On the Distribution of Cycle Lengths in Graphs

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ABSTRACT

The set of different cycle lengths of a graph G is denoted by $C(G)$. We study how the distribution of $C(G)$ depends on the minimum degree of G. We prove two results indicating that $C(G)$ is dense in some sense. These results lead to the solution of a conjecture of Erdös and Hajnal stating that for suitable positive constants a, b the following holds:

$$\sum_{i \in C(G)} i^{-1} \geq a \log[\delta(G)], \qquad \delta(G) \geq b,$$

where $\delta(G)$ denotes the minimum degree of G.

1. INTRODUCTION

The set of different cycle lengths of a graph G is denoted by $C(G)$. There exist many theorems which assert that $C(G)$ is "dense" provided that the number of edges in G is substantially larger than the number of vertices. Here, dense may mean various things, for example, $C(G) = \{3, 4, \ldots, |V(G)|\}$ (G is pancyclic); $C(G)$ contains a large sequence of consecutive integers; $C(G)$ contains a large sequence of consecutive even integers. A survey of such results can be found in [1].

There are important families of graphs not satisfying the above assumption but still suspected of having a set of cycles whose lengths are

Journal of Graph Theory, Vol. 8 (1984) 441–462

CCC 0364-9024/84/040441-22$04.00

dense in some sense. Graphs of chromatic number k, graphs of minimum degree δ, and graphs of density α are examples of such families. All these families contain graphs of arbitrarily large girth; therefore small integers do not necessarily appear as cycle lengths. A measure of density for these families was proposed in a paper of Erdös [2], who asked about the behavior of $L(G)$, the sum of reciprocals of elements of $C(G)$, as a function of the edge density of G. He introduced the function

$$f(\alpha) = \inf\{L(G) : |E(G)| \geqslant \alpha|V(G)|\}$$

where $V(G)$ and $E(G)$ denote the sets of vertices and edges of G. It is obvious that $f(\alpha) = 0$ for $\alpha \leqslant 1$. The complete bipartite graphs with equal color classes show that $f(\alpha) \leqslant c \log(\alpha)$. It was conjectured by Erdös and Hajnal that $f(\alpha) \geqslant c' \log(\alpha)$ but it was not even known whether $f(\alpha)$ tends to infinity with α ([1], p. 168). In this paper we prove that $f(\alpha) \geqslant a \log(\alpha)$ if $\alpha \geqslant b$, for suitable constants a and b (Theorem 4$'$). The behavior of $f(\alpha)$ for $\alpha \in (1, 1 + \varepsilon)$ seems to be an interesting problem.*

Throughout the paper we shall study the distribution of $C(G)$ for graphs of fixed minimum degree $\delta(G)$. Since a graph of chromatic number k contains a subgraph of minimum degree at least $k - 1$ and a graph of density α contains a subgraph of minimum degree at least $\lfloor \alpha \rfloor + 1$, our results can be applied both to graphs of fixed chromatic number and to graphs of fixed density.

We have two results on the distribution of $C(G)$, both of which express certain density properties of $C(G)$.

The first density result on $C(G)$ is Theorem 1$'$ in Section 3: if G is a graph such that $\delta(G) \geqslant c_1$ then there exists an integer n (depending on G) such that n tends to infinity with $\delta(G)$ and $|C(G) \cap [3, 4n]| \geqslant \frac{1}{8}n$; moreover, n can be chosen to satisfy $n \geqslant \log[\varepsilon\delta(G)]$. Theorem 1$'$ immediately implies that $C(G)$ has positive upper density if G is an infinite graph having finite subgraphs H_i such that $\delta(H_i)$ tends to infinity with i (Corollary 2). As a special case, $C(G)$ has positive upper density if G is a graph of infinite chromatic number. This was also conjectured by Erdös and Hajnal ([3], p. 37).

The second density result on $C(G)$ is Theorem 2$'$ in Section 4: if G is a graph satisfying $\delta(G) \geqslant c_2$ and $\delta(G) \geqslant 4\varepsilon^{-1} \log^2[|V(G)|]$ then $C(G)$ contains almost all even integers in the interval $[2[\varepsilon\delta(G)]^{1/2} + 2, 3^{-1}[\varepsilon\delta(G) - 1]]$. Here, "almost all" means "with the possible exception of the multiples of $2t$ for some integer $t \geqslant 2$."

The above density results on $C(G)$ will be proved in slightly stronger forms (Theorem 1 and Theorem 2), in which the function $\varepsilon\delta(G)$ is replaced by the conceptually more complicated function

* A result on this problem will appear in [4].

$$p(G, \varepsilon) = \min\{p(H) : H \subset G, \delta(H) \geqslant \varepsilon\delta(G)\}$$

where $p(H)$ denotes the number of vertices in a longest path of H. The reason for doing this is purely technical, as will be apparent in Section 5 where corollaries of Theorem 1 and Theorem 2 are applied to derive lower bounds on $L(G)$. The main result of Section 5 is Theorem 4: for suitable constants a and b, $L(G) \geqslant a \log[\delta(G)]$ if $\delta(G) \geqslant b$.

In Section 2 we introduce the notion of $\frac{3}{2}$-trees, which play a fundamental role in the proofs of Theorem 1 and Theorem 2.

The variables a, b, c, ε will denote positive constants; ε will be used for constants less than 1.

2. THE $\frac{3}{2}$-TREES AND THEIR CROWNS

We use the well-studied concept of rooted trees. Let T be a rooted tree. The vertices whose distance in T from the root is i are at *level* i for $i = 0, 1, \ldots$. We can visualize a rooted tree with its root at the bottom and the different levels above each other in increasing order. The *height* of a tree T is its maximal level number; it is denoted by $h(T)$. The set of vertices of T at level i is denoted by L_i; if $i = h(T)$ we use the more expressive notation *top(T)*. For $x, y \in V(T)$ we write $x \leqslant y$ if the path in T between y and the root of T contains x; $\leqslant$ is a partial ordering on $V(T)$. For $x \in V(T)$ the *cone* of x, denoted by $C(x)$, is the subtree of T containing the vertices $y \in V(T)$ such that $y \geqslant x$. The *root* of $C(x)$ is x. The *cone distance* between $x, y \in V(T)$ is defined as the minimal height of a cone containing both x and y. The cone distances between vertices of T are $0, 1, \ldots, h(T)$.

A $\frac{3}{2}$-*tree* is a tree T such that for every i between 0 and $h(T) - 1$, at least $\frac{3}{4}|L_i|$ vertices of L_i have at least two neighbors in L_{i+1}. Note that a $\frac{3}{2}$-tree has at least three vertices, by definition.

In a $\frac{3}{2}$-tree T obviously

$$|L_{i+1}| \geqslant \tfrac{3}{2}|L_i|, \qquad 0 \leqslant i < h(T), \tag{1}$$

justifying its name. Equally straightforward from (1) is

$$|L_h| \geqslant \tfrac{3}{2}^h \tag{2}$$

where h is the height of T. A slight variation of (1) is the property

$$|L_i| \leqslant |\mathrm{top}(T)|\, \tfrac{2}{3}^{h-i}, \qquad 1 \leqslant i \leqslant h, \tag{3}$$

and the summation of (3) gives

$$|V(T)| \leqslant 3|\mathrm{top}(T)|. \tag{4}$$

A $\frac{3}{2}$-tree $T \subset G$ is called *maximal* in G if T is a $\frac{3}{2}$-tree and T cannot be extended to a $\frac{3}{2}$-tree of height $h(T) + 1$ by the addition of edges of G connecting vertices of $\mathrm{top}(T)$ and $V(G) - V(T)$.

We shall use two types of maximal $\frac{3}{2}$-trees. In Section 3 we use binary maximal $\frac{3}{2}$-trees. (A tree T is *binary* if for every i, $0 \leqslant i < h(T)$, and for every $x \in L_i$, x is connected in T to at most two vertices of L_{i+1}.) In Section 4 saturated maximal $\frac{3}{2}$-trees are used. A tree $T \subset G$ is *saturated* in G if for every i, $0 \leqslant i \leqslant h(T)$, L_i contains all vertices of G whose distance in G from the root of T is i. It is very easy to see that a graph G contains a saturated maximal $\frac{3}{2}$-tree with root x for every $x \in V(G)$ if $\delta(G) \geqslant 2$.

If G is a graph and $x \in V(G)$ then the degree of x in G is denoted by $d_G(x)$. The following technical lemma will be used many times in the paper.

Lemma 1. Let k be a positive integer, let c be a real number. Assume that G is a graph and $M \subset V(G)$ such that $|V(G)| \leqslant c|M|$ and $d_G(x) \geqslant k$ for all $x \in M$. Then there exists a subgraph $H \subset G$ such that $\delta(H) \geqslant k/4c$ and $|V(H) \cap M| \geqslant \frac{1}{4}|M|$.

