On the Distribution of Cycle Lengths in Graphs

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ABSTRACT

The set of different cycle lengths of a graph $G$ is denoted by $C(G)$. We study how the distribution of $C(G)$ depends on the minimum degree of $G$. We prove two results indicating that $C(G)$ is dense in some sense. These results lead to the solution of a conjecture of Erdős and Hajnal stating that for suitable positive constants $a, b$ the following holds:

$$ \sum_{i \in C(G)} i^{-1} \geq a \log |\delta(G)|, \quad \delta(G) \geq b, $$

where $\delta(G)$ denotes the minimum degree of $G$.

1. INTRODUCTION

The set of different cycle lengths of a graph $G$ is denoted by $C(G)$. There exist many theorems which assert that $C(G)$ is “dense” provided that the number of edges in $G$ is substantially larger than the number of vertices. Here, dense may mean various things, for example, $C(G) = \{3, 4, \ldots, |V(G)|\}$ ($G$ is pancyclic); $C(G)$ contains a large sequence of consecutive integers; $C(G)$ contains a large sequence of consecutive even integers. A survey of such results can be found in [1].

There are important families of graphs not satisfying the above assumption but still suspected of having a set of cycles whose lengths are
dense in some sense. Graphs of chromatic number \( k \), graphs of minimum degree \( \delta \), and graphs of density \( \alpha \) are examples of such families. All these families contain graphs of arbitrarily large girth; therefore small integers do not necessarily appear as cycle lengths. A measure of density for these families was proposed in a paper of Erdős [2], who asked about the behavior of \( L(G) \), the sum of reciprocals of elements of \( C(G) \), as a function of the edge density of \( G \). He introduced the function

\[
f(\alpha) = \inf \{ L(G) : |E(G)| \geq \alpha |V(G)| \}
\]

where \( V(G) \) and \( E(G) \) denote the sets of vertices and edges of \( G \). It is obvious that \( f(\alpha) = 0 \) for \( \alpha \leq 1 \). The complete bipartite graphs with equal color classes show that \( f(\alpha) \leq c \log(\alpha) \). It was conjectured by Erdős and Hajnal that \( f(\alpha) \geq c' \log(\alpha) \) but it was not even known whether \( f(\alpha) \) tends to infinity with \( \alpha \) ([1], p. 168). In this paper we prove that \( f(\alpha) \geq a \log(\alpha) \) if \( \alpha \geq b \), for suitable constants \( a \) and \( b \) (Theorem 4'). The behavior of \( f(\alpha) \) for \( \alpha \in (1, 1 + \varepsilon) \) seems to be an interesting problem.*

Throughout the paper we shall study the distribution of \( C(G) \) for graphs of fixed minimum degree \( \delta(G) \). Since a graph of chromatic number \( k \) contains a subgraph of minimum degree at least \( k - 1 \) and a graph of density \( \alpha \) contains a subgraph of minimum degree at least \( \lfloor \alpha \rfloor + 1 \), our results can be applied both to graphs of fixed chromatic number and to graphs of fixed density.

We have two results on the distribution of \( C(G) \), both of which express certain density properties of \( C(G) \).

The first density result on \( C(G) \) is Theorem 1' in Section 3: if \( G \) is a graph such that \( \delta(G) \geq c_1 \) then there exists an integer \( n \) (depending on \( G \)) such that \( n \) tends to infinity with \( \delta(G) \) and \( |C(G) \cap [3, 4n]| \geq \frac{1}{2} n \); moreover, \( n \) can be chosen to satisfy \( n \geq \log[\varepsilon \delta(G)] \). Theorem 1' immediately implies that \( C(G) \) has positive upper density if \( G \) is an infinite graph having finite subgraphs \( H_i \) such that \( \delta(H_i) \) tends to infinity with \( i \) (Corollary 2). As a special case, \( C(G) \) has positive upper density if \( G \) is a graph of infinite chromatic number. This was also conjectured by Erdős and Hajnal ([3], p. 37).

The second density result on \( C(G) \) is Theorem 2' in Section 4: if \( G \) is a graph satisfying \( \delta(G) \geq c_2 \) and \( \delta(G) \geq 4\varepsilon^{-1} \log^2[|V(G)|] \) then \( C(G) \) contains almost all even integers in the interval \( [2|\varepsilon \delta(G)|^{1/2} + 2, 3^{-1}|\varepsilon \delta(G)| - 1] \). Here, "almost all" means "with the possible exception of the multiples of \( 2t \) for some integer \( t \geq 2 \)."

The above density results on \( C(G) \) will be proved in slightly stronger forms (Theorem 1 and Theorem 2), in which the function \( \varepsilon \delta(G) \) is replaced by the conceptually more complicated function

* A result on this problem will appear in [4].
\[ p(G, \varepsilon) = \min\{p(H) : H \subset G, \delta(H) \geq \varepsilon \delta(G)\} \]

where \( p(H) \) denotes the number of vertices in a longest path of \( H \). The reason for doing this is purely technical, as will be apparent in Section 5 where corollaries of Theorem 1 and Theorem 2 are applied to derive lower bounds on \( L(G) \). The main result of Section 5 is Theorem 4: for suitable constants \( a \) and \( b \), \( L(G) \geq a \log[\delta(G)] \) if \( \delta(G) \geq b \).

In Section 2 we introduce the notion of 3-trees, which play a fundamental role in the proofs of Theorem 1 and Theorem 2.

The variables \( a, b, c, \varepsilon \) will denote positive constants; \( \varepsilon \) will be used for constants less than 1.

### 2. THE 3-TREES AND THEIR CROWNS

We use the well-studied concept of rooted trees. Let \( T \) be a rooted tree. The vertices whose distance in \( T \) from the root is \( i \) are at level \( i \) for \( i = 0, 1, \ldots \). We can visualize a rooted tree with its root at the bottom and the different levels above each other in increasing order. The height of a tree \( T \) is its maximal level number; it is denoted by \( h(T) \). The set of vertices of \( T \) at level \( i \) is denoted by \( L_i \); if \( i = h(T) \) we use the more expressive notation \( \text{top}(T) \). For \( x, y \in V(T) \) we write \( x \sim y \) if the path in \( T \) between \( y \) and the root of \( T \) contains \( x \); \( \sim \) is a partial ordering on \( V(T) \). For \( x \in V(T) \) the cone of \( x \), denoted by \( C(x) \), is the subtree of \( T \) containing the vertices \( y \in V(T) \) such that \( y \succ x \). The root of \( C(x) \) is \( x \). The cone distance between \( x, y \in V(T) \) is defined as the minimal height of a cone containing both \( x \) and \( y \). The cone distances between vertices of \( T \) are 0, 1, ..., \( h(T) \).

A 3-tree is a tree \( T \) such that for every \( i \) between 0 and \( h(T) - 1 \), at least \( \frac{3}{2}|L_i| \) vertices of \( L_i \) have at least two neighbors in \( L_{i+1} \). Note that a 3-tree has at least three vertices, by definition.

In a 3-tree \( T \) obviously

\[ |L_{i+1}| \geq \frac{3}{2}|L_i|, \quad 0 \leq i < h(T), \quad (1) \]

justifying its name. Equally straightforward from (1) is

\[ |L_h| \geq \frac{3^h}{2} \quad (2) \]

where \( h \) is the height of \( T \). A slight variation of (1) is the property

\[ |L_i| \leq |\text{top}(T)| \frac{3^h}{2} - i, \quad 1 \leq i \leq h, \quad (3) \]

and the summation of (3) gives

\[ |V(T)| \leq 3|\text{top}(T)|. \quad (4) \]
A $\frac{3}{2}$-tree $T \subseteq G$ is called maximal in $G$ if $T$ is a $\frac{3}{2}$-tree and $T$ cannot be extended to a $\frac{3}{2}$-tree of height $h(T) + 1$ by the addition of edges of $G$ connecting vertices of $\text{top}(T)$ and $V(G) - V(T)$.

