathcal{T}(l+k)| =$$

$$= |A_{l+k-1}| + 2|B_{l+k-1}| \geqq \frac{2\alpha^k}{2-\alpha^{k-1}}|\mathcal{T}(l)| + \frac{2(\alpha^{k-1}-1)}{\alpha^{k-1}-2}|B_{l+k-1}| \geqq \frac{2\alpha^k}{2-\alpha^{k-1}}|\mathcal{T}(l)|,$$

as again $0<\alpha<1$. Thus assertion (4.1) is valid.

Let us mention that in fact the sets A_{l+i} and B_{l+i} can be chosen in such a way that equality holds in (4.1) occasionally.

It remains to define the set GOOD and the mapping s: GOOD $\to T$ such that (4.2) and (4.3) are satisfied. Put

$$\text{GOOD} = \mathcal{T}(N) \setminus (A_N \cup B_N).$$

According to (4.6) then (4.2) is valid. Due to the maximality of B_N for every $x \in$ GOOD there exists some vertex $s(x) \in T = T_{N+1}$ satisfying (4.3). ∎

For the remainder of this proof fix a rooted ($\leqq 2$)-tree $\mathcal{T}=(T, r, F)$ of height N or $N+1$ and a set GOOD $\subseteq \mathcal{T}(N)$ as well as a mapping s: GOOD $\to T$ such that the assertions (4.1), (4.2) and (4.3) are satisfied.

(5) Observation. *There exists a set* EQP $\subseteq$ GOOD *and there exists an integer* $n \geqq g$ *such that*

(5.1) $$\operatorname{dist}(x, s(x)) = n \quad \text{for every} \quad x \in \text{EQP},$$

(5.2) $$|\text{EQP}| \geqq |\text{GOOD}|/n^2,$$

(5.3) $$|\text{EQP}| \geqq \frac{1-\alpha}{n^2}\left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{N-k+1}.$$

Proof. Recall that $g = \text{girth}(G)-1$. Consider the mapping Δ: GOOD $\to \mathbf{N} \setminus \{0, \ldots, g-1\}$ which is defined by $\Delta(x) = \operatorname{dist}(x, s(x))$ for every $x \in$ GOOD. Then $|\text{GOOD}| = \sum_{n=g}^{\infty} |\Delta^{-1}(n)|$ and assuming that (5.2) fails for every $n \geqq g$ and EQP $= \Delta^{-1}(n)$ it follows that

$$|\text{GOOD}| = \sum_{n=g}^{\infty} |\Delta^{-1}(n)| < |\text{GOOD}| \sum_{n=2}^{\infty} \frac{1}{n^2} = |\text{GOOD}|(\pi^2/6-1) < |\text{GOOD}|,$$

an obvious contradiction.

Hence for some $n \geqq g$ the set $\Delta^{-1}(n)$ satisfies (5.1) and (5.2). Assertion (5.3) is an obvious consequence from (4.2) and (4.1) and (5.2). ∎

Fix an integer $n \geqq g$ and a set $\text{EQP} \subseteqq \text{GOOD}$ satisfying (5.1), (5.2) and (5.3). Put $n^* = \lfloor n/2 - 1 \rfloor$.

(6) Notation. For vertices $z \in T$ let

$$C(z) = \{y \in \mathcal{T}(N) | \text{dist}(z, y) = N - \text{dist}(r, y)\}$$

be the vertices in the N-th level of $\mathcal{T}$ belonging to the cone generated by z.

(7) Observation. *Let* $0 < \beta < 1$ *be a real number such that*

$$\frac{1-\alpha}{2g^2} \geqq \beta^{(g-4)/2}. \tag{7.1}$$

There exists a vertex $z \in \mathcal{T}(N-n^*)$ *such that*

$$|C(z) \cap \text{EQP}| \geqq \frac{1-\alpha}{2n^2} \beta^{n^*} \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{n^*-k+1}. \tag{7.2}$$

Proof. Call a vertex $z \in \mathcal{T}(N-n^*)$ *small* iff

$$|C(z)| < \beta^{n^*} \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{n^*-k+1}. \tag{7.3}$$