Proof. It is sufficient to prove the lemma for a graph G which is minimal in the following sense: the removal of any edge of G destroys the property $d_G(x) \geqslant k$ for at least one $x \in M$.

We define $H_1, H_2, \ldots, H_m$ as follows. Let H_1 be defined as G. If H_i is already defined and $d_{H_i}(v) < k/4c$ for some $v \in V(H_i)$ then $H_{i+1} = H_i - v$, otherwise $m = i$. Since G has at least $\frac{1}{2}k|M|$ edges of which fewer than $|V(G)|(k/4c) \leqslant c|M|(k/4c) = \frac{1}{4}k|M|$ are missing from $E(H_m)$, we get $|E(H_m)| > \frac{1}{4}k|M|$. The minimality of G implies that every edge of H_m is incident to a vertex of M of degree k in G. Therefore $|V(H_m) \cap M|k \geqslant |E(H_m)| > \frac{1}{4}k|M|$, and the graph $H = H_m$ satisfies the requirements. ∎

We need the next lemma for the definition of the crown.

Lemma 2. Let G be a graph satisfying $\delta(G) \geqslant 3$ and let T be a maximal $\frac{3}{2}$-tree in G. Then there exists a subgraph $F = F(G, T)$ of G with the following properties:

(a) $E(F) \cap E(T) = \emptyset$,
(b) $\mathrm{top}(T)$ meets every edge of F,
(c) $\delta(F) \geqslant \frac{1}{72}[\delta(G) - 2]$,
(d) $|V(F) \cap \mathrm{top}(T)| \geqslant \frac{1}{16}|\mathrm{top}(T)|$,
(e) $|V(F) \cap \mathrm{top}(T)| \geqslant \frac{1}{72}|V(F)|$.

Proof. Let $A \subset \mathrm{top}(T)$, $Y \subset V(G) - V(T)$, $|Y| = 2|A|$. A tree T' is an (A, Y)-extension of T if $V(T') = V(T) \cup Y$ and $E(T') = E(T) \cup E'$ where E' contains edges of G between A and Y and exactly two edges of E' are incident to x for all $x \in A$. Let us choose T' as an (A, Y)-extension of T for which $|A|$ is maximal. (Note that $A = \emptyset$ is possible, in which case $Y = \emptyset$ and $T' = T$.) Let $B = \text{top}(T) - A$ and let F' be the subgraph of G spanned by the edge-set $\{(x, y) : (x, y) \in E(G) - E(T), x \in B, y \in V(T')\}$. The maximality of A implies that every $x \in B$ is connected to at most one vertex of $V(G) - V(T')$ in G. Moreover, every $x \in B$ is incident to one edge of T. Therefore

$$\delta_{F'}(x) \geq \delta(G) - 2 \quad \text{for all } x \in B. \tag{5}$$

Since T is a maximal $\frac{3}{2}$-tree, $|V(T)| \leq 3|\text{top}(T)|$ by property (4) of $\frac{3}{2}$-trees. On the other hand, $|Y| < \frac{3}{2}|\text{top}(T)|$ and $|\text{top}(T)| < 4|B|$ by the maximality of T. Thus

$$|V(F')| \leq |Y| + |V(T)| < \tfrac{3}{2}|\text{top}(T)| + 3|\text{top}(T)| < 18|B|. \tag{6}$$

Now (5) and (6) ensure that we can apply Lemma 1 with F' in the role of G, B in the role of M, $\delta(G) - 2$ in the role of k ($\delta(G) - 2 \geq 1$), and 18 in the role of c. The subgraph $F \subset F'$ guaranteed by Lemma 1 satisfies properties (a)–(e). ∎

If T is a maximal $\frac{3}{2}$-tree of a graph G then any graph $F = F(G, T)$ defined by Lemma 2 is called a *crown* of T. Let F be a crown of a maximal $\frac{3}{2}$-tree T of G, let $X = V(F) \cap \text{top}(T)$, $Y = V(F) - V(T)$. If $x \in X$ and $t \in V(T)$ then an *(x, t)-join* is either the edge $(x, t) \in E(F)$ or a path (x, y, t) such that $y \in Y$ and $(x, y), (y, t) \in E(F)$. In the latter case the (x, t)-join is called an *out-of-tree join* with *center* y. If (x, t) is an out-of-tree join then $t \in \text{top}(T)$ by the definition of F. Let $x \in X$ and $t \in V(T)$. We say that t is an *n-neighbor* of x if there exists an (x, t)-join and the cone distance between x and t is n. The concepts of join and n-neighbor are illustrated in Figure 1, where the edges of F are heavy lines and the edges of T are dotted lines. There are (x, t_i)-joins in the figure for $1 \leq i \leq 8$. For $i = 5$ and $i = 8$ these are out-of-tree joins. The sets $\{t_1\}$, $\{t_2, t_3, t_4\}$, $\{t_5, t_6\}$, $\{t_7, t_8\}$ are the 1-, 2-, 3-, 4-neighbors of x, respectively.

3. INTERVALS CONTAINING MANY CYCLE LENGTHS

Theorem 1. For suitable constants c_1, ε_1 the following holds. If G is a graph such that $\delta(G) \geq c_1$ then there exists a positive integer $n \geq \log[\frac{1}{4}p(G, \varepsilon_1)]$ such that $|C(G) \cap [3, 4n]| \geq \frac{1}{8}n$.

Theorem 1 immediately implies that the sum of reciprocals in $C(G) \cap [\frac{1}{16}n, 4n]$ is at least $\frac{1}{16}n(1/4n) = \frac{1}{64}$. Thus we get the following corollary.

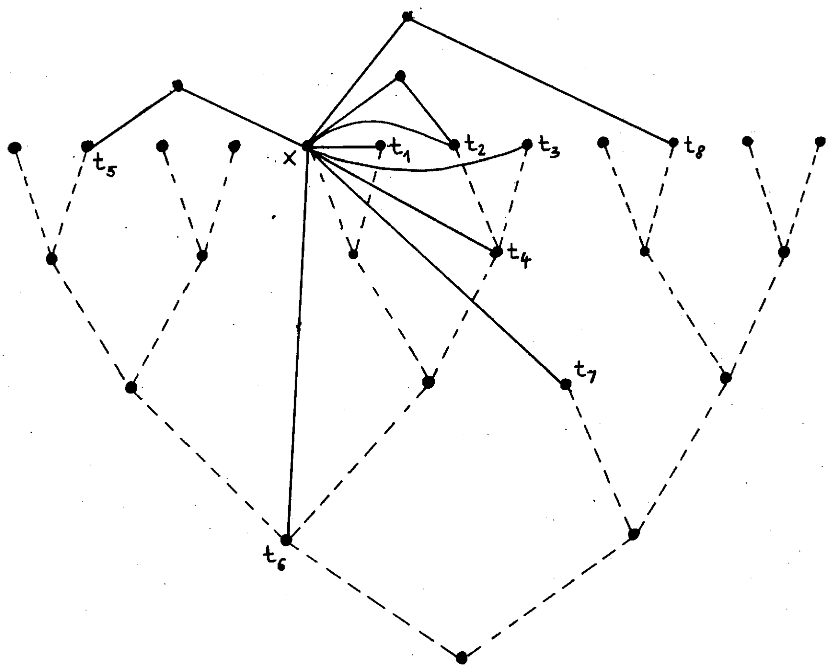

FIGURE 1

Corollary 1. If $\delta(G) \geq c_1$ then the sum of reciprocals in $C(G) \cap [(16)^{-1}\log[\frac{1}{4}p(G, \varepsilon_1)], |V(G)|]$ is at least $\frac{1}{64}$.

Since $p(G, \varepsilon_1) \geq \varepsilon_1\delta(G)$, we can formulate Theorem 1 in a slightly weaker but simpler form.

Theorem 1′. For every graph G satisfying $\delta(G) \geq c_1$, there exists a positive integer $n \geq \log[\frac{1}{4}\varepsilon_1\delta(G)]$ such that $|C(G) \cap [3, 4n]| \geq \frac{1}{8}n$.

Observing that n tends to infinity with $\delta(G)$ in Theorem 1′, we get a result for infinite graphs.

Corollary 2. Let G be an infinite graph containing finite subgraphs $H_1, H_2, \ldots, H_i, \ldots,$ such that $\delta(H_i)$ tends to infinity with i. Then the upper density of $C(G)$ is at least $\frac{1}{32}$.

A graph G of infinite chromatic number obviously satisfies the condition of Corollary 2. Therefore $C(G)$ has positive upper density in this case, as conjectured by Erdös an Hajnal ([3], p. 37).