We shall use two types of maximal $\frac{3}{2}$-trees. In Section 3 we use binary maximal $\frac{3}{2}$-trees. (A tree $T$ is binary if for every $i$, $0 \leq i < h(T)$, and for every $x \in L_i$, $x$ is connected in $T$ to at most two vertices of $L_{i+1}$.) In Section 4 saturated maximal $\frac{3}{2}$-trees are used. A tree $T \subseteq G$ is saturated in $G$ if for every $0 \leq i \leq h(T)$, $L_i$ contains all vertices of $G$ whose distance in $G$ from the root of $T$ is $i$. It is very easy to see that a graph $G$ contains a saturated maximal $\frac{3}{2}$-tree with root $x$ for every $x \in V(G)$ if $\delta(G) \geq 2$.

If $G$ is a graph and $x \in V(G)$ then the degree of $x$ in $G$ is denoted by $d_G(x)$. The following technical lemma will be used many times in the paper.

Lemma 1. Let $k$ be a positive integer, let $c$ be a real number. Assume that $G$ is a graph and $M \subseteq V(G)$ such that $|V(G)| \leq c|M|$ and $d_G(x) \geq k$ for all $x \in M$. Then there exists a subgraph $H \subseteq G$ such that $\delta(H) \geq k/4c$ and $|V(H) \cap M| \geq \frac{1}{4}|M|$.

Proof. It is sufficient to prove the lemma for a graph $G$ which is minimal in the following sense: the removal of any edge of $G$ destroys the property $d_G(x) \geq k$ for at least one $x \in M$.

We define $H_1$, $H_2$, ..., $H_m$ as follows. Let $H_1$ be defined as $G$. If $H_i$ is already defined and $d_{H_i}(v) < k/4c$ for some $v \in V(H_i)$ then $H_{i+1} = H_i - v$, otherwise $m = i$. Since $G$ has at least $\frac{1}{4}k|M|$ edges of which fewer than $|V(G)|(k/4c) \leq c|M|(k/4c) = \frac{1}{4}k|M|$ are missing from $E(H_m)$, we get $|E(H_m)| > \frac{1}{4}k|M|$. The minimality of $G$ implies that every edge of $H_m$ is incident to a vertex of $M$ of degree $k$ in $G$. Therefore $|V(H_m) \cap M| \geq |E(H_m)| > \frac{1}{4}k|M|$, and the graph $H = H_m$ satisfies the requirements. 

We need the next lemma for the definition of the crown.

Lemma 2. Let $G$ be a graph satisfying $\delta(G) \geq 3$ and let $T$ be a maximal $\frac{3}{2}$-tree in $G$. Then there exists a subgraph $F = F(G, T)$ of $G$ with the following properties:

(a) $E(F) \cap E(T) = \emptyset$,
(b) $\text{top}(T)$ meets every edge of $F$,
(c) $\delta(F) \geq \frac{1}{2}[\delta(G) - 2]$,
(d) $|V(F) \cap \text{top}(T)| \geq \frac{1}{10}|\text{top}(T)|$,
(e) $|V(F) \cap \text{top}(T)| \geq \frac{1}{2}|V(F)|$.

Proof. Let $A \subseteq \text{top}(T)$, $Y \subseteq V(G) - V(T)$, $|Y| = 2|A|$. A tree $T'$ is an $(A, Y)$-extension of $T$ if $V(T') = V(T) \cup Y$ and $E(T') = E(T) \cup E'$ where $E'$ contains edges of $G$ between $A$ and $Y$ and exactly two edges
of $E'$ are incident to $x$ for all $x \in A$. Let us choose $T'$ as an $(A, Y)$-extension of $T$ for which $|A|$ is maximal. (Note that $A = \emptyset$ is possible, in which case $Y = \emptyset$ and $T' = T$.) Let $B = \text{top}(T) - A$ and let $F'$ be the subgraph of $G$ spanned by the edge-set $\{(x, y) : (x, y) \in E(G) - E(T), x \in B, y \in V(T')\}$. The maximality of $A$ implies that every $x \in B$ is connected to at most one vertex of $V(G) - V(T')$ in $G$. Moreover, every $x \in B$ is incident to one edge of $T$. Therefore

$$\delta_{F'}(x) \geq \delta(G) - 2 \quad \text{for all } x \in B. \quad (5)$$

Since $T$ is a maximal $\frac{3}{2}$-tree, $|V(T)| \leq 3|\text{top}(T)|$ by property (4) of $\frac{3}{2}$-trees. On the other hand, $|Y| < \frac{3}{2}|\text{top}(T)|$ and $|\text{top}(T)| < 4|B|$ by the maximality of $T$. Thus

$$|V(F')| \leq |Y| + |V(T)| < \frac{3}{2}|\text{top}(T)| + 3|\text{top}(T)| < 18|B|. \quad (6)$$

Now (5) and (6) ensure that we can apply Lemma 1 with $F'$ in the role of $G$, $B$ in the role of $M$, $\delta(G) - 2$ in the role of $k (\delta(G) - 2 \geq 1)$, and 18 in the role of $c$. The subgraph $F \subset F'$ guaranteed by Lemma 1 satisfies properties (a)-(e).

If $T$ is a maximal $\frac{3}{2}$-tree of a graph $G$ then any graph $F = F(G, T)$ defined by Lemma 2 is called a crown of $T$. Let $F$ be a crown of a maximal $\frac{3}{2}$-tree $T$ of $G$, let $X = V(F) \cap \text{top}(T)$, $Y = V(F) - V(T)$. If $x \in X$ and $t \in V(T)$ then an $(x, t)$-join is either the edge $(x, t) \in E(F)$ or a path $(x, y, t)$ such that $y \in Y$ and $(x, y), (y, t) \in E(F)$. In the latter case the $(x, t)$-join is called an out-of-tree join with center $y$. If $(x, t)$ is an out-of-tree join then $t \in \text{top}(T)$ by the definition of $F$. Let $x \in X$ and $t \in V(T)$. We say that $t$ is an $n$-neighbor of $x$ if there exists an $(x, t)$-join and the cone distance between $x$ and $t$ is $n$. The concepts of join and $n$-neighbor are illustrated in Figure 1, where the edges of $F$ are heavy lines and the edges of $T$ are dotted lines. There are $(x, t_i)$-joins in the figure for $1 \leq i \leq 8$. For $i = 5$ and $i = 8$ these are out-of-tree joins. The sets $\{t_1\}, \{t_2, t_3, t_4\}, \{t_5, t_6\}, \{t_7, t_8\}$ are the 1-, 2-, 3-, 4-neighbors of $x$, respectively.

3. INTERVALS CONTAINING MANY CYCLE LENGTHS

**Theorem 1.** For suitable constants $c_1, e_1$ the following holds. If $G$ is a graph such that $\delta(G) \geq c_1$, then there exists a positive integer $n \geq \log[\frac{1}{2} p(G, e_1)]$ such that $|C(G) \cap [3, 4n]| \geq \frac{1}{8} n$.

Theorem 1 immediately implies that the sum of reciprocals in $C(G) \cap [\frac{1}{16} n, 4n]$ is at least $\frac{1}{16} n(1/4n) = \frac{1}{64}$. Thus we get the following corollary.
Corollary 1. If $\delta(G) \geq c_1$ then the sum of reciprocals in $C(G) \cap \left\{ (16)^{-1} \log[p(G, e_1)], |V(G)| \right\}$ is at least $\frac{1}{4}$.