Without loss of generality we can assume that $C(z) \neq \emptyset$ for every $z \in \mathcal{T}(N-n^*)$. Hence it follows from (4.1) that

$$\begin{aligned} \Big| \bigcup_{\substack{z \in \mathcal{T}(N-n^*) \\ z \text{ small}}} C(z) \Big| &< |\mathcal{T}(N-n^*)| \beta^{n^*} \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{n^*-k+1} \\ &\leqq |\mathcal{T}(N)| \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{k-1-n^*} \beta^{n^*} \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{n^*-k+1} \\ &= |\mathcal{T}(N)| \beta^{n^*}. \end{aligned} \tag{7.4}$$

Note that particularly $g \geqq 2$ (since girth (G) is at least 3). From $n \geqq g \geqq 2$ and (7.1) it follows that

$$\frac{1-\alpha}{2n^2} \geqq \beta^{n^*}. \tag{7.5}$$

Denote by EQP* the set of vertices in EQP which are not covered by small cones. According to (7.4), (5.2), (4.2), (7.1) and since $n \geqq g$ it follows that

$$\begin{aligned} |\text{EQP}^*| &\geqq |\text{EQP}| - \Big| \bigcup_{\substack{z \in \mathcal{T}(N-n^*) \\ z \text{ small}}} C(z) \Big| \geqq |\text{EQP}| - |T(N)| \beta^{n^*} \\ &\geqq |\mathcal{T}(N)| \frac{1-\alpha}{n^2} - |\mathcal{T}(N)| \beta^{n^*} = |\mathcal{T}(N)| \left(\frac{1-\alpha}{n^2} - \beta^{n^*}\right) \\ &\geqq |\mathcal{T}(N)| \frac{1-\alpha}{2n^2}. \end{aligned} \tag{7.6}$$

Note that $C(z)\cap C(z')=\emptyset$ for different vertices z and z' in $\mathcal{T}(N-n^*)$. Hence there exist at most

$$|\mathcal{T}(N)|/\beta^{n^*}\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{n^*-k+1}$$

vertices $z\in\mathcal{T}(N-n^*)$ which are not small. Assuming that every such "large" vertex fails to satisfy (7.2) yields

$$|\mathrm{EQP}^*| < \frac{1-\alpha}{2n^2}|\mathcal{T}(N)|,$$

contradicting (7.6). Hence the observation is proved. ∎

Fix a vertex $z\in\mathcal{T}(N-n^*)$ satisfying (7.2).

(8) Notation. Let $m<N$. A vertex $x\in\mathcal{T}(m)$ is a *cycle vertex* iff there exist two vertices y_0 and y_1 in $\mathcal{T}(m+1)$ such that $\{x, y_0\}\in F$ and $\{x, y_1\}\in F$ (i.e. x possesses two immediate successors in $\mathcal{T}$, viz. y_0 and y_1) satisfying

(8.1) $$C(y_0)\cap\mathrm{EQP}\neq\emptyset \quad\text{and}\quad C(y_1)\cap\mathrm{EQP}\neq\emptyset.$$

(9) Observation. *Let $\gamma>0$ be a real number and assume that α, β, g (as introduced above) and γ satisfy*

(9.1) $$\log\left(\frac{1-\alpha}{2n^2}\right)+\left(\frac{n}{2}-1\right)\log\beta+\left(\frac{n}{2}-k-1\right)\log\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)\geqq n\gamma\log 2$$

for all $n\geqq g$. Then there exists an ascending path $z=x_0, x_1, \ldots, x_{n^*-1}$ in $\mathcal{T}$ (i.e. $\{x_i, x_{i+1}\}\in F$ and $x_i\in\mathcal{T}(N-n^*+i)$) which contains at least γn cycle vertices.