Proof of Theorem 1. Let G be a graph satisfying $\delta(G) \geq 2$, let T be a binary maximal $\frac{3}{2}$-tree of G of height h, let F be a crown of T, and let

$X = \mathrm{top}(T) \cap V(F)$. The set $\{x \in X : x$ has an i-neighbor$\}$ is denoted by X_i $(1 \leqslant i \leqslant h)$.

Claim 1. For suitable constants d_1, ε_1 the following holds. If $\delta(G) \geqslant d_1$ and $m = \lfloor \log[\frac{1}{4}p(G, \varepsilon_1)] \rfloor$ then $\bigcup_{i=m}^{h} |X_i| \geqslant \frac{1}{2}|X|$.

Proof. We choose ε_1 and d_1 as follows: $\varepsilon_1 = (2 \times 72 \times 4 \times 144)^{-1}$, $d_1 > 2/\varepsilon_1$. Let $A = X - \dot{\bigcup}_{i=m}^{h} X_i$ and assume indirectly that $|A| > \frac{1}{2}|X|$.

Now $|V(F)| \leqslant 72|X| < 144|A|$ by property (e) of crowns and by the indirect assumption. On the other hand $d_F(x) \geqslant \frac{1}{72}[\delta(G) - 2]$ for all $x \in A$ by property (c) of crowns. We apply Lemma 1 with F in the role of G, A in the role of M, 144 in the role of c, $\lfloor \frac{1}{72}[\delta(G) - 2] \rfloor$ in the role of k. Lemma 1 gives a subgraph of $F_1 \subset F$ such that

$$\begin{aligned}\delta(F_1) &\geqslant \left\lfloor [\tfrac{1}{72}\delta(G) - 2] \right\rfloor / (4 \times 144) \\ &> \delta(G)/(2 \times 72 \times 4 \times 144) > \varepsilon_1 \delta(G).\end{aligned}$$

Let S be a longest path of F_1. Since $\delta(F_1) > \varepsilon_1\delta(G)$, $|S| \geqslant p(G, \varepsilon_1)$. Therefore, because every edge of F_1 is incident to a vertex of A, $|S \cap A| \geqslant \frac{1}{2}(|S| - 1) \geqslant \frac{1}{2}[p(G, \varepsilon_1) - 1]$. On the other hand, the choice of A implies that $S \cap A \subset \mathrm{top}(C)$ for some cone C of T of height m. Since T is binary, $|S \cap A| \leqslant 2^m$. Comparing the lower and upper bounds of $|S \cap A|$ and noting that $p(G, \varepsilon_1) \geqslant \varepsilon_1\delta(G) \geqslant \varepsilon_1 d_1 > 2$, we get $m \geqslant \log\{\frac{1}{2}[p(G, \varepsilon_1) - 1]\} > \log[\frac{1}{4}p(G, \varepsilon_1)] \geqslant m$, a contradiction. ∎

Claim 2. Let $\delta(G) \geqslant d_2 = 8d_1$. There exists an integer $n \geqslant m = \lfloor \log[\frac{1}{4}p(G, \varepsilon_1)] \rfloor$ and a $Z_n \subset X$ such that $|Z_n| \geqslant |X|/[2n^2(n + 1)]$ and every vertex of Z_n has an n-neighbor at the same level of T.

Proof. Let $X' = \bigcup_{i=m}^{h} X_i$. We prove that for some n, $m \leqslant n \leqslant h$, $|X_n| \geqslant |X'|/n^2$ holds. The reasoning is indirect. If $|X_i| < |X'|/i^2$ for all i, $m \leqslant i \leqslant h$, then

$$|X'| \leqslant \sum_{i=m}^{h} |X_i| < \sum_{i=m}^{h} \frac{|X'|}{i^2} = |X'| \sum_{i=m}^{h} i^{-2} < |X'| \sum_{i=m}^{\infty} i^{-2},$$

from which we get $1 \leqslant \sum_{i=m}^{h} i^{-2} = \frac{1}{6}\pi^2 - \sum_{i=1}^{m-1} i^{-2}$. However, $m = \lfloor \log[\frac{1}{4}p(G, \varepsilon_1)] \rfloor \geqslant \lfloor \log[4^{-1}\varepsilon_1\delta(G)] \rfloor \geqslant \lfloor \log(4^{-1}\varepsilon_1 8d_1) \rfloor > 2$ since $d_1 > 2/\varepsilon_1$. Thus we get the contradiction $1 \leqslant \frac{1}{6}\pi^2 - 1$.

The n-neighbors of the vertices of X_n are distributed at $n + 1$ different levels of T (at levels h, $h - 1$, ..., $h - n$). Therefore the pigeonhole principle gives $|X_n|/(n + 1) \geqslant |X'|/n^2(n + 1)$ vertices of X with n-neighbors at the same level of T. Since $|X'| \geqslant \frac{1}{2}|X|$ by Claim 1, the proof of Claim 2 is complete. ∎

From now on we use m, n, Z_n as defined in Claim 2, for every graph G satisfying $\delta(G) \geq d_2$.

Claim 3. For a suitable constant $d_3 \geq d_2$, the following holds. If G is a graph satisfying $\delta(G) \geq d_3$ then there exists a cone S of T with height $n - 1$ satisfying the properties

(a) $|\mathrm{top}(S)| \geq \frac{4}{3}^{n-1}$,
(b) $|\mathrm{top}(S) \cap Z_n| > |\mathrm{top}(S)|/33n^2(n + 1)$.

Proof. A cone S of height $n - 1$ in T is called *large* if it satisfies (a), otherwise it is *small*. The number of small cones is at most $|L_{h-n+1}|$ (the number of vertices at level $h - n + 1$ of T) and they contain fewer than $|L_{h-n+1}|\frac{4}{3}^{n-1}$ vertices of top(T); therefore the subset $Z \subset Z_n$ containing vertices above the large cones is large:

$$|Z| > |Z_n| - |L_{h-n+1}|\tfrac{4}{3}^{n-1} \geq |X|/2n^2(n + 1) - |L_{h-n+1}|\tfrac{4}{3}^{n-1}.$$

Observing that $|X| \geq \frac{1}{16}|\mathrm{top}(T)|$ by property (d) of crowns and $|L_{h-n+1}| \leq |\mathrm{top}(T)|\frac{2}{3}^{n-1}$ by property (3) of $\frac{3}{2}$-trees, we get the following lower bound for $|Z|$:

$$|Z| > |\mathrm{top}(T)|\{[32n^2(n + 1)]^{-1} - \tfrac{8}{9}^{n-1}\} = |\mathrm{top}(T)|g(n). \tag{7}$$

We choose an n_0 such that $g(n) \geq [33n^2(n + 1)]^{-1}$ for $n \geq n_0$. If we choose $d_3 \geq 4\varepsilon_1^{-1}2^{n_0}$ then $n \geq m = \lfloor \log[\frac{1}{4}p(G, \varepsilon_1)] \rfloor \geq \lfloor \log[4^{-1}\varepsilon_1\delta(G)] \rfloor \geq n_0$ and (7) implies

$$|Z| > |\mathrm{top}(T)|[33n^2(n + 1)]^{-1}. \tag{8}$$

If B denotes the union of the tops of large cones then $|B|/|Z| \leq |\mathrm{top}(T)|/|Z| < 33n^2(n + 1)$ because of (8). Thus $|\mathrm{top}(S)|/|\mathrm{top}(S) \cap Z| < 33n^2(n + 1)$ for some large cone S, proving the claim with $d_3 = \max\{d_2, 4\varepsilon_1^{-1}2^{n_0}\}$. ∎

For $x \in V(T)$, $L(x)$ and $R(x)$ denote the left and right subtree of T above x, respectively. Note that either $L(x)$ or $R(x)$ or both may be empty (e.g., both are empty if $x \in \mathrm{top}(T)$). A vertex x of the cone S is called a *branching vertex* if neither $L(x) \cap Z_n$ nor $R(x) \cap Z_n$ is empty.

Claim 4. For a suitable constant $d_4 \geq d_3$ the following holds. If G is a graph satisfying $\delta(G) \geq d_4$ then there exists a path $P = (x_1, x_2, \ldots, x_n)$ in the cone S such that $x_1 = s$, the root of S, $x_n \in \mathrm{top}(S)$ and P contains at least $\frac{1}{4}n$ branching vertices.