Since $p(G, e_1) \geq e_1 \delta(G)$, we can formulate Theorem 1 in a slightly weaker but simpler form.

**Theorem 1'**. For every graph $G$ satisfying $\delta(G) \geq c_1$, there exists a positive integer $n \geq \log[\frac{1}{2}e_1 \delta(G)]$ such that $|C(G) \cap [3, 4n]| \geq \frac{1}{8}n$.

Observing that $n$ tends to infinity with $\delta(G)$ in Theorem 1', we get a result for infinite graphs.

Corollary 2. Let $G$ be an infinite graph containing finite subgraphs $H_1, H_2, ..., H_i, ..., $ such that $\delta(H_i)$ tends to infinity with $i$. Then the upper density of $C(G)$ is at least $\frac{1}{12}$.

A graph $G$ of infinite chromatic number obviously satisfies the condition of Corollary 2. Therefore $C(G)$ has positive upper density in this case, as conjectured by Erdős an Hajnal ([3], p. 37).

**Proof of Theorem 1**. Let $G$ be a graph satisfying $\delta(G) \geq 2$, let $T$ be a binary maximal $\frac{3}{2}$-tree of $G$ of height $h$, let $F$ be a crown of $T$, and let
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\[ X = \text{top}(T) \cap V(F). \] The set \( \{ x \in X : x \text{ has an } i\text{-neighbor} \} \) is denoted by \( X_i \) (1 ≤ i ≤ h).

**Claim 1.** For suitable constants \( d_1, \varepsilon_1 \) the following holds. If \( \delta(G) \geq d_1 \) and \( m = \lceil \log[4^4 p(G, \varepsilon_1)] \rceil \) then \( \bigcup_{i=m}^{h} X_i \geq \frac{1}{3} |X| \).

**Proof.** We choose \( \varepsilon_1 \) and \( d_1 \) as follows: \( \varepsilon_1 = (2 \times 72 \times 4 \times 144)^{-1} \), \( d_1 > 2/\varepsilon_1 \). Let \( A = X - \bigcup_{i=m}^{h} X_i \) and assume indirectly that \( |A| > \frac{1}{3} |X| \).

Now \( |V(F)| \leq 72|X| < 144|A| \) by property (e) of crowns and by the indirect assumption. On the other hand \( d_F(x) \geq \frac{1}{2}|\delta(G) - 2| \) for all \( x \in A \) by property (c) of crowns. We apply Lemma 1 with \( F \) in the role of \( G \), \( A \) in the role of \( M \), 144 in the role of \( c \), \( \log[4^4 \delta(G) - 2] \) in the role of \( k \). Lemma 1 gives a subgraph of \( F_1 \subset F \) such that

\[
\delta(F_1) \geq \left\lfloor \frac{1}{2^7} \delta(G) - 2 \right\rfloor / (4 \times 144) \\
> \delta(G) / (2 \times 72 \times 4 \times 144) > \varepsilon_1 \delta(G).
\]

Let \( S \) be a longest path of \( F_1 \). Since \( \delta(F_1) > \varepsilon_1 \delta(G) \), \( |S| \geq p(G, \varepsilon_1) \). Therefore, because every edge of \( F_1 \) is incident to a vertex of \( A \), \( |S \cap A| \geq \frac{1}{2}(|S| - 1) \geq \frac{1}{2}[p(G, \varepsilon_1) - 1] \). On the other hand, the choice of \( A \) implies that \( S \cap A \subset \text{top}(C) \) for some cone \( C \) of \( T \) of height \( m \). Since \( T \) is binary, \( |S \cap A| < 2^m \). Comparing the lower and upper bounds of \( |S \cap A| \) and noting that \( p(G, \varepsilon_1) \geq \varepsilon_1 \delta(G) \geq \varepsilon_1 d_1 > 2 \), we get \( m \geq \log[\frac{1}{2} [p(G, \varepsilon_1) - 1]] > \log[4^4 p(G, \varepsilon_1)] \geq m \), a contradiction.

**Claim 2.** Let \( \delta(G) \geq d_2 = 8d_1 \). There exists an integer \( n \geq m = \lfloor \log[4^4 p(G, \varepsilon_1)] \rfloor \) and a \( Z_n \subset X \) such that \( |Z_n| \geq |X| / [2n^2(n + 1)] \) and every vertex of \( Z_n \) has an \( n \)-neighbor at the same level of \( T \).

**Proof.** Let \( X' = \bigcup_{i=m}^{h} X_i \). We prove that for some \( n \), \( m \leq n \leq h \), \( |X_n| \geq |X'/n^2 \) holds. The reasoning is indirect. If \( |X_i| < |X'|/i^2 \) for all \( i, m \leq i \leq h \), then

\[
|X'| \leq \sum_{i=m}^{h} |X_i| < \sum_{i=m}^{h} \frac{|X'|}{i^2} = |X'| \sum_{i=m}^{h} \frac{1}{i^2} < |X'| \sum_{i=m}^{\infty} \frac{1}{i^2},
\]

from which we get \( 1 \leq \sum_{i=m}^{h} \frac{1}{i^2} = \frac{1}{6} \pi^2 - \sum_{i=1}^{m-1} \frac{1}{i^2} \). However, \( m = \lfloor \log[4^4 p(G, \varepsilon_1)] \rfloor \geq \lfloor \log[4^4 \varepsilon_1 \delta(G)] \rfloor \geq \lfloor \log[4^4 \varepsilon_1 \delta(G)] \rfloor > 2 \) since \( d_1 > 2/\varepsilon_1 \). Thus we get the contradiction \( 1 \leq \frac{1}{6} \pi^2 - 1 \).

The \( n \)-neighbors of the vertices of \( X_n \) are distributed at \( n + 1 \) different levels of \( T \) (at levels \( h, h - 1, ..., h - n \)). Therefore the pigeonhole principle gives \( |X_n| / (n + 1) \geq |X'|/n^2(n + 1) \) vertices of \( X \) with \( n \)-neighbors at the same level of \( T \). Since \( |X'| \geq \frac{1}{3} |X| \) by Claim 1, the proof of Claim 2 is complete.
From now on we use $m, n, Z_n$ as defined in Claim 2, for every graph $G$ satisfying $\delta(G) \geq d_2$.

**Claim 3.** For a suitable constant $d_3 \geq d_2$, the following holds. If $G$ is a graph satisfying $\delta(G) \geq d_3$ then there exists a cone $S$ of $T$ with height $n - 1$ satisfying the properties

(a) $|\text{top}(S)| \geq \frac{4}{3} n^{-1}$,
(b) $|\text{top}(S) \cap Z_n| > |\text{top}(S)|/33n^2(n + 1)$.

**Proof.** A cone $S$ of height $n - 1$ in $T$ is called large if it satisfies (a), otherwise it is small. The number of small cones is at most $|L_{h-n+1}|$ (the number of vertices at level $h - n + 1$ of $T$) and they contain fewer than $|L_{h-n+1}| \frac{4}{3} n^{-1}$ vertices of $\text{top}(T)$; therefore the subset $Z \subset Z_n$ containing vertices above the large cones is large:

$$|Z| > |Z_n| - |L_{h-n+1}| \frac{4}{3} n^{-1} \geq |X|/2n^2(n + 1) - |L_{h-n+1}| \frac{4}{3} n^{-1}.$$

Observing that $|X| \geq \frac{1}{16}|\text{top}(T)|$ by property (d) of crowns and $|L_{h-n+1}| \leq |\text{top}(T)| \frac{2}{3} n^{-1}$ by property (3) of $\frac{3}{2}$-trees, we get the following lower bound for $|Z|$:

$$|Z| > |\text{top}(T)|\left[\frac{32n^2(n + 1)}{4} - \frac{4}{3} n^{-1}\right] = |\text{top}(T)|g(n). \quad (7)$$

We choose an $n_0$ such that $g(n) \geq |33n^2(n + 1)|^{-1}$ for $n \geq n_0$. If we choose $d_3 \geq 4\varepsilon_1^{-1}2^{n_0}$ then $n \geq m = \log[1/p(G, \varepsilon_1)] \geq \varepsilon_1^{-1}\log\delta(G)] \geq n_0$ and (7) implies

$$|Z| > |\text{top}(T)|\left[33n^2(n + 1)\right]^{-1}. \quad (8)$$

If $B$ denotes the union of the tops of large cones then $|B|/|Z| \leq |\text{top}(T)|/|Z|$ $< 33n^2(n + 1)$ because of (8). Thus $|\text{top}(S)|/|\text{top}(S) \cap Z| < 33n^2(n + 1)$ for some large cone $S$, proving the claim with $d_3 = \max\{d_2, 4\varepsilon_1^{-1}2^{n_0}\}$.