Proof. Construct $x_1, \ldots, x_{n^*-1}$ by a greedy procedure. If there exists precisely one vertex $y\in\mathcal{T}(N-n^*+m+1)$ such that $\{x_m, y\}\in F$, put $x_{m+1}=y$. If there exist two vertices y_0 and y_1 in $\mathcal{T}(N-n^*+m+1)$ such that $\{x_m, y_0\}\in F$ and $\{x_m, y_1\}\in F$, put

$$\begin{aligned} x_{m+1} &= y_0 \quad\text{iff}\quad |C(y_0)\cap\mathrm{EQP}|\geqq|C(y_1)\cap\mathrm{EQP}| \\ &= y_1 \quad\text{iff}\quad |C(y_0)\cap\mathrm{EQP}|<|C(y_1)\cap\mathrm{EQP}|. \end{aligned}$$

We show that $z=x_0, \ldots, x_{n^*-1}$ contains at least γn cycle vertices. Let $x_{m_0}, \ldots, x_{m_{t-1}}$, where $m_0<m_1<\ldots<m_{t-1}$, be the cycle vertices amongst $x_0, \ldots, x_{n^*-1}$.

According to the construction and (8.1) it follows that

(9.2) $$|C(x_{m_{t-1}})\cap\mathrm{EQP}|\leqq 2,$$

(9.3) $$|C(x_{m_{l-1}})\cap\mathrm{EQP}|\leqq 2|C(x_{m_l})\cap\mathrm{EQP}| \quad\text{for}\quad 0<l<t.$$

Hence

(9.4) $$|C(x_{m_0})\cap\mathrm{EQP}|\leqq 2^t.$$

Note that $|C(x_{m_0})\cap\mathrm{EQP}|=|C(z)\cap\mathrm{EQP}|$, hence it follows from (7.2) that

(9.5) $$2^t\geqq\frac{1-\alpha}{2n^2}\beta^{n^*}\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)^{n^*-k+1}$$

i. e.

(9.6) $$t \geqq \frac{1}{\log 2}\left(\log\left(\frac{1-\alpha}{2n^2}\right)+n^*\log\beta+(n^*-k+1)\log\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)\right)$$
$$\geqq \frac{1}{\log 2}\left(\log\left(\frac{1-\alpha}{2n^2}\right)+\left(\frac{n}{2}-1\right)\log\beta+\left(\frac{n}{2}-k-1\right)\log\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)\right)$$
$$\geqq \gamma n$$

according to (9.1) and the fact that $n \geqq g$. ∎

Fix a path $x_0, x_1, \ldots, x_{n^*-1}$ containing the cycle vertices $x_{m_0}, \ldots, x_{m_{t-1}}$, where $m_0 < m_1 < \ldots < m_{t-1}$ and $t \geqq \gamma n$. Fix a vertex $x_{m_t} \in C(x_{m_{t-1}}) \cap \mathrm{EQP}$. For every vertex x_{m_i} let $y_{m_i} \in \mathrm{EQP}$ be such that

(10) $y_{m_i} \in C(x_{m_i}) \cap \mathrm{EQP} \setminus C(x_{m_i+1})$ *for every* $0 \leqq i < t$.

Put $y_{m_t} = x_{m_t}$. Such vertices y_{m_i} exist as the vertices x_{m_i} are cycle vertices.

(11) Notation. For vertices x and y in T denote by $P(x, y)$ the (uniquely determined) path in $\mathscr{T}$ joining x and y. In order to avoid ambiguties say that $P(x, y)$ consists of edges (in F), determining the path between x and y. Note that $|P(x, y)| = \mathrm{dist}(x, y)$.

(12) Observation. *Let $z^* \in T$ denote the vertex satisfying $\{z^*, z\} \in P(r, z)$, i.e. z^* is the (uniquely determined) predecessor of z in T. Then $\{z^*, z\} \in P(y_{m_i}, s(y_{m_i}))$ for every $0 \leqq i \leqq t$.*

Proof. Denote by $\bar{C}(z) = \bigcup_{0 \leqq i \leqq n^*+1} \{y \in \mathscr{T}(N-n^*+i) \mid \mathrm{dist}(y, z) = i\}$ the (upper) cone generated by z. Obviously

(12.1) $$\mathrm{MAX}\{\mathrm{dist}(y, z) \mid y \in \bar{C}(z)\} \leqq n^*+1,$$

hence

(12.2) $$\mathrm{MAX}\{\mathrm{dist}(x, y) \mid y \in C(z),\, x \in \bar{C}(z)\} \leqq n^*+n^*+1 = 2\left\lfloor\frac{n}{2}-1\right\rfloor+1 \leqq n-1.$$