Proof. The path P is defined by starting from s and continuing always in the subtree containing the larger number of vertices of Z_n. More formally, let $x_1 = s$, and if $x_1, x_2, \ldots, x_r$ are already defined for $r < n$ then

$$x_{r+1} = \begin{cases} \text{the root of } L(x_r) \text{ if } |L(x_r) \cap Z_n| \geqslant |R(x_r) \cap Z_n|, \\ \text{the root of } R(x_r) \text{ if } |R(x_r) \cap Z_n| > |L(x_r) \cap Z_n|. \end{cases}$$

Let $y_1, y_2, \ldots, y_t$ denote the branching vertices of $P = (x_1, \ldots, x_n)$ indexed in their natural order as they appear in P. The existence of at least one branching vertex is guaranteed if $|\text{top}(S) \cap Z_n| \geqslant 2$. Claim 3 shows that this is ensured if $n \geqslant n_1$, where n_1 is the smallest integer satisfying $\frac{4}{3}^{n_1-1}[33n_1^2(n_1 + 1)]^{-1} \geqslant 2$. The definition of P shows that $|C(y_i) \cap Z_n| \leqslant 2|C(y_{i+1}) \cap Z_n|$ $(i = 1, \ldots, t - 1)$, and the definition of t shows that $|C(y_t) \cap Z_n| = 2$. Therefore, by Claim 3,

$$2^t \geqslant |C(y_1) \cap Z_n| = |\text{top}(S) \cap Z_n| > \tfrac{4}{3}^{n-1}[33n^2(n + 1)]^{-1}.$$

From this inequality we get

$$t > (n - 1)[2 - \log(3)] - \log[33n^2(n + 1)] = n[2 - \log(3)] - o(n),$$

showing that $t > \frac{1}{4}n$ if $n \geqslant n_2$, since $2 - \log(3) > \frac{1}{4}$. Let $d_4 = \max\{d_3, 4\varepsilon_1^{-1}2^{n_3}\}$ where $n_3 = \max\{n_1, n_2\}$. In this case $n \geqslant m = \lfloor \log[\frac{1}{4}p(G, \varepsilon_1)] \rfloor \geqslant \lfloor \log[4^{-1}\varepsilon_1\delta(G)] \rfloor \geqslant n_3$, which validates our arguments. ∎

Without restricting the generality of the proof we may assume that the path P goes to the left, i.e., $x_{i+1} \in L(x_i)$ for $i = 1, 2, \ldots, n - 1$.

Let I denote the index-set of the branching vertices of P. For every $i \in I$, we can choose a $z_i \in Z_n$ and a path P_i connecting x_i and z_i in S such that $P_i - x_i \subset R(x_i)$. The tree $P \cup (\cup_{i \in I} P_i)$ is denoted by U. Clearly top(U) contains $|I| + 1$ vertices: z_i for $i \in I$ and x_n. We define z_n as x_n for technical reasons. It is also obvious from the definition that top(U) $\subset Z_n$.

Let U' denote the subtree of T whose vertices are at cone distance n from top(S). The definition of U ensures that every vertex in top(U) has an n-neighbor at the same level of U'; this level of U' is denoted by R. For $i \in I \cup n$, $R(i)$ denotes the set of n-neighbors of z_i in R. We note that $R \subset \text{top}(T)$ when there exists an $i \in I \cup n$ and a $v \in R(i)$ such that the join (z_i, v) is an out-of-tree join. (Figure 2 illustrates P_i, U, U', and R.)

Now we shall define a cycle $H(i)$ in G for all $i \in I$. The definition is given by the following procedure.

Procedure Cycles

C0. Let i be the smallest index in I.

C1. Consider triples (j, a, b) where $j \in I \cup n$, $j > i$, $a \in R(i)$, $b \in R(j)$. Let $U(z_i, z_j)$ be the path connecting z_i and z_j in U. Let

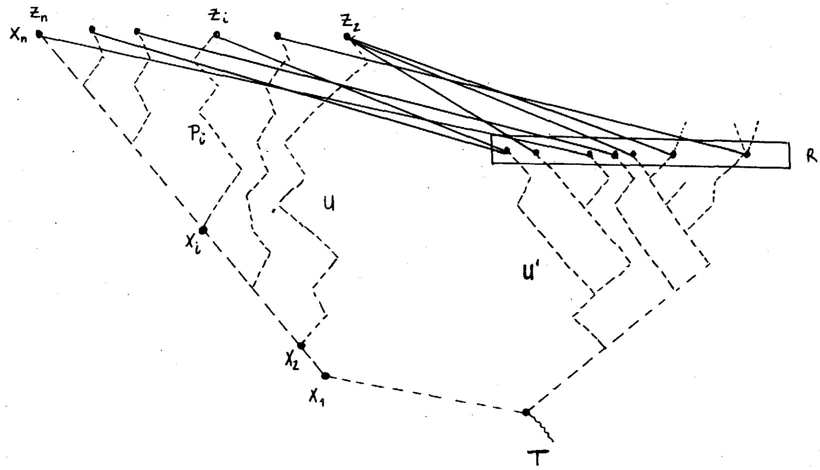

FIGURE 2

$U'(a, b)$ be the path connecting a and b in U'. (Observe that $U(z_i, z_j)$ contains $2(n - i) + 1$ vertices.) Let J_1 be a (z_i, a)-join and let J_2 be a (z_j, b)-join. The union of the joins J_1, J_2 with the paths $U(z_i, z_j)$, $U'(a, b)$ gives a cycle of G unless J_1 and J_2 are out-of-tree joins with the same center $v \in V(G) - V(T)$. Let $\mathscr{H}(i)$ denote the set of cycles in this form, where all triples (j, a, b) and all pairs J_1, J_2 are considered. If $\mathscr{H}(i) = \emptyset$ then continue at C2, otherwise continue at C3.

C2. There is a vertex $v \in V(G) - V(T)$ such that for all triples (j, a, b) and for all J_1, J_2, the joins J_1 and J_2 are out-of-tree joins with center v. We define the cycle $H(j)$ for all $j \in I, j \geq i$, as the union of $U(z_n, z_j)$ with the edges (z_n, v), $(z_j, v) \in E(F)$. The length of $H(j)$ is $2(n - j) + 2$. Stop. (See Fig. 3.)

C3. Define $H(i)$ as an element of $\mathscr{H}(i)$ belonging to a triple $(j(i), a(i), b(i))$ for which the number of vertices in $U'(a, b)$ is maximum. If the number of vertices in $U'(a(i), b(i))$ is denoted by $r(i)$ then the length of $H(i)$ can be expressed as $2(n - i) + 1 + r(i) + e(i)$ where $e(i)$ is defined as 0, 1, or 2 according to the number of out-of-tree joins in $\{J_1, J_2\}$. (See Fig. 4.)

If there are some unprocessed indices in I then i is redefined as the next index in the natural order of I. Continue at C1. If every index is processed then Stop.
End of procedure cycles.

The length of the cycle $H(i)$ can be written in the following form:

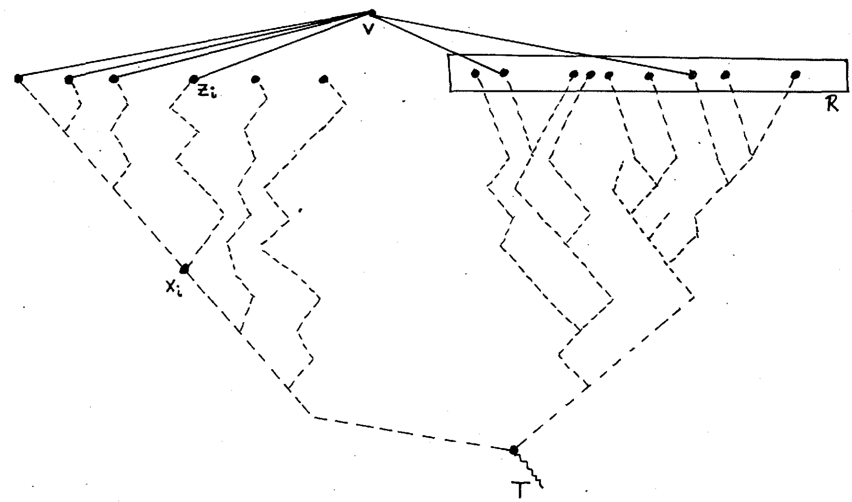

FIGURE 3

$|H(i)| = 2(n - i) + 1 + r(i) + e(i)$ where $r(i) = 1$, $e(i) = 0$ if $H(i)$ was defined in C2.

Claim 5. If $i, i' \in I$ and $i' > i$ then $r(i') \leqslant r(i)$.

Proof. If $H(i')$ was defined in C2 then $r(i') = 1$ and if $H(i)$ was defined in C2 then $r(i') = r(i) = 1$. In both cases, the claim is true.

So assume that both $H(i)$ and $H(i')$ were defined in C3. Let C be the cone of minimal height in U' containing both $a(i)$ and $b(i)$. We shall prove that $a(i'), b(i') \in$ top (C).

