AKADÉMIAI KIADÓ

"LESS" STRONG CHROMATIC INDICES AND THE (7, 4)-CONJECTURE

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ABSTRACT

A proper edge coloring of a graph G is *strong* if the union of any two color classes does not contain a path with three edges (i.e. the color classes are *induced matchings*). The *strong chromatic index* $q(G)$ is the smallest number of colors needed for a strong coloring of G. One form of the famous $(6, 3)$-theorem of Ruzsa and Szemerédi (solving the $(6, 3)$-conjecture of Brown–Erdős–Sós) states that $q(G)$ cannot be linear in n for a graph G with n vertices and cn^2 edges. Here we study two refinements of $q(G)$ arising from the analogous $(7, 4)$-conjecture. The first is $q_A(G)$, the smallest number of colors needed for a proper edge coloring of G such that the union of any two color classes does not contain a path or cycle with four edges, we call it an *A-coloring*. The second is $q_B(G)$, the smallest number of colors needed for a proper edge coloring of G such that all four-cycles are colored with four different colors, we call it a *B-coloring*. These notions lead to two stronger and one equivalent form of the $(7, 4)$-conjecture in terms of $q_A(G), q_B(G)$ where G is a balanced bipartite graph. Since these are questions about graphs, perhaps they will be easier to handle than the original $(7, 4)$-conjecture. In order to understand the behavior of $q_A(G)$ and $q_B(G)$, we study these parameters for some special graphs.

We note that $q_A(G)$ has already been extensively studied from various motivations. However, as far as we know the behavior of $q_B(G)$ is studied here for the first time.

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1. INTRODUCTION

The celebrated $(6, 3)$-theorem of Ruzsa and Szemerédi [32] states that if a 3-uniform hypergraph (or shortly 3-graph) H on n vertices does not contain 3 edges on at most 6 vertices then it has $o(n^2)$ edges. This answers the first special case of the famous Brown–Erdős–Sós conjecture (BES-conjecture, [3], [11]) claiming that if a 3-graph H on n vertices does not contain k edges on at most $k+3$ vertices then it has $o(n^2)$ edges. In their proof, Ruzsa and Szemerédi used Szemerédi's Regularity Lemma [36] to prove an auxiliary result which is now known as the Triangle Removal Lemma (see [6]). This area have been studied extensively since then for example in [1, 6, 7, 16, 19, 23, 34]. The next case of the BES-conjecture, the famous $(7, 4)$-conjecture, is still wide open (throughout the paper we will refer to this conjecture as the $(7, 4)$-problem):

CONJECTURE 1.1. *Assume that a* 3-*graph H on n vertices does not contain 4 edges on at most 7 vertices (briefly we say that H is $(7, 4)$-free). Then $|E(H)| = o(n^2)$.*

Conjecture 1.1 can be equivalently stated using two well-known reductions. The first (see for example Lemma 1.1 in [33]) is the observation that in a $(7, 4)$-free 3-graph H any edge can intersect at most two other edges in two vertices, thus we can keep at least the third of the edges of H so that they form a *linear* 3-graph, i.e. any two edges intersect in at most one vertex. A 3-graph H is 3-partite, if there exists a partition of the vertex set V into 3 classes V_1, V_2, V_3 such that every edge intersects each class in exactly one vertex. In addition, we say that H is balanced if $|V_1| = |V_2| = |V_3|$. Then the second reduction is applying the well-known lemma of Erdős and Kleitman ([13]) allowing to keep at least $\frac{2}{9}$-th proportion of the edges of H forming a balanced 3-partite 3-graph. Thus it is enough to prove Conjecture 1.1 for linear balanced 3-partite 3-graphs. One can easily check (or use the list of four-edge configurations in [8, Figure 13.1]) that in this case the $(7, 4)$-free property means the exclusion of two 3-graphs, C_{14} and C_{16} as shown in Figure 1 (edges are represented by straight line segments). These 3-graphs (especially C_{16} known as the Pasch configuration) are well studied in Steiner triple systems, see [8] for basic facts. It is worth noting that there are C_{14}-free and there are C_{16}-free Steiner triple systems, thus forbidding both is essential in Conjecture 1.1. Moreover, every Steiner triple system contains either C_{14} or C_{16} since extending a triangle with an edge containing two of its midpoints is a C_{14} or a C_{16}.