From (5.1) it follows that $s(y_{m_i}) \notin \bar{C}(z)$ for any $0 \leqq i \leqq t$. Hence the assertion follows. ∎

(13) Observation. *Let $0 \leqq i < j \leqq t$. Then*

(13.1) $$P(y_{m_i}, y_{m_j}) \cap P(s(y_{m_i}), s(y_{m_j})) = \emptyset.$$

(13.2) $$C_{i,j} = P(y_{m_i}, y_{m_j}) \cup P(s(y_{m_i}), s(y_{m_j})) \cup \{\{y_{m_i}, s(y_{m_i})\}, \{y_{m_j}, s(y_{m_j})\}\}$$

forms a cycle of length at most 2n.

Proof. As the only way of entering or leaving the cone $\bar{C}(z)$ is to use the edge $\{z, z^*\}$, assertion (13.1) follows from (12). Consequently C_{ij} is a cycle. As $y_{m_i} \in C(x_{m_i}) \setminus C(x_{m_i+1})$ and $\{x_{m_i}, x_{m_i+1}\} \in F$ it follows from $i < j$ that

(13.3) $$\mathrm{dist}(y_{m_i}, y_{m_j}) = 2(n^*-m_i).$$

From (12) and (5.1) it follows that

(13.4) $$\operatorname{dist}(z^*, s(y_{m_i})) = \operatorname{dist}(y_{m_i}, s(y_{m_i}))\text{-}\operatorname{dist}(y_{m_i}, z^*) = n-n^*-1,$$

hence by the triangle inequality

(13.5) $$\operatorname{dist}(s(y_{m_i}), s(y_{m_j})) \leqq 2(n-n^*-1),$$

this yields

(13.6) $$\left|P(y_{m_i}, y_{m_j}) \cup P(s(y_{m_i}), s(y_{m_j})) \cup \{\{y_{m_i}, s(y_{m_i})\}, \{y_{m_j}, s(y_{m_j})\}\}\right| \leqq 2(n^*-m_i)+2(n-n^*-1)+2 = 2n-2m_i \leqq 2n$$

showing that C_{ij} is a cycle of length at most $2n$. ∎

(14) Fact. *Let* $\mathcal{T}^*=(T^*, r^*, F^*)$ *be a rooted tree with distance function* dist*. *For every nonnegative integer* m *and every vertex* $\hat{x}\in\mathcal{T}^*(m)$ *it follows that*

(14.1) $$\operatorname{MAX}\{\operatorname{dist}^*(\hat{x}, y) \mid y\in\mathcal{T}^*(m)\} = \operatorname{MAX}\{\operatorname{dist}^*(y, y') \mid y, y'\in\mathcal{T}^*(m)\}.$$

Proof. Obvious by induction on m. ∎

(15) Observation. *For nonnegative integers* $0\leqq i<t$ *let* $i<\xi(i)\leqq t$ *be such that*

(15.1) $$\operatorname{dist}(s(y_{m_i}), s(y_{m_{\xi(i)}})) = \operatorname{MAX}\{\operatorname{dist}(s(y_{m_i}), s(y_{m_j})) \mid i<j\leqq t\}.$$

Then

(15.2) $$\operatorname{dist}(s(y_{m_{i+1}}), s(y_{m_{\xi(i+1)}})) \leqq \operatorname{dist}(s(y_{m_i}), s(y_{m_{\xi(i)}})).$$

Proof. Consider

$$T^* = \{y\in T \mid \operatorname{dist}(z^*, y) < n-n^*-1\}\setminus\bar{C}(z)$$

and for every $0\leqq i<t$ let

$$T_i^* = T^*\cup\{s(y_{m_j}) \mid i\leqq j\leqq t\}.$$

Denote by $\mathcal{T}_i^*$ the rooted subtree of $\mathcal{T}$ consisting of vertices T_i^* with root z^*. According to (12) and (13.4) then