Let c denote one of $a(i')$ and $b(i')$. There exists a $k \in I \cup n$, $k > i$, such that $c \in R(k)$, since for $c = a(i')$, $k = i'$ is a good choice and for $c = b(i')$, $k = j(i')$ is a good choice. If we assume indirectly that $c \notin \text{top}(C)$ then the path P_2 connecting c and $a(i)$ in U' and the path P_3 connecting c and $b(i)$ in U' are longer than the path P_1 connecting $a(i)$ and $b(i)$ in U'. Let J_1, J_2 denote the joins in the definition of $H(i)$.

If the triple $(k, a(i), c)$ defines a cycle in G with the $(z_i, a(i))$-join J_1 and with some (z_k, c)-join J_2' then $U'(a(i), c) = P_2$, contradicting the maximality of $U'(a(i), b(i)) = P_1$.

We may therefore assume that J_1 and J_2' are out-of-tree joins with a common center $v \in V(G) - V(T)$ (see Fig. 5). Thus (z_i, v) and (v, c) are edges of F and $J_1' = (z_i, v) \cup (v, c)$ is a (z_i, c)-join. Therefore the triple $(j(i), c, b(i))$ defines a cycle in G with the joins J_1' and J_2. (Note that J_2 does not contain v because J_1 contains v.) Now $U'(c, b(i)) = P_3$, contradicting the maximality of $U'(a(i), b(i)) = P_1$.

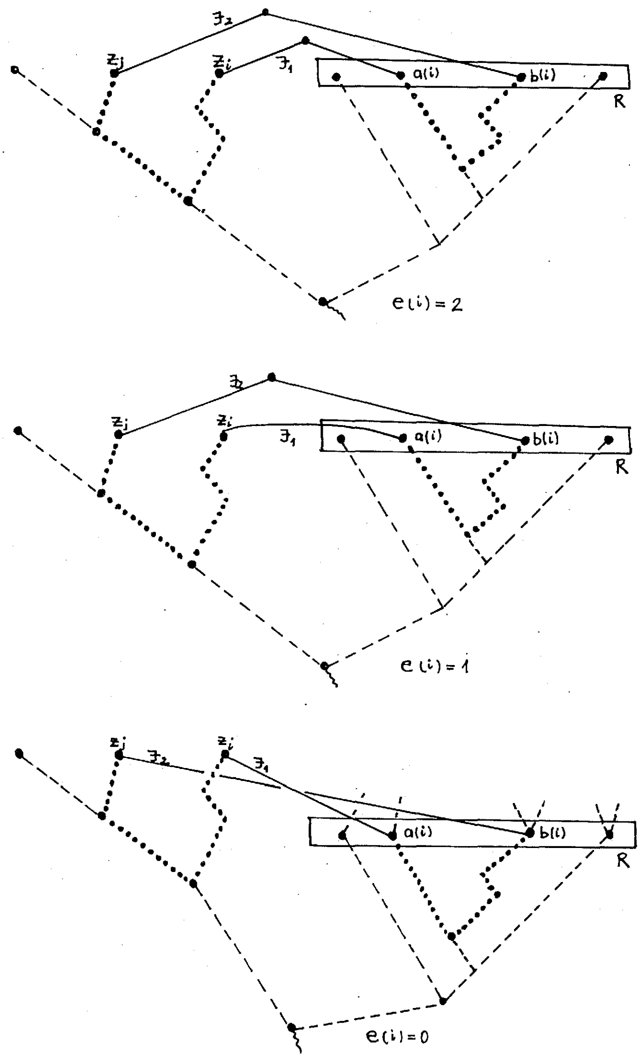

FIGURE 4

We have proved that $a(i'), b(i') \in \operatorname{top}(C)$. This implies $r(i') \leqslant r(i)$. ∎

If $i_1 < i_2 < i_3$ are three consecutive indices of I then

$$|H(i_1)| - |H(i_3)| = 2(i_3 - i_1) + r(i_1) - r(i_3) + e(i_1) - e(i_3) \geqslant 2$$

since $i_3 - i_1 \geqslant 2$, $e(i_1) \geqslant 0$, $e(i_3) \leqslant 2$, and $r(i_1) - r(i_3) \geqslant 0$ by Claim 5. This argument shows that the set $H = \{|H(i)| : i \in I\}$ has at least $\frac{1}{2}|I|$ elements. Since $|I| \geqslant \frac{1}{4}n$ by Claim 4, we get $|H| \geqslant \frac{1}{8}n$. On the other hand, $|H(i)| = 2(n - i) + 1 + r(i) + e(i) \leqslant 2(n - 1) + 1 + 2n - 1 + 2 =$

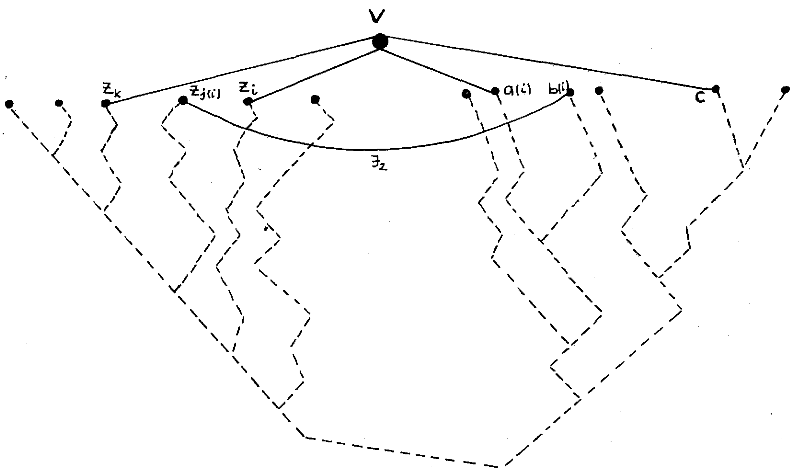

FIGURE 5

$4n$, for every $i \in I$. The proof of Theorem 1 is complete if we choose $c_1 = d_4$ and ε_1 as defined in Claim 1.

4. INTERVALS CONTAINING ALMOST ALL EVEN CYCLE LENGTHS

Let $[A, B]$ denote an interval of positive real numbers. We say that $C(G)$ contains *almost all* even integers in $[A, B]$ if there exists an integer $t \geqslant 2$ such that $C(G)$ contains all even integers of $[A, B]$ with the possible exception of the multiples of $2t$. The proof of the next proposition is omitted since it requires straightforward elementary calculations.

Proposition 1. There is a positive constant a_1 with the following property. If G is a graph and $[A, B]$ is an interval such that $C(G) \cap [A, B] \neq \emptyset$ and $C(G)$ contains almost all even integers in $[A, B]$ then $L(G) \geqslant a_1 \log(B/A)$.

Theorem 2. For suitable constants c_2, ε_2 the following holds. If G is a graph satisfying $\delta(G) \geqslant c_2$ and $p(G, \varepsilon_2) \geqslant 4 \log^2[|V(G)|]$ then $C(G)$ contains almost all even integers in the interval $[4 \log[|V(G)|] + 2, 3^{-1}[p(G, \varepsilon_2) - 1]]$.

Now we formulate Theorem 2′, a slightly weaker but simpler version of Theorem 2.

Theorem 2′. For suitable constants c_3, ε_2 the following holds. If G is a graph and $\delta(G) \geqslant \max\{c_3, 4\varepsilon_2^{-1} \log^2[|V(G)|]\}$ then $C(G)$ contains almost all even integers in the interval

$$[2[\varepsilon_2\delta(G)]^{1/2} + 2, 3^{-1}[\varepsilon_2\delta(G) - 1]].$$

Proof. If $c_3 \geqslant c_2$ then we can apply Theorem 2 for G since $p(G, \varepsilon_2) \geqslant \varepsilon_2\ \delta(G) \geqslant 4\log^2[|V(G)|]$. Theorem 2 ensures that $C(G)$ contains almost all even integers in the interval $[A_1, B_1] = [4\log[|V(G)|] + 2, 3^{-1}[p(G, \varepsilon_2) - 1]]$. Now $A_2 = 2[\varepsilon_2\delta(G)]^{1/2} + 2 \geqslant A_1$ and $B_2 = 3^{-1}[\varepsilon_2\delta(G) - 1] \leqslant B_1$. Therefore $C(G)$ contains almost all even integers from $[A_2, B_2]$ if $A_2 < B_2$ is ensured by choosing c_3 sufficiently large. ∎

Theorem 2 and Proposition 1 easily yield the following result, which we need in Section 5.

Corollary 3. For suitable positive constants a_2, c_4 the following holds. If G is a graph satisfying $\delta(G) \geqslant c_4$ and $p(G, \varepsilon_2) \geqslant 4\log^2[|V(G)|]$ then $L(G) \geqslant a_2\log[\delta(G)]$.