For $x \in V(T)$, $L(x)$ and $R(x)$ denote the left and right subtree of $T$ above $x$, respectively. Note that either $L(x)$ or $R(x)$ or both may be empty (e.g., both are empty if $x \in \text{top}(T)$). A vertex $x$ of the cone $S$ is called a branching vertex if neither $L(x) \cap Z_n$ nor $R(x) \cap Z_n$ is empty.

**Claim 4.** For a suitable constant $d_4 \geq d_3$ the following holds. If $G$ is a graph satisfying $\delta(G) \geq d_4$ then there exists a path $P = (x_1, x_2, ..., x_n)$ in the cone $S$ such that $x_1 = s$, the root of $S$, $x_n \in \text{top}(S)$ and $P$ contains at least $\frac{1}{4} n$ branching vertices.

**Proof.** The path $P$ is defined by starting from $s$ and continuing always in the subtree containing the larger number of vertices of $Z_n$. More
formally, let $x_1 = s$, and if $x_1, x_2, \ldots, x_r$ are already defined for $r < n$ then

$$x_{r+1} = \begin{cases} 	ext{the root of } L(x_r) \text{ if } |L(x_r) \cap Z_n| \geq |R(x_r) \cap Z_n|, \\ 
\text{the root of } R(x_r) \text{ if } |R(x_r) \cap Z_n| > |L(x_r) \cap Z_n|. 
\end{cases}$$

Let $y_1, y_2, \ldots, y_t$ denote the branching vertices of $P = (x_1, \ldots, x_n)$ indexed in their natural order as they appear in $P$. The existence of at least one branching vertex is guaranteed if $|\text{top}(S) \cap Z_n| \geq 2$. Claim 3 shows that this is ensured if $n \geq n_i$, where $n_i$ is the smallest integer satisfying $\frac{3}{4}n_i - 1[33n_i^2(n_i + 1)]^{-1} \geq 2$. The definition of $P$ shows that $|C(y_i) \cap Z_n| \leq 2|C(y_{i+1}) \cap Z_n|$ (for $i = 1, \ldots, t - 1$), and the definition of $t$ shows that $|C(y_t) \cap Z_n| = 2$. Therefore, by Claim 3,

$$2^t \geq |C(y_1) \cap Z_n| = |\text{top}(S) \cap Z_n| > \frac{3}{4}n - 1[33n^2(n + 1)]^{-1}.$$

From this inequality we get

$$t > (n - 1)[2 - \log(3)] - \log[33n^2(n + 1)] = n[2 - \log(3)] - o(n),$$

showing that $t > \frac{1}{4}n$ if $n \geq n_2$, since $2 - \log(3) > \frac{1}{4}$. Let $d_4 = \max\{d_3, 4e_1^{-1}2n_3\}$ where $n_3 = \max\{n_1, n_2\}$. In this case $n \geq m = \log[4p(G, \varepsilon_1)] \geq \log[4^{-1}e_1\delta(G)] \geq n_3$, which validates our arguments.

Without restricting the generality of the proof we may assume that the path $P$ goes to the left, i.e., $x_{i+1} \in L(x_i)$ for $i = 1, 2, \ldots, n - 1$.

Let $I$ denote the index-set of the branching vertices of $P$. For every $i \in I$, we can choose $z_i \in Z_n$ and a path $P_i$ connecting $x_i$ and $z_i$ in $S$ such that $P_i - x_i \subset R(x_i)$. The tree $P \cup (U_{i \in I} P_i)$ is denoted by $U$. Clearly $\text{top}(U)$ contains $|I| + 1$ vertices: $z_i$ for $i \in I$ and $x_n$. We define $z_n$ as $x_n$ for technical reasons. It is also obvious from the definition that $\text{top}(U) \subset Z_n$.

Let $U'$ denote the subtree of $T$ whose vertices are at cone distance $n$ from $\text{top}(S)$. The definition of $U$ ensures that every vertex in $\text{top}(U)$ has an $n$-neighbor at the same level of $U'$; this level of $U'$ is denoted by $R$. For $i \in I \cup n$, $R(i)$ denotes the set of $n$-neighbors of $z_i$ in $R$. We note that $R \subset \text{top}(T)$ when there exists an $i \in I \cup n$ and a $v \in R(i)$ such that the join $(z_i, v)$ is an out-of-tree join. (Figure 2 illustrates $P_i$, $U$, $U'$, and $R$.)

Now we shall define a cycle $H(i)$ in $G$ for all $i \in I$. The definition is given by the following procedure.

**Procedure Cycles**

C0. Let $i$ be the smallest index in $I$.

C1. Consider triples $(j, a, b)$ where $j \in I \cup n, j > i, a \in R(i), b \in R(j)$. Let $U(z_i, z_j)$ be the path connecting $z_i$ and $z_j$ in $U$. Let
$U'(a, b)$ be the path connecting $a$ and $b$ in $U'$. (Observe that $U(z_i, z_j)$ contains $2(n - i) + 1$ vertices.) Let $J_1$ be a $(z_i, a)$-join and let $J_2$ be a $(z_j, b)$-join. The union of the joins $J_1, J_2$ with the paths $U(z_i, z_j), U'(a, b)$ gives a cycle of $G$ unless $J_1$ and $J_2$ are out-of-tree joins with the same center $v \in V(G) - V(T)$. Let $\mathcal{H}(i)$ denote the set of cycles in this form, where all triples $(j, a, b)$ and all pairs $J_1, J_2$ are considered. If $\mathcal{H}(i) = \emptyset$ then continue at C2, otherwise continue at C3.

C2. There is a vertex $v \in V(G) - V(T)$ such that for all triples $(j, a, b)$ and for all $J_1, J_2$, the joins $J_1$ and $J_2$ are out-of-tree joins with center $v$. We define the cycle $H(j)$ for all $j \in I, j \geq i$, as the union of $U(z_n, z_j)$ with the edges $(z_n, v), (z_j, v) \in E(F)$. The length of $H(j)$ is $2(n - j) + 2$. Stop. (See Fig. 3.)

C3. Define $H(i)$ as an element of $\mathcal{H}(i)$ belonging to a triple $(j(i), a(i), b(i))$ for which the number of vertices in $U'(a, b)$ is maximum. If the number of vertices in $U'(a(i), b(i))$ is denoted by $r(i)$ then the length of $H(i)$ can be expressed as $2(n - i) + 1 + r(i) + e(i)$ where $e(i)$ is defined as 0, 1, or 2 according to the number of out-of-tree joins in $\{J_1, J_2\}$. (See Fig. 4.)

If there are some unprocessed indices in $I$ then $i$ is redefined as the next index in the natural order of $I$. Continue at C1. If every index is processed then Stop.

End of procedure cycles.