(15.3) $$\mathcal{T}_i^*(n-n^*-1) = \{s(y_{m_j}) \mid i\leqq j\leqq t\}.$$

As $T_i^*\supseteqq T_{i+1}^*$ it follows that

(15.4) $$\operatorname{MAX}\{\operatorname{dist}_{\mathcal{T}_{i+1}^*}(y, y') \mid y, y'\in\mathcal{T}_{i+1}(n-n^*-1)\} \leqq \operatorname{MAX}\{\operatorname{dist}_{\mathcal{T}_i^*}(y, y') \mid y, y'\in\mathcal{T}_i(n-n^*-1)\}.$$

According to (14.1) then

(15.5) $$\operatorname{dist}(s(y_{m_{i+1}}), s(y_{m_{\xi(i+1)}})) = \operatorname{dist}_{\mathcal{T}_{i+1}^*}(s(y_{m_{i+1}}), s(y_{m_{\xi(i+1)}})) \leqq \operatorname{dist}_{\mathcal{T}_i^*}(s(y_{m_i}), s(y_{m_{\xi(i)}})) = \operatorname{dist}(s(y_{m_i}), s(y_{m_{\xi(i)}})).$$ ∎

(16) Observation. $|C_{i+1,\xi(i+1)}|<|C_{i,\xi(i)}|$ *for every* $0\leqq i<t$.

Proof. According to (13.3) and since $m_i<m_{i+1}$ this follows from (15.2). ∎

(17) Observation. *Let the real numbers* $0<\alpha<1$, $0<\beta<1$ *and* $\gamma>0$ *satisfy* (7.1) *and* (9.1). *Then*

$$\mathcal{L}(G) \geqq \gamma/2. \quad \blacksquare$$

(18) Observation. *Put* $\alpha=2^{-1/2k}$; $\beta=\left(\frac{2}{2\sqrt{2}-\sqrt[2k]{2}}\right)^{-1/2k}$; $\gamma=0.0011/k$. *Then* (9.1) *is satisfied, provided that* $g>300k\log k$, *and* $k\geqq n_0$, *where* n_0 *is sufficiently large.*

Proof. We write (9.1) for $n=g$ equivalently as

$$\text{(18.1)} \qquad \log\left(\frac{1-\alpha}{2g^2}\right)+(g/2-1)\log\left(\alpha\beta\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)-k\log\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)\geqq g\gamma\log 2$$

and evaluate each of the three summands on the left hand side of (18.1) separately.

$$\text{(18.2)} \qquad \log\frac{1-\alpha}{2g^2}=\log(1-2^{-1/2k})-\log 2-2\log g\geqq\log\left(\frac{\log 2}{2k}\right)-\log 2-2\log g$$
$$\geqq-(2\log 2+\log k+2\log g).$$

$$\text{(18.3)} \qquad \alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}=\sqrt[k]{\frac{2}{2\sqrt{2}-\sqrt[2k]{2}}}.$$

$$\text{(18.4)} \qquad \alpha\beta\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}=\sqrt[2k]{\frac{2}{2\sqrt{2}-\sqrt[2k]{2}}},$$

$$\text{(18.5)} \qquad (g/2-1)\log\left(\alpha\beta\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)=(g/2-1)\log\left(\sqrt[2k]{\frac{2}{2\sqrt{2}-\sqrt[2k]{2}}}\right)$$
$$=\frac{1}{2k}(g/2-1)\log\left(\frac{2}{2\sqrt{2}-\sqrt[2k]{2}}\right)$$
$$\geqq\frac{g}{4k}0.088-\frac{1}{2k}0.088$$
$$=g0.022/k-\frac{0.044}{k},$$

$$\text{(18.6)} \qquad -k\log\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)\geqq\log(2\sqrt{2}-1)-\log 2\geqq-\log 2.$$

Now, putting (18.2), (18.5), and (18.6) together shows that

$$\log\left(\frac{1-\alpha}{2g^2}\right)+(g/2-1)\log\left(\alpha\beta\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)-k\log\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)$$
$$\geqq g0.022/k-\left(\frac{0.044}{k}+3\log 2+\log k\right)-2\log g$$
$$=g0.011/k+g\cdot 0.011/k-\left(\frac{0.044}{k}+3\log 2+\log k\right)-2\log g$$

and thus it suffices to show that

$$g\cdot 0.011/k-\left(\frac{0.044}{k}+3\log 2+\log k\right)-2\log g \geqq 0. \tag{18.7}$$