Proof. First, we choose c_4 so that $c_4 \geqslant c_2$, and apply Theorem 2 to deduce that $C(G)$ contains almost all even integers in $[A_1, B_1]$ where $A_1 = 4\log[|V(G)|] + 2$ and $B_1 = 3^{-1}[p(G, \varepsilon_2)]^{1/2} + 2$. Then $A_2 \geqslant A_1$, and so $C(G)$ contains almost all even integers in $[A_2, B_1]$. Now note that $p(G, \varepsilon_2)$ can be made arbitrarily large by an appropriate choice of c_4, because

$$p(G, \varepsilon_2) \geqslant \varepsilon_2\delta(G) \geqslant \varepsilon_2 c_4.$$

We may therefore choose c_4 so that $[A_2, B_1] \cap C(G) \neq \emptyset$. By Proposition 1, $L(G) \geqslant a_1\log(B_1/A_2)$. Similarly, we can choose c_4 so that $B_1/A_2 \geqslant 12^{-1}p(G, \varepsilon_2)^{1/2}$ and $\delta(G) \geqslant \varepsilon_2^{-2}12^4$. Therefore $L(G) \geqslant a_1\log(B_1/A_2) \geqslant a_1\log[12^{-1}p(G, \varepsilon_2)^{1/2}] \geqslant a_1\log\{12^{-1}[\varepsilon_2\delta(G)]^{1/2}\} \geqslant 4^{-1}a_1\log[\delta(G)]$ and the corollary holds with $a_2 = \frac{1}{4}a_1$. ∎

For the proof of Theorem 2 we need the following lemma.

Lemma 3. Let (U, V) be a partition of $\{1, 2, \ldots, n\}$ into two nonempty sets U, V and let

$$D(U, V) = \{d : d = |u - v|, u \in U, v \in V\}.$$

Then there exists an integer $m \geqslant 2$ such that $D(U, V)$ contains all natural numbers not greater than $\frac{1}{3}n$ and not divisible by m.

Proof. If $D(U, V) = \{1, 2, \ldots, n - 1\}$ then the lemma is obviously true. Let m be the smallest natural number not in $D(U, V)$. Clearly $m \geqslant 2$, since $m = 1$ would imply that U or V is empty. Let $U_m = U \cap \{1, 2, \ldots, m\}$ and let $V_m = V \cap \{1, 2, \ldots, m\}$. Then $m \notin D(U, V)$

implies that

$$U = \{u + im : u \in U_m, u + im \leqslant n\},$$
$$V = \{v + jm : v \in V_m, v + jm \leqslant n\}. \tag{9}$$

Let $k = rm + s$, where $0 \leqslant r$, $1 \leqslant s \leqslant m - 1$, and $k \leqslant \frac{1}{3}n$. We prove that $k \in D(U, V)$.

A. If $r = 0$ then $k \in D(U, V)$ by the minimum property of m. Moreover, if we have a pair $u \in U$, $v \in V$ satisfying $|u - v| = s$ then, by the periodicity of U and V expressed in (9), we can find $u(s) \in U$, $v(s) \in V$ such that $|u(s) - v(s)| = |u - v| = s$ and $1 \leqslant u(s), v(s) \leqslant 2m - 1$.

B. Assuming $r \geqslant 1$, we consider $u(s)$ and $v(s)$ as defined in *A*. We define u' and v' as follows: if $u(s) > v(s)$ then $u' = u(s) + rm$ and $v' = v(s)$, otherwise $u' = u(s)$ and $v' = v(s) + rm$. Since $rm < k \leqslant \frac{1}{3}n$, $m < n/3r$. Therefore, $\max(u', v') = rm + max\{u(s), v(s)\} < rm + 2m = (r + 2)m < (r + 2)n/3r \leqslant n$. By (9), $u' \in U$ and $v' \in V$; moreover, $|u' - v'| = rm + s = k$. Thus $k \in D(U, V)$.

Proof of Theorem 2. Assume that $\delta(G) \geqslant 3$. Let T be a saturated maximal $\frac{3}{2}$-tree of G. We apply Lemma 2 to define F, the crown of T. Because T is saturated, there is no edge of F from top(T) to L_i, for $0 \leqslant i \leqslant h(T) - 2$. Since $\delta(F) \geqslant \frac{1}{72}[\delta(G) - 2] = d$ by property (c) of crowns, each vertex of $X = \text{top}(T) \cap V(F)$ belongs to at least one of the following three sets:

$A_1 = \{x \in X$: at least $\frac{1}{3}d$ edges of F go from x to $B_1 = V(F) - V(T)\}$,
$A_2 = \{x \in X$: at least $\frac{1}{3}d$ edges of F go from x to $B_2 = \text{top}(T)\}$,
$A_3 = \{x \in X$: at least $\frac{1}{3}d$ edges of F go from x to $B_3 = L_{h(T)-1}\}$.

We can choose an i $(1 \leqslant i \leqslant 3)$ such that $|A_i| \geqslant \frac{1}{3}|X|$. We define the tree T' as follows: $T' = T$ if $i = 1$ or $i = 2$ and $T' = T - \text{top}(T)$ if $i = 3$.

Assuming that $\frac{1}{3}d \geqslant 1$, i.e.,

$$\delta(G) \geqslant 218, \tag{10}$$

we apply Lemma 1 for the graph spanned by the edges of F which join vertices of A_i to vertices of B_i. The set A_i plays the role of M, $\lfloor\frac{1}{3}d\rfloor$ plays the role of k and $3 \times 72 = 216$ plays the role of c. ($|V(F)| \leqslant 72|X| \leqslant 72 \times 3|A_i|$ holds by property (e) of crowns and by the definition of A_i.) Lemma 1 gives a subgraph $F_1 \subset F$ such that $\delta(F_1) \geqslant \lfloor[\delta(G) - 2]/3 \times 72\rfloor/4 \times 216 \geqslant \delta(G)/2 \times 3 \times 72 \times 4 \times 216$ if $\delta(G) \geqslant 2 \times 3 \times 72 + 2$. If we choose

$$\varepsilon_2 = (2 \times 3 \times 72 \times 4 \times 216)^{-1} \tag{11}$$

then $\delta(F_1) \geqslant \varepsilon_2\delta(G)$. Therefore F_1 contains a path of $p(G, \varepsilon_2)$ vertices. By removing zero, one, or two end-vertices of this path, we obtain a path $P = (x_1, x_2, \ldots, x_r)$ of F_1 with the following properties:

$$\begin{aligned}
&r \text{ is odd and } r \geqslant p(G, \varepsilon_2) - 2,\\
&x_j \in \operatorname{top}(T') \text{ if } j \text{ is odd and } 1 \leqslant j \leqslant r,\\
&x_j \notin V(T') - \operatorname{top}(T') \text{ for } 1 \leqslant j \leqslant r.
\end{aligned}$$

The set $\{x_1, x_3, x_5, \ldots, x_r\}$ is denoted by Y. The tree T' and the path P are shown in Figure 6 for $i = 1, 2, 3$.

Let U' denote the subtree of T' for which $\operatorname{top}(U') = Y$ and $V(U')$ is as small as possible. The root of U' contains at least two branches if $|Y| \geqslant 2$; this can be ensured by setting

$$c_2 = 5\varepsilon_2^{-1}, \tag{12}$$

since $r + 2 \geqslant p(G, \varepsilon_2) \geqslant \varepsilon_2\delta(G) \geqslant \varepsilon_2 c_2$. One branch from the root of U' and the union of the other branches divide Y into two nonempty sets Y_1 and Y_2 with the property that for every pair $x_p \in Y_1$, $x_q \in Y_2$, the number of vertices in the path P_1 connecting x_p and x_q in U' is $2h' + 1$, where $h' = h(U')$. On the other hand, x_p and x_q are connected by a subpath P_2 of P of $|p - q| + 1$ vertices. We define the cycle $H(p, q)$ for every $x_p \in Y_1$, $x_q \in Y_2$ as the union of P_1 and P_2. Clearly $|H(p, q)| = 2h' + |p - q|$ (U' and $H(p, q)$ are shown in Fig. 7).