The length of the cycle $H(i)$ can be written in the following form:
DISTRIBUTION OF CYCLE LENGTHS IN GRAPHS

\[ |H(i)| = 2(n - i) + 1 + r(i) + e(i) \] where \( r(i) = 1, e(i) = 0 \) if \( H(i) \) was defined in C2.

**Claim 5.** If \( i, i' \in I \) and \( i' > i \) then \( r(i') \leq r(i) \).

**Proof.** If \( H(i') \) was defined in C2 then \( r(i') = 1 \) and if \( H(i) \) was defined in C2 then \( r(i') = r(i) = 1 \). In both cases, the claim is true.

So assume that both \( H(i) \) and \( H(i') \) were defined in C3. Let \( C \) be the cone of minimal height in \( U' \) containing both \( a(i) \) and \( b(i) \). We shall prove that \( a(i'), b(i') \in \text{top}(C) \).

Let \( c \) denote one of \( a(i') \) and \( b(i') \). There exists a \( k \in I \cup n, k > i \), such that \( c \in R(k) \), since for \( c = a(i') \), \( k = i' \) is a good choice and for \( c = b(i') \), \( k = j(i') \) is a good choice. If we assume indirectly that \( c \notin \text{top}(C) \) then the path \( P_2 \) connecting \( c \) and \( a(i) \) in \( U' \) and the path \( P_3 \) connecting \( c \) and \( b(i) \) in \( U' \) are longer than the path \( P_1 \) connecting \( a(i) \) and \( b(i) \) in \( U' \). Let \( J_1, J_2 \) denote the joins in the definition of \( H(i) \).

If the triple \((k, a(i), c)\) defines a cycle in \( G \) with the \((z_i, a(i))-join J_1\) and with some \((z_k, c)-join J'_2\) then \( U'(a(i), c) = P_2 \), contradicting the maximality of \( U'(a(i), b(i)) = P_1 \).

We may therefore assume that \( J_1 \) and \( J'_2 \) are out-of-tree joins with a common center \( v \in V(G) - V(T) \) (see Fig. 5). Thus \((z_i, v)\) and \((v, c)\) are edges of \( F \) and \( J'_1 = (z_i, v) \cup (v, c) \) is a \((z_i, c)-join. Therefore the triple \((j(i), c, b(i))\) defines a cycle in \( G \) with the joins \( J'_1 \) and \( J_2 \). (Note that \( J_2 \) does not contain \( v \) because \( J_1 \) contains \( v \).) Now \( U'(c, b(i)) = P_3 \), contradicting the maximality of \( U'(a(i), b(i)) = P_1 \).
We have proved that \( a(i'), b(i') \in \text{top}(C) \). This implies \( r(i') \leq r(i) \).

If \( i_1 < i_2 < i_3 \) are three consecutive indices of \( I \) then

\[
|H(i_1)| - |H(i_3)| = 2(i_3 - i_1) + r(i_1) - r(i_3) + e(i_1) - e(i_3) \geq 2
\]

since \( i_3 - i_1 \geq 2 \), \( e(i_1) \geq 0 \), \( e(i_3) \leq 2 \), and \( r(i_1) - r(i_3) \geq 0 \) by Claim 5. This argument shows that the set \( H = \{ |H(i) : i \in I \} \) has at least \( \frac{1}{2} |I| \) elements. Since \( |I| \geq \frac{1}{2} n \) by Claim 4, we get \( |H| \geq \frac{1}{2} n \). On the other hand,

\[
|H(i)| = 2(n - i) + 1 + r(i) + e(i) \leq 2(n - 1) + 1 + 2n - 1 + 2 =
\]
4n, for every \( i \in I \). The proof of Theorem 1 is complete if we choose \( c_1 = d_4 \) and \( \varepsilon_1 \) as defined in Claim 1.

4. INTERVALS CONTAINING ALMOST ALL EVEN CYCLE LENGTHS

Let \([A, B]\) denote an interval of positive real numbers. We say that \( C(G) \) contains \textit{almost all} even integers in \([A, B]\) if there exists an integer \( t \geq 2 \) such that \( C(G) \) contains all even integers of \([A, B]\) with the possible exception of the multiples of \( 2t \). The proof of the next proposition is omitted since it requires straightforward elementary calculations.

**Proposition 1.** There is a positive constant \( a_1 \) with the following property. If \( G \) is a graph and \([A, B]\) is an interval such that \( C(G) \cap [A, B] \neq \emptyset \) and \( C(G) \) contains almost all even integers in \([A, B]\) then \( L(G) \geq a_1 \log(B/A) \).

**Theorem 2.** For suitable constants \( c_2, \varepsilon_2 \) the following holds. If \( G \) is a graph satisfying \( \delta(G) \geq c_2 \) and \( p(G, \varepsilon_2) \geq 4 \log^2(|V(G)|) \) then \( C(G) \) contains almost all even integers in the interval \([4 \log(|V(G)|) + 2, 3^{-1}[p(G, \varepsilon_2) - 1]]\).

Now we formulate Theorem 2', a slightly weaker but simpler version of Theorem 2.

**Theorem 2'.** For suitable constants \( c_3, \varepsilon_2 \) the following holds. If \( G \) is a graph and \( \delta(G) \geq \max\{c_3, 4\varepsilon_2^{-1} \log^2(|V(G)|)\} \) then \( C(G) \) contains almost
all even integers in the interval
\[ [2[e_2 \delta(G)]^{1/2} + 2, 3^{-1}[e_2 \delta(G) - 1]]. \]

**Proof.** If \( c_3 \geq c_2 \) then we can apply Theorem 2 for \( G \) since \( p(G, e_2) \geq e_2 \delta(G) \geq 4 \log^2[|V(G)|] \). Theorem 2 ensures that \( C(G) \) contains almost all even integers in the interval \([A_1, B_1] = [4 \log[|V(G)|] + 2, 3^{-1}[p(G, e_2) - 1]] \). Now \( A_2 = 2[e_2 \delta(G)]^{1/2} + 2 \geq A_1 \) and \( B_2 = 3^{-1}[e_2 \delta(G) - 1] \leq B_1 \). Therefore \( C(G) \) contains almost all even integers from \([A_2, B_2] \) if \( A_2 < B_2 \) is ensured by choosing \( c_3 \) sufficiently large. \( \blacksquare \)

Theorem 2 and Proposition 1 easily yield the following result, which we need in Section 5.

**Corollary 3.** For suitable positive constants \( a_2, c_4 \) the following holds. If \( G \) is a graph satisfying \( \delta(G) \geq c_4 \) and \( p(G, e_2) \geq 4 \log^2[|V(G)|] \) then \( L(G) \geq a_2 \log[\delta(G)] \).

**Proof.** First, we choose \( c_4 \) so that \( c_4 \geq c_2 \), and apply Theorem 2 to deduce that \( C(G) \) contains almost all even integers in \([A_1, B_1] \) where \( A_1 = 4 \log[|V(G)|] + 2 \) and \( B_1 = 3^{-1}[p(G, e_2)]^{1/2} + 2 \). Then \( A_2 \geq A_1 \), and so \( C(G) \) contains almost all even integers in \([A_2, B_2] \). Now note that \( p(G, e_2) \) can be made arbitrarily large by an appropriate choice of \( c_4 \), because

\[ p(G, e_2) \geq e_2 \delta(G) \geq e_2 c_4. \]

We may therefore choose \( c_4 \) so that \( [A_2, B_1] \cap C(G) \neq \emptyset \). By Proposition 1, \( L(G) \geq a_1 \log(B_1/A_2) \). Similarly, we can choose \( c_4 \) so that \( B_1/A_2 \geq 12^{-1} p(G, e_2)^{1/2} \) and \( \delta(G) \geq e_2^{-2} 12^4 \). Therefore \( L(G) \geq a_1 \log(B_1/A_2) \geq a_1 \log[12^{-1} p(G, e_2)^{1/2}] \geq a_1 \log[12^{-1}[e_2 \delta(G)]^{1/2}] \geq a_1 \log[12^{-1} \delta(G)] \) and the corollary holds with \( a_2 = \frac{1}{2} a_1 \). \( \blacksquare \)

For the proof of Theorem 2 we need the following lemma.