This follows immediately according to the choices of g and k. To show (9.1) for arbitrary $n\geqq g$ it is enough to see

$$\frac{1}{2}\log\beta+\frac{1}{2}\log\left(\alpha\sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right)-\gamma\log 2-2\log\left(\frac{g+1}{g}\right)\geqq 0 \tag{18.8}$$

but this also holds by the choices of α, β, γ, g and k. ∎

(19) Observation. *Put α and β as before, then* (7.1) *is satisfied, provided* $g\geqq 300k\log k$ *and* $k\geqq n_0$ *for some sufficiently large* n_0.

Proof.

$$\log(\beta^{g-4/2}) = \frac{g-4}{2}\frac{(-1)}{2k}\log\frac{2}{2\sqrt{2}-\sqrt[2k]{2}} \leqq \frac{g-4}{2}\frac{(-1)}{2k}\log 2 \tag{19.1}$$

thus, according to (18.2) and (19.1), it suffices to show that

$$\frac{g-4}{4k}\log 2 \geqq (2\log 2+\log k+2\log g),$$

this follows immediately, as $g\geqq 300k\log k$ and $k\geqq n_0$. ∎

(20) Observation. *There exists a positive integer n_0 such that for every graph $G=(V,E)$ with* $\frac{|E|}{|V|}\geqq\frac{k+1}{k}$, *where* $k\geqq n_0$, *it follows that* $\mathscr{L}(G)\geqq\min\left\{\frac{1}{300k\log k}, 0.00055/k\right\}$.

Proof. Recall (17), (18) and (19). ∎

3. Concluding Remarks

We have the following upper bound for $f(1+\varepsilon)$:

Theorem 2. $f\left(\frac{k+1}{k}\right)\leqq\frac{1}{k+1}\frac{77}{72}$ *for all positive integers* $k\equiv 2 \bmod 15$.

Proof. For the positive integers l and n denote by $K^l_{n,n}$ the graph resulting from the complete bipartite graph $K_{n,n}$ by inserting l additional vertices on each edge of $K_{n,n}$. Clearly

$$|V(K^l_{n,n})| = 2n+n^2 l,\ |E(K^l_{n,n})| = n^2(l+1),$$

$$\frac{|E(K^l_{n,n})|}{|V(K^l_{n,n})|} = \frac{l+1}{l+\frac{2}{n}},\quad \mathscr{L}(K^l_{n,n}) = \frac{1}{l+1}\left(\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\ldots+\frac{1}{2n}\right).$$

Putting $l=\frac{k(n-2)-2}{n}$ shows that $f\left(\frac{k+1}{k}\right)\leqq\frac{1}{k+1}\frac{n}{n-2}\left(\frac{1}{4}+\frac{1}{6}+\ldots+\frac{1}{2n}\right)$ and this expression is minimized for $n=5$, hence $f\left(\frac{k+1}{k}\right)\leqq\frac{1}{k+1}\frac{77}{72}$, provided $k\equiv 2$ mod 15. ∎

A careful inspection of the proof of our main theorem suggests the following definition:

Definition. For positive reals α and g let

$$f^*(g,\alpha)=\inf\left\{\mathscr{L}(G)\,\middle|\,\frac{|E(G)|}{|V(G)|}\geqq\alpha \quad\text{and}\quad \operatorname{girth}(G)\geqq g\right\}.$$

Corollary. *There exists a positive integer n_0 such that*

$$f^*\left(300k\log k,\ \frac{k+1}{k}\right)\geqq\frac{0.00055}{k}$$

for all positive integers $k\geqq n_0$.

Problem. Fix $\alpha>1$, is $f^*(g,\alpha)$ bounded as g tends to infinity?

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A. Gyárfás

Computer and Automation Institute of the Hungarian Academy of Sciences Kende u. 13—17, 1111-Hungary

H. J. Prömel and B. Voigt

Fakultät für Mathematik Universität Bielefeld Bielefeld 1 4800, West Germany

E. Szemerédi

Mathematical Institute of the Hungarian Academy of Sciences Budapest, P.O.B. 127 1364, Hungary