Now we apply Lemma 3, where (U, V) is the partition of $\{1, 2, \ldots, \frac{1}{2}(r + 1)\}$ defined by

$$i \in U, \quad \text{if } x_{2i-1} \in Y_1, \qquad i \in V, \quad x_{2i-1} \in Y_2.$$

If $d \in D(U, V)$, i.e., $d = |u - v|$ for some $u \in U$, $v \in V$, then

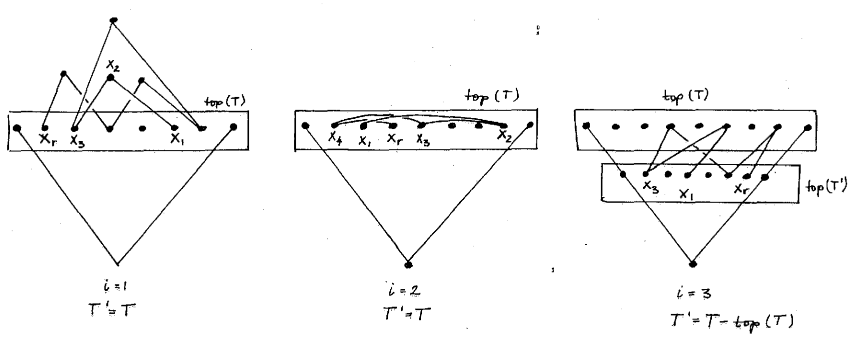

FIGURE 6

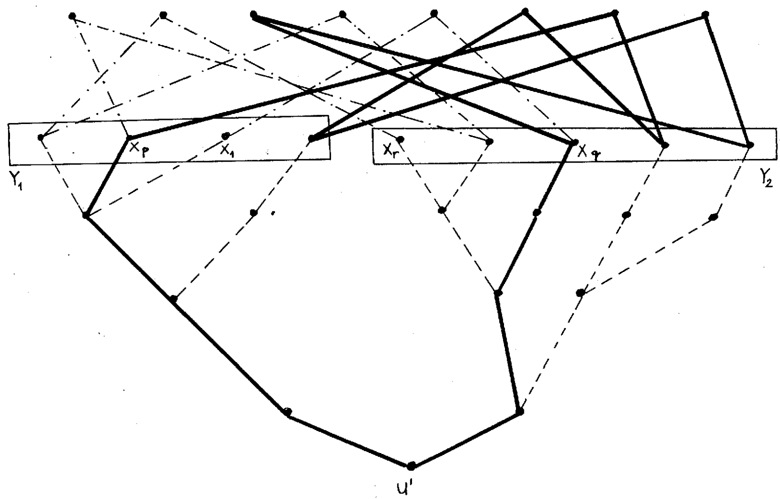

FIGURE 7

$x_{2u-1} \in Y_1$ and $x_{2v-1} \in Y_2$. Therefore G contains the cycle $H(2u - 1, 2v - 1)$ of length $2h' + |2u - 1 - (2v - 1)| = 2h' + 2d$. Since d takes on all values in $[1, \frac{1}{6}(r + 1)]$ except, possibly, the multiples of m, $2h' + 2d$ takes on all even values in $I = [2h' + 2, 2h' + 2 + \frac{1}{3}(r + 1)]$ except, possibly, the multiples of $2m$, where $m \geqslant 2$. In other words, $C(G)$ contains almost all even integers in I.

Since $\frac{3}{2}^{h(T)} \leqslant |\text{top}(T)| \leqslant |V(G)|$ by property (2) of $\frac{3}{2}$-trees,

$$h' \leqslant h(T) \leqslant \log|V(G)|/(\log 3 - 1) \leqslant 2 \log[|V(G)|]. \tag{13}$$

On the other hand, $r + 1 \geqslant p(G, \varepsilon_2) - 1$. Therefore,

$$I \supset [2h' + 2, \tfrac{1}{3}(r + 1)] \supset [4 \log[|V(G)|] + 2, 3^{-1}[p(G, \varepsilon_2) - 1]] = I',$$

and $C(G)$ contains almost all even integers in I', as stated in the theorem, provided that ε_2 and c_2 are chosen according to (11) and (12). ∎

5. THE SUM OF RECIPROCALS OF CYCLE LENGTHS

Let H be a subgraph of a graph G. A vertex $x \in V(H)$ is an *inner vertex* of H if $d_H(x) = d_G(x)$.

Lemma 4. Every graph G has a subgraph H such that $|V(H)| \leqslant 2p(G)$ and H has at least $p(G)$ inner vertices.

Proof. Let $S = (x_1, x_2, \ldots, x_p)$ be a longest path of G. We define paths $S_1, S_2, \ldots, S_p$ and vertices $y_1, y_2, \ldots, y_p$ inductively as follows: $S_1 = S$, $y_1 = x_p$. If $S_t = (z_1, z_2, \ldots z_{m_t})$ and $Y_t = \{y_1, y_2, \ldots, y_t\}$ are already defined, where $t < p$, we define S_{t+1} and y_{t+1} by the rule $S_{t+1} = (z_1, z_2, \ldots, z_{m_{t+1}})$ is a longest path of $G - Y_t$ starting with the vertices $z_1, z_2, \ldots, z_{m_t}$; $y_{t+1} = z_{m_{t+1}}$. Figure 8 shows an example in which G is a tree with $p(G) = 12$.

Let H be the subgraph of G spanned by the set $V(S_1) \cup V(S_2) \cup \cdots \cup V(S_p) \cup Y_p$. Since $V(S_t) \subset V(S_{t+1}) \cup Y_t$ for $1 \leqslant t \leqslant p - 1$, $|V(H)| \leqslant |V(S_p)| + |Y_p| \leqslant 2p$. On the other hand, Y_p is a subset of $V(H)$ with p vertices and, for every t, $1 \leqslant t \leqslant p$, y_t can be joined in G only to vertices of S_t and Y_t. Therefore $d_H(y_t) = d_G(y_t)$ for $1 \leqslant t \leqslant p$. ∎

We need the following corollary of Lemma 4.

Corollary 4. Let G be a graph satisfying $\delta(G) \geqslant 1$. There exists a subgraph $G' \subset G$ such that $\delta(G') \geqslant \frac{1}{8}\delta(G)$ and $|V(G')| \leqslant 2p(G)$.

Proof. We choose $H \subset G$ according to Lemma 4. Let X denote the set of inner vertices of H. Since $|V(H)| \leqslant 2|X|$ and $d_H(x) = d_G(x) \geqslant \delta(G)$ for all $x \in X$, we can apply Lemma 1 for H with $k = \delta(G)$, $c = 2$, $M = X$. Lemma 1 gives a subgraph $G' \subset H$ such that $\delta(G') \geqslant k/4c = \frac{1}{8}\delta(G)$. Moreover, $|V(G')| \leqslant |V(H)| \leqslant 2p(G)$. ∎

We prepare the main result of this section with the following (rather technical) theorem.

Theorem 3. There are constants c_5, a_3, and ε_3 with the following property. If G is a graph satisfying $\delta(G) \geqslant c_5$ then either $L(G) \geqslant a_3 \log[\delta(G)]$ or there exists a subgraph $H \subset G$ such that $\delta(H) \geqslant \varepsilon_3\delta(G)$ and $|V(H)| < 16^{-1} \log[\frac{1}{4}p(G, \varepsilon_1)]$. (The constant ε_1 comes from Theorem 1.)

Proof. The proof is a simple calculation based on repeated applications of Corollary 3 and Corollary 4. Let c_4, a_2, ε_2 be the constants appearing in Corollary 3. We define

$$\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}. \tag{14}$$

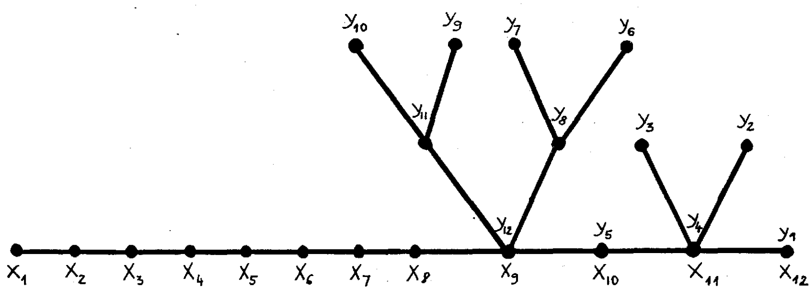

FIGURE 8

The definition of $p(G, \varepsilon)$ ensures a subgraph $G_1 \subset G$ such that

$$\delta(G_1) \geqslant \varepsilon\delta(G), \qquad p(G_1) = p(G, \varepsilon). \tag{15}$$

Applying Corollary 4 for G_1, we get $G_2 \subset G_1$, satisfying

$$\delta(G_2) \geqslant \tfrac{1}{8}\delta(G_1), \qquad |V(G_2)| \leqslant 2p(G_1) = 2p(G, \varepsilon). \tag{16}$$

Assuming that

$$c_5 \geqslant 8\varepsilon^{-1}c_4, \tag{17}$$

$\delta(G_2) \geqslant c_4$ follows from the first parts of (16) and (15). If $p(G_2, \varepsilon) \geqslant 4 \log^2|V(G_2)|$ then we can apply Corollary 3 for G_2, which gives $L(G) \geqslant L(G_2) \geqslant a_2 \log[\delta(G_2)] \geqslant a_2 \log[8^{-1}\varepsilon\delta(G)] \geqslant \frac{1}{2}a_2 \log[\delta(G)]$, where the last inequality holds if

$$c_5 \geqslant 8^2\varepsilon^{-2}. \tag{18}$$

Therefore the first alternative of Theorem 3 holds with $a_3 = \frac{1}{2}a_2$ if (17) and (18) are satisfied.