**Lemma 3.** Let \( (U, V) \) be a partition of \( \{1, 2, \ldots, n\} \) into two nonempty sets \( U, V \) and let

\[ D(U, V) = \{d : d = |u - v|, u \in U, v \in V\}. \]

Then there exists an integer \( m \geq 2 \) such that \( D(U, V) \) contains all natural numbers not greater than \( \frac{3}{2} n \) and not divisible by \( m \).

**Proof.** If \( D(U, V) = \{1, 2, \ldots, n - 1\} \) then the lemma is obviously true. Let \( m \) be the smallest natural number not in \( D(U, V) \). Clearly \( m \geq 2 \), since \( m = 1 \) would imply that \( U \) or \( V \) is empty. Let \( U_m = U \cap \{1, 2, \ldots, m\} \) and let \( V_m = V \cap \{1, 2, \ldots, m\} \). Then \( m \notin D(U, V) \).
implies that

\[ U = \{ u + im : u \in U_m, u + im \leq n \}, \]
\[ V = \{ v + jm : v \in V_m, v + jm \leq n \}. \]  

(9)

Let \( k = rm + s \), where \( 0 \leq r, 1 \leq s \leq m - 1 \), and \( k \leq \frac{1}{3}n \). We prove that \( k \in D(U, V) \).

A. If \( r = 0 \) then \( k \in D(U, V) \) by the minimum property of \( m \). Moreover, if we have a pair \( u \in U, v \in V \) satisfying \( |u - v| = s \) then, by the periodicity of \( U \) and \( V \) expressed in (9), we can find \( u(s) \in U \), \( v(s) \in V \) such that \( |u(s) - v(s)| = |u - v| = s \) and \( 1 \leq u(s), v(s) \leq 2m - 1 \).

B. Assuming \( r \geq 1 \), we consider \( u(s) \) and \( v(s) \) as defined in A. We define \( u' \) and \( v' \) as follows: if \( u(s) > v(s) \) then \( u' = u(s) + rm \) and \( v' = v(s) \); otherwise \( u' = u(s) \) and \( v' = v(s) + rm \). Since \( rm < k \leq \frac{1}{3}n \), \( m < n/3r \). Therefore, \( \max(u', v') = rm + \max\{u(s), v(s)\} < rm + 2m = (r + 2)m < (r + 2)n/3r \leq n \). By (9), \( u' \in U \) and \( v' \in V \); moreover, \( |u' - v'| = rm + s = k \). Thus \( k \in D(U, V) \).

**Proof of Theorem 2.** Assume that \( \delta(G) \geq 3 \). Let \( T \) be a saturated maximal \( \frac{3}{2} \)-tree of \( G \). We apply Lemma 2 to define \( F \), the crown of \( T \). Because \( T \) is saturated, there is no edge of \( F \) from \( \text{top}(T) \) to \( Li, \) for \( 0 \leq i \leq h(T) - 2 \). Since \( \delta(F) \geq \frac{1}{12}[\delta(G) - 2] \) by property (c) of crowns, each vertex of \( X = \text{top}(T) \cap V(F) \) belongs to at least one of the following three sets:

\[ A_1 = \{ x \in X : \text{at least } \frac{1}{3}d \text{ edges of } F \text{ go from } x \text{ to } B_1 = V(F) - V(T) \}, \]
\[ A_2 = \{ x \in X : \text{at least } \frac{1}{3}d \text{ edges of } F \text{ go from } x \text{ to } B_2 = \text{top}(T) \}, \]
\[ A_3 = \{ x \in X : \text{at least } \frac{1}{3}d \text{ edges of } F \text{ go from } x \text{ to } B_3 = L_{h(T) - 1} \}. \]

We can choose an \( i \) (\( 1 \leq i \leq 3 \)) such that \( |A_i| \geq \frac{1}{72}|X| \). We define the tree \( T' \) as follows: \( T' = T \) if \( i = 1 \) or \( i = 2 \) and \( T' = T - \text{top}(T) \) if \( i = 3 \).

Assuming that \( \frac{1}{3}d \geq 1 \), i.e.,

\[ \delta(G) \geq 218, \]  

(10)

we apply Lemma 1 for the graph spanned by the edges of \( F \) which join vertices of \( A_i \) to vertices of \( B_j \). The set \( A_i \) plays the role of \( M \), \( \frac{1}{2}d \) plays the role of \( k \) and \( 3 \times 72 = 216 \) plays the role of \( c \). \((|V(F)| \leq 72 \times |X| \leq 216 \times 3|A_i| \) holds by property (e) of crowns and by the definition of \( A_i \).) Lemma 1 gives a subgraph \( F_i \subset F \) such that \( \delta(F_i) \geq \frac{1}{2}[\delta(G) - 2]/3 \times 72/4 \times 216 \geq \delta(G)/2 \times 3 \times 72 \times 4 \times 216 \) if \( \delta(G) \geq 2 \times 3 \times 72 + 2 \). If we choose
\[\varepsilon_2 = (2 \times 3 \times 72 \times 4 \times 216)^{-1} \quad (11)\]

then \(\delta(F_1) \geq \varepsilon_2 \delta(G)\). Therefore \(F_1\) contains a path of \(p(G, \varepsilon_2)\) vertices. By removing zero, one, or two end-vertices of this path, we obtain a path \(P = (x_1, x_2, \ldots, x_r)\) of \(F_1\) with the following properties:

- \(r\) is odd and \(r \geq p(G, \varepsilon_2) - 2\),
- \(x_j \in \text{top}(T')\) if \(j\) is odd and \(1 \leq j \leq r\),
- \(x_j \notin \text{V}(T') - \text{top}(T')\) for \(1 \leq j \leq r\).

The set \(\{x_1, x_3, x_5, \ldots, x_r\}\) is denoted by \(Y\). The tree \(T'\) and the path \(P\) are shown in Figure 6 for \(i = 1, 2, 3\).

Let \(U'\) denote the subtree of \(T'\) for which \(\text{top}(U') = Y\) and \(\text{V}(U')\) is as small as possible. The root of \(U'\) contains at least two branches if \(|Y| \geq 2\); this can be ensured by setting

\[c_2 = 5\varepsilon_2^{-1}, \quad (12)\]

since \(r + 2 \geq p(G, \varepsilon_2) \geq \varepsilon_2 \delta(G) \geq \varepsilon_2 c_2\). One branch from the root of \(U'\) and the union of the other branches divide \(Y\) into two nonempty sets \(Y_1\) and \(Y_2\) with the property that for every pair \(x_p \in Y_1, x_q \in Y_2\), the number of vertices in the path \(P_1\) connecting \(x_p\) and \(x_q\) in \(U'\) is \(2h' + 1\), where \(h' = h(U')\). On the other hand, \(x_p\) and \(x_q\) are connected by a subpath \(P_2\) of \(P\) of \(|p - q| + 1\) vertices. We define the cycle \(H(p, q)\) for every \(x_p \in Y_1, x_q \in Y_2\) as the union of \(P_1\) and \(P_2\). Clearly \(|H(p, q)| = 2h' + |p - q|\) (\(U'\) and \(H(p, q)\) are shown in Fig. 7).