We may assume $p(G_2, \varepsilon) < 4 \log^2|V(G_2)|$. The second part of (16) implies

$$p(G_2, \varepsilon) < 4 \log^2|V(G_2)| \leqslant 4 \log^2[2p(G, \varepsilon)]. \tag{19}$$

Now the previous argument is repeated. The definition of $p(G_2, \varepsilon)$ ensures $G_3 \subset G_2$ such that

$$\delta(G_3) \geqslant \varepsilon\delta(G_2), \qquad p(G_3) = p(G_2, \varepsilon). \tag{20}$$

Applying Corollary 4 for G_3, we get $G_4 \subset G_3$ satisfying

$$\delta(G_4) \geqslant \tfrac{1}{8}\delta(G_3), \qquad |V(G_4)| \leqslant 2p(G_3) = 2p(G_2, \varepsilon). \tag{21}$$

We try to apply Corollary 3 for G_4 as we did before for G_2. The same argument shows that the condition

$$c_5 \geqslant \max\{8^2\varepsilon^{-2}c_4, 8^4\varepsilon^{-4}\} \tag{22}$$

either ensures the first alternative of Theorem 3 with $a_3 = \frac{1}{2}a_2$ or $p(G_4, \varepsilon) < 4 \log^2|V(G_4)|$. In the latter case the second parts of (21) and (19) imply

$$p(G_4, \varepsilon) < 4 \log^2|V(G_4)| < 4 \log^2\{8 \log^2[2p(G, \varepsilon)]\}. \tag{23}$$

We start the same argument a third time. The definition of $p(G_4, \varepsilon)$ ensures $G_5 \subset G_4$ such that

$$\delta(G_5) \geq \varepsilon\delta(G_4), \qquad p(G_5) = p(G_4, \varepsilon). \tag{24}$$

Applying Corollary 4 for G_5, we get $H \subset G_5$, satisfying

$$\delta(H) \geq \tfrac{1}{8}\delta(G_5), \qquad |V(H)| \leq 2p(G_5) = 2p(G_4, \varepsilon). \tag{25}$$

We prove that H satisfies the second alternative of Theorem 3 if $\varepsilon_3 = 8^{-3}\varepsilon^3$ and c_5 is sufficiently large. The condition $\delta(H) \geq \varepsilon_3\delta(G)$ is guaranteed by the first parts of (25), (24), (21), (20), (16), (15). On the other hand, $\varepsilon \leq \varepsilon_1$ by (14). Therefore $p(G, \varepsilon) \leq p(G, \varepsilon_1)$ and it is sufficient to show that $|V(H)| < 16^{-1} \log[\frac{1}{4}p(G, \varepsilon)]$. Since $|V(H)|$ can be estimated by the second part of (25) and by (23), we have to prove that $8 \log^2\{8 \log^2[2p(G, \varepsilon)]\} < 16^{-1} \log[\frac{1}{4}p(G, \varepsilon)]$, which obviously holds if $p(G, \varepsilon)$ is large enough, say $p(G, \varepsilon) \geq c_0$. This can be ensured by requiring $c_5 \geq c_0\varepsilon^{-1}$.

We conclude that Theorem 3 holds with the following constants: $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$, $c_5 = \max\{8^2\varepsilon^{-2} c_4, 8^4\varepsilon^{-4}, c_0\varepsilon^{-1}\}$, $\varepsilon_3 = 8^{-3}\varepsilon^3$, $a_3 = \frac{1}{2}a_2$. ∎

Theorem 4. For suitable constants a and b the following holds. If G is a graph satisfying $\delta(G) \geq b$ then $L(G) \geq a \log[\delta(G)]$.

Proof. Let c_1, ε_1 be the constants defined in Corollary 1. Let c_5, a_3, ε_3 be the constants defined in Theorem 3. We define the graphs $G_0, G_1, G_2, \ldots, G_t$ inductively as follows. Let $G_0 = G$. Assume that $G_0, G_1, \ldots, G_i$ are already defined for some i, $i \geq 0$. The next procedure either defines G_{i+1} or stops with $G_t = G_i$.

Procedure G_{i+1}.
Step 1. If $\delta(G_i) < c_5$ then $G_t = G_i$. Stop.
Step 2. Apply Theorem 3 for G_i. If the first alternative holds in Theorem 3 then $G_t = G_i$. Stop.
Step 3. If $\delta(G_i) < c_1$ then $G_t = G_i$. Stop.
Step 4. Let G_{i+1} be the subgraph H of G_i ensured by the second alternative in Theorem 3.
End of procedure G_{i+1}.

Since G_{i+1} is a proper subgraph of G_i, the definition eventually ends with some G_t. The definition of G_{i+1} implies that $\delta(G_{i+1}) \geq \varepsilon_3\delta(G_i)$ for all i, $0 \leq i < t$; thus

$$\delta(G_t) \geq \varepsilon_3^t\delta(G). \tag{26}$$

On the other hand, $\delta(G_i) \geqslant c_1$ and $|V(G_{i+1})| < 16^{-1} \log[\frac{1}{4}p(G_i, \varepsilon_1)]$ for all i, $0 \leqslant i < t$. Corollary 1 implies that the sum of reciprocals in $C(G) \cap [|V(G_{i+1})|, |V(G_i)|]$ is at least $\frac{1}{64}$ for all i, $0 \leqslant i < t$. Therefore

$$L(G) \geqslant \tfrac{1}{64}t. \tag{27}$$

Now we need an easy discussion of cases. If G_t is defined in Step 1 then $c_5 > \delta(G_t) \geqslant \varepsilon_3^t\delta(G)$ by (26), which gives $t > \{\log[\delta(G)] - \log(c_5)\}/\log(\varepsilon_3^{-1})$. This estimation and (27) imply

$$L(G) > \frac{\log\delta(G)}{64\log\varepsilon_3^{-1}} - \frac{\log c_5}{64\log\varepsilon_3^{-1}}. \tag{28}$$

If G_t is defined in Step 3 then the previous argument can be followed. We get

$$L(G) > \frac{\log\delta(G)}{64\log\varepsilon_3^{-1}} - \frac{\log c_1}{64\log\varepsilon_3^{-1}}. \tag{29}$$

If G_t is defined in Step 2 then by (27) and by Theorem 3

$$L(G) \geqslant \tfrac{1}{64}t + a_3\log[\delta(G_t)]. \tag{30}$$

If $t \leqslant \log[\delta(G)]/[2\log(\varepsilon_3^{-1})]$ then $L(G)$ can be estimated by the second term of (30):

$$L(G) \geqslant a_3\log[\delta(G_t)] \geqslant a_3\log[\varepsilon_3^t\delta(G)] \geqslant \tfrac{1}{2}a_3\log[\delta(G)]. \tag{31}$$

If $t > \log[\delta(G)]/[2\log(\varepsilon_3^{-1})]$ then $L(G)$ can be estimated by the first term of (30):

$$L(G) \geqslant \tfrac{1}{64}t > [2\log(\varepsilon_3^{-1})64]^{-1}\log[\delta(G)]. \tag{32}$$

In all cases [(28), (29), (31), (32)] we have a lower bound on $L(G)$ of the form $a_i'\log[\delta(G)] - b_i'$, $i = 1, 2, 3, 4$. Let $a' = \min\{a_1', a_2', a_3', a_4'\}$, $a = \frac{1}{2}a'$, $b' = \max\{b_1', b_2', b_3', b_4'\}$. Then

$$L(G) \geqslant a'\log[\delta(G)] - b' \geqslant \tfrac{1}{2}a'\log[\delta(G)] = a\log[\delta(G)]$$

if $\log[\delta(G)] \geqslant 2b'/a'$, which can be ensured by choosing $b = 2^{2b'/a'}$. ∎

Because a graph of density α contains a subgraph of minimum degree $\lfloor\alpha\rfloor + 1 > \alpha$, Theorem 4 has the following consequence.

Theorem 4′. Let a and b be the constants defined in Theorem 4. If G is a graph of density α, where $\alpha \geqslant b$, then $L(G) \geqslant a\log(\alpha)$.

ACKNOWLEDGMENT

The authors thank one of the referees for his many helpful suggestions leading to a thorough revision of the paper.

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