Now we apply Lemma 3, where \((U, V)\) is the partition of \(\{1, 2, \ldots, \frac{1}{2}(r + 1)\}\) defined by

\[i \in U, \quad \text{if } x_{2i-1} \in Y_1, \quad i \in V, \quad x_{2i-1} \in Y_2.\]

If \(d \in \mathcal{D}(U, V)\), i.e., \(d = |u - v|\) for some \(u \in U, v \in V\), then
\( x_{2u-1} \in Y_1 \) and \( x_{2v-1} \in Y_2 \). Therefore \( G \) contains the cycle \( H(2u - 1, 2v - 1) \) of length \( 2h' + |2u - 1 - (2v - 1)| = 2h' + 2d \). Since \( d \) takes on all values in \([1, \frac{1}{2}(r + 1)]\) except, possibly, the multiples of \( m \), \( 2h' + 2d \) takes on all even values in \( I = [2h' + 2, 2h' + 2 + \frac{1}{2}(r + 1)] \) except, possibly, the multiples of \( 2m \), where \( m \geq 2 \). In other words, \( C(G) \) contains almost all even integers in \( I \).

Since \( \frac{3}{2}h(T) \leq |\text{top}(T)| \leq |V(G)| \) by property (2) of \( \frac{3}{2} \)-trees,

\[
h' \leq h(T) \leq \log|V(G)|/(\log 3 - 1) \leq 2 \log|V(G)|. \tag{13}
\]

On the other hand, \( r + 1 \geq p(G, e_2) - 1 \). Therefore,

\[
I \supseteq [2h' + 2, \frac{1}{2}(r + 1)] \supseteq [4 \log|V(G)|] + 2, 3^{-1}[p(G, e_2) - 1] = I',
\]

and \( C(G) \) contains almost all even integers in \( I' \), as stated in the theorem, provided that \( e_2 \) and \( c_2 \) are chosen according to (11) and (12). \( \blacksquare \)

5. THE SUM OF RECIPROCALs OF CYCLE LENGTHS

Let \( H \) be a subgraph of a graph \( G \). A vertex \( x \in V(H) \) is an inner vertex of \( H \) if \( d_H(x) = d_G(x) \).

Lemma 4. Every graph \( G \) has a subgraph \( H \) such that \( |V(H)| \leq 2p(G) \) and \( H \) has at least \( p(G) \) inner vertices.
Proof. Let $S = (x_1, x_2, \ldots, x_p)$ be a longest path of $G$. We define paths $S_1, S_2, \ldots, S_p$ and vertices $y_1, y_2, \ldots, y_p$ inductively as follows: $S_1 = S$, $y_1 = x_p$. If $S_t = (z_1, z_2, \ldots, z_m)$ and $Y_t = \{y_1, y_2, \ldots, y_t\}$ are already defined, where $t < p$, we define $S_{t+1}$ and $y_{t+1}$ by the rule $S_{t+1} = (z_1, z_2, \ldots, z_{m+1})$ is a longest path of $G - Y_t$ starting with the vertices $z_1, z_2, \ldots, z_m, y_{t+1} = z_{m+1}$. Figure 8 shows an example in which $G$ is a tree with $p(G) = 12$.

Let $H$ be the subgraph of $G$ spanned by the set $V(S_1) \cup V(S_2) \cup \cdots \cup V(S_p) \cup Y_p$. Since $V(S_t) \subseteq V(S_{t+1}) \cup Y_t$ for $1 \leq t \leq p - 1$, $|V(H)| \leq |V(S_p)| + |Y_p| \leq 2p$. On the other hand, $Y_p$ is a subset of $V(H)$ with $p$ vertices and, for every $t$, $1 \leq t \leq p$, $y_t$ can be joined in $G$ only to vertices of $S_t$ and $Y_t$. Therefore $d_H(y_t) = d_G(y_t)$ for $1 \leq t \leq p$.

We need the following corollary of Lemma 4.

**Corollary 4.** Let $G$ be a graph satisfying $\delta(G) \geq 1$. There exists a subgraph $G' \subseteq G$ such that $\delta(G') \geq \frac{1}{2} \delta(G)$ and $|V(G')| \leq 2p(G)$.

**Proof.** We choose $H \subseteq G$ according to Lemma 4. Let $X$ denote the set of inner vertices of $H$. Since $|V(H)| \leq 2|X|$ and $d_H(x) = d_G(x) \geq \delta(G)$ for all $x \in X$, we can apply Lemma 1 for $H$ with $k = \delta(G)$, $c = 2$, $M = X$. Lemma 1 gives a subgraph $G' \subseteq H$ such that $\delta(G') \geq k/4c = \frac{1}{2} \delta(G)$. Moreover, $|V(G')| \leq |V(H)| \leq 2p(G)$.

We prepare the main result of this section with the following (rather technical) theorem.

**Theorem 3.** There are constants $c_5$, $a_3$, and $\varepsilon_3$ with the following property. If $G$ is a graph satisfying $\delta(G) \geq c_5$ then either $L(G) \geq a_3 \log[\delta(G)]$ or there exists a subgraph $H \subseteq G$ such that $\delta(H) \geq \varepsilon_3 \delta(G)$ and $|V(H)| < 16^{-1} \log[\frac{1}{2}p(G, \varepsilon_1)]$. (The constant $\varepsilon_1$ comes from Theorem 1.)

**Proof.** The proof is a simple calculation based on repeated applications of Corollary 3 and Corollary 4. Let $c_4$, $a_2$, $\varepsilon_2$ be the constants appearing in Corollary 3. We define

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}.$$ (14)

![Figure 8](image)
The definition of $p(G, \varepsilon)$ ensures a subgraph $G_1 \subset G$ such that
\begin{equation}
\delta(G_1) \geq \varepsilon \delta(G), \quad p(G_1) = p(G, \varepsilon).
\end{equation}

Applying Corollary 4 for $G_1$, we get $G_2 \subset G_1$, satisfying
\begin{equation}
\delta(G_2) \geq \frac{1}{2} \delta(G_1), \quad |V(G_2)| \leq 2p(G_1) = 2p(G, \varepsilon).
\end{equation}

Assuming that
\begin{equation}
c_5 \geq 8\varepsilon^{-1}c_4,
\end{equation}
$\delta(G_2) \geq c_4$ follows from the first parts of (16) and (15). If $p(G_2, \varepsilon) \geq 4 \log^2|V(G_2)|$, then we can apply Corollary 3 for $G_2$, which gives $L(G) \geq L(G_2) \geq a_2 \log[\delta(G_2)] \geq a_2 \log[8^{-1} \varepsilon \delta(G)] \geq \frac{1}{2} a_2 \log[\delta(G)]$, where the last inequality holds if
\begin{equation}
c_5 \geq 8^2 \varepsilon^{-2}.
\end{equation}

Therefore the first alternative of Theorem 3 holds with $a_3 = \frac{1}{2} a_2$ if (17) and (18) are satisfied.

We may assume $p(G_2, \varepsilon) < 4 \log^2|V(G_2)|$. The second part of (16) implies
\begin{equation}
p(G_2, \varepsilon) < 4 \log^2|V(G_2)| \leq 4 \log^2[2p(G, \varepsilon)].
\end{equation}

Now the previous argument is repeated. The definition of $p(G_2, \varepsilon)$ ensures $G_3 \subset G_2$ such that
\begin{equation}
\delta(G_3) \geq \varepsilon \delta(G_2), \quad p(G_3) = p(G_2, \varepsilon).
\end{equation}

Applying Corollary 4 for $G_3$, we get $G_4 \subset G_3$ satisfying
\begin{equation}
\delta(G_4) \geq \frac{1}{2} \delta(G_3), \quad |V(G_4)| \leq 2p(G_3) = 2p(G_2, \varepsilon).
\end{equation}

We try to apply Corollary 3 for $G_4$ as we did before for $G_2$. The same argument shows that the condition
\begin{equation}
c_5 \geq \max\{8^2 \varepsilon^{-2}c_4, 8^4 \varepsilon^{-4}\}
\end{equation}
either ensures the first alternative of Theorem 3 with $a_3 = \frac{1}{2} a_2$ or $p(G_4, \varepsilon) < 4 \log^2|V(G_4)|$. In the latter case the second parts of (21) and (19) imply
\begin{equation}
p(G_4, \varepsilon) < 4 \log^2|V(G_4)| < 4 \log^2[8 \log^3[2p(G, \varepsilon)]].
\end{equation}
We start the same argument a third time. The definition of \( p(G_4, \varepsilon) \) ensures \( G_5 \subset G_4 \) such that
\[
\delta(G_5) \geq \varepsilon \delta(G_4), \quad p(G_5) = p(G_4, \varepsilon).
\] (24)

Applying Corollary 4 for \( G_5 \), we get \( H \subset G_5 \), satisfying
\[
\delta(H) \geq \frac{1}{k} \delta(G_5), \quad |V(H)| \leq 2p(G_5) = 2p(G_4, \varepsilon).
\] (25)

We prove that \( H \) satisfies the second alternative of Theorem 3 if \( \varepsilon_3 = 8^{-3} \varepsilon^3 \) and \( c_5 \) is sufficiently large. The condition \( \delta(H) \geq \varepsilon_3 \delta(G) \) is guaranteed by the first parts of (25), (24), (21), (20), (16), (15). On the other hand, \( \varepsilon \leq \varepsilon_1 \) by (14). Therefore \( p(G, \varepsilon) \leq p(G, \varepsilon_1) \) and it is sufficient to show that \( |V(H)| < 16^{-1} \log[\frac{1}{2} p(G, \varepsilon)] \). Since \( |V(H)| \) can be estimated by the second part of (25) and by (23), we have to prove that \( 8 \log^2[8 \log^2[2p(G, \varepsilon)]] < 16^{-1} \log[\frac{1}{2} p(G, \varepsilon)] \), which obviously holds if \( p(G, \varepsilon) \) is large enough, say \( p(G, \varepsilon) \geq c_9 \). This can be ensured by requiring \( c_5 \geq c_0 \varepsilon^{-1} \).

We conclude that Theorem 3 holds with the following constants: \( \varepsilon = \min\{\varepsilon_1, \varepsilon_2\} \), \( c_5 = \max\{8^2 \varepsilon^{-2}, c_4, 8^4 \varepsilon^{-4}, c_9 \varepsilon^{-1}\} \), \( \varepsilon_3 = 8^{-3} \varepsilon^3 \), \( a_3 = \frac{1}{2} a_2 \). [\( \blacksquare \)]

**Theorem 4.** For suitable constants \( a \) and \( b \) the following holds. If \( G \) is a graph satisfying \( \delta(G) \geq b \) then \( L(G) \geq a \log[\delta(G)] \).

**Proof.** Let \( c_1, \varepsilon_1 \) be the constants defined in Corollary 1. Let \( c_5, a_3, \varepsilon_3 \) be the constants defined in Theorem 3. We define the graphs \( G_0, G_1, G_2, \ldots, G_i \) inductively as follows. Let \( G_0 = G \). Assume that \( G_0, G_1, \ldots, G_i \) are already defined for some \( i, i \geq 0 \). The next procedure either defines \( G_{i+1} \) or stops with \( G_t = G_i \).

**Procedure** \( G_{i+1} \).

**Step 1.** If \( \delta(G_i) < c_5 \) then \( G_t = G_i \). Stop.

**Step 2.** Apply Theorem 3 for \( G_i \). If the first alternative holds in Theorem 3 then \( G_t = G_i \). Stop.

**Step 3.** If \( \delta(G_i) < c_1 \) then \( G_t = G_i \). Stop.

**Step 4.** Let \( G_{i+1} \) be the subgraph \( H \) of \( G_i \) ensured by the second alternative in Theorem 3.

**End of procedure** \( G_{i+1} \).

Since \( G_{i+1} \) is a proper subgraph of \( G_i \), the definition eventually ends with some \( G_t \). The definition of \( G_{i+1} \) implies that \( \delta(G_{i+1}) \geq \varepsilon_3 \delta(G_i) \) for all \( i, 0 \leq i < t \); thus
\[
\delta(G_i) \geq \varepsilon_3^i \delta(G).
\] (26)
On the other hand, \( \delta(G_i) \geq c_1 \) and \(|V(G_{i+1})| < 16^{-1}\log[^{1/4} p(G_i, e_i)]\) for all \( i, 0 \leq i < t \). Corollary 1 implies that the sum of reciprocals in \( C(G) \cap \{[V(G_{i+1})], [V(G_i)]\} \) is at least \( \frac{1}{c_1} \) for all \( i, 0 \leq i < t \). Therefore

\[
L(G) \geq \frac{1}{c_1} t. \tag{27}
\]

Now we need an easy discussion of cases. If \( G_t \) is defined in Step 1 then \( c_3 > \delta(G_t) \geq e^{-1}_3 \delta(G_t) \) by (26), which gives \( t > \frac{\log[\delta(G)] - \log(c_3)}/\log(e^{-1}_3) \). This estimation and (27) imply

\[
L(G) > \frac{\log[\delta(G)]}{64 \log(e^{-1}_3)} - \frac{\log(c_3)}{64 \log(e^{-1}_3)}. \tag{28}
\]

If \( G_t \) is defined in Step 3 then the previous argument can be followed. We get

\[
L(G) > \frac{\log[\delta(G)]}{64 \log(e^{-1}_3)} - \frac{\log(c_1)}{64 \log(e^{-1}_3)}. \tag{29}
\]

If \( G_t \) is defined in Step 2 then by (27) and by Theorem 3

\[
L(G) \geq \frac{1}{c_4} t + a_1 \log[\delta(G_t)]. \tag{30}
\]

If \( t \leq \log[\delta(G)]/[2 \log(e^{-1}_3)] \) then \( L(G) \) can be estimated by the second term of (30):

\[
L(G) \geq a_3 \log[\delta(G_t)] \geq a_3 e^{-1}_3 \delta(G_t) \geq \frac{1}{2} a_3 \log[\delta(G)]. \tag{31}
\]

If \( t > \log[\delta(G)]/[2 \log(e^{-1}_3)] \) then \( L(G) \) can be estimated by the first term of (30):

\[
L(G) \geq \frac{1}{c_4} t > [2 \log(e^{-1}_3)64]^{-1} \log[\delta(G)]. \tag{32}
\]

In all cases [(28), (29), (31), (32)] we have a lower bound on \( L(G) \) of the form \( a'_i \log[\delta(G)] - b'_i, i = 1, 2, 3, 4 \). Let \( a' = \min\{a'_i, a'_2, a'_3, a'_4\}, b' = \max\{b'_i, b'_2, b'_3, b'_4\} \). Then

\[
L(G) \geq a' \log[\delta(G)] - b' \geq \frac{1}{2} a' \log[\delta(G)] = a \log[\delta(G)]
\]

if \( \log[\delta(G)] \geq 2b'/a' \), which can be ensured by choosing \( b = 2^{2b'/a'} \).

Because a graph of density \( \alpha \) contains a subgraph of minimum degree \( \lceil \alpha \rceil + 1 > \alpha \), Theorem 4 has the following consequence.

**Theorem 4'**. Let \( a \) and \( b \) be the constants defined in Theorem 4. If \( G \) is a graph of density \( \alpha \), where \( \alpha \geq b \), then \( L(G) \geq a \log(\alpha) \).
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