An interesting test case for Conjecture 1.1 is the sets of lines in $PG(n, 2)$, the projective space of dimension n over $\mathbb{F}_2$. In this case no four lines form a C_{14}, thus Conjecture 1.1 states that any positive proportion of lines in $PG(n, 2)$ contain a Pasch configuration for a large enough n. This is a corollary of a result of Solymosi [33] who proved, relying on the Removal Lemma for 3-graphs proved by Frankl and Rödl [18], that Conjecture 1.1 is true for 3-graphs defined by groups. (We wonder whether any positive proportion of lines in $PG(n, 2)$ contain subspaces as well, in particular a Fano plane.) Recently Solymosi's result was extended and strengthened by Nenadov, Sudakov and Tyomkyn [30] and independently by Long [25] (see also Wong [37]) for the full BES-conjecture in groups (in fact, much denser configurations were found in these special cases). Other recent results on the BES-conjecture include an approximative result of
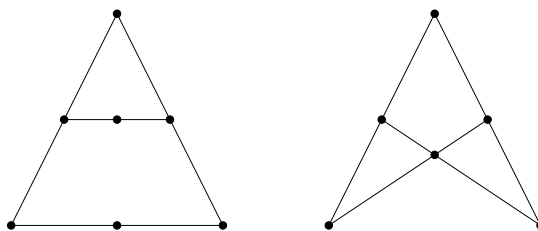
FIGURE 1. Configurations C_{14} and C_{16} (Pasch configuration).

Conlon, Gishboliner, Levanzov and Shapira [7] (improving [34]), a Ramsey variant due to Shapira and Tyomkyn [35] and a proof for large uniformity by Keevash and Long [21].

Our aim here is to investigate proper edge colorings of balanced bipartite graphs emerging from Conjecture 1.1. An edge coloring of a graph G is *proper* if the color classes form *matchings* (i.e., contain pairwise disjoint edges) and the *chromatic index* is the minimum number of colors needed for a proper coloring of G. The proof technique of [32] (see also [22]) applied the Regularity Lemma for proper edge colorings of graphs such that the union of any two color classes does not contain a path with three edges (i.e. the color classes are so-called *induced matchings*). These colorings reappeared later in several applications and were called *strong colorings*. The *strong chromatic index* $q(G)$ of a graph G is the minimum m for which G has a strong m-coloring.

We shall show below how the $(7, 4)$-problem leads to the following less restrictive proper colorings. A proper edge coloring of a graph G is called an *A-coloring* if the union of any two color classes does not contain paths or cycles with four edges; it is called a *B-coloring* if the edges of every four-cycle must be colored with four different colors. Finally, a *C-coloring* satisfies both conditions. We define $q_A(G), q_B(G), q_C(G)$ as the minimum number of colors needed in A-, B-, C-colorings of the edge set of G. In the next subsection we use these notions to formulate two stronger and one equivalent form of Conjecture 1.1 in terms of $q_A(G), q_B(G), q_C(G)$ where G is a balanced bipartite graph.

We note that $q_A(G)$ has already been extensively studied from different motivations. Erdős and the first author [12] studied it in connection with $(9, 6)$-colorings of complete graphs. Dvořak, Mohar and Šámal [10] initiated its systematic study motivated by the star chromatic number and introduced the term *star chromatic index* for $q_A(G)$. However, as far as we know the behavior of $q_B(G)$ is studied here for the first time.

Based on the discussion above Conjecture 1.1 can be rephrased to $\{C_{14}, C_{16}\}$-free *balanced* 3-partite 3-graphs H. We may assume that $V(H) = X \cup Y \cup Z$ where X, Y, Z are pairwise disjoint, $|X| = |Y| = |Z| = n$ (and any two edges of H intersect in at most one vertex). The next standard step is the *projection* (appeared already in [32]), representing H as a balanced bipartite graph

$G = (X \cup Y, E(G))$ with a proper coloring using n colors: for any $z \in Z$ define the color class

$$M_z = \{(x, y) : (x, y, z) \in E(H)\}.$$

Since G is a graph, it might be easier to handle than the original hypergraph H. How to translate the $\{C_{14}, C_{16}\}$-free property of H to G in the projection? The symmetry of C_{16} implies that it appears as an *alternating four-cycle* in G as a subgraph of $M_{z_1} \cup M_{z_2}$. On the other hand, C_{14} can appear in G in two ways. The first is a 3-*colored four-cycle* as a subgraph of $M_{z_1} \cup M_{z_2} \cup M_{z_3}$, the second is an *alternating path* with four edges as a subgraph of $M_{z_1} \cup M_{z_2}$. The projections are shown in Figure 2. These considerations lead to two different questions.
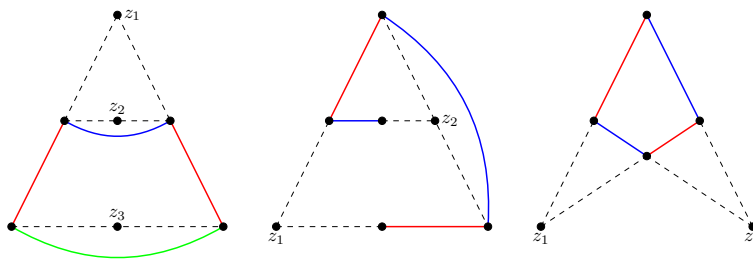

FIGURE 2. Configurations C_{14} and C_{16} with their projections.

QUESTION 1.2 (A-variant). Assume that $G = (X \cup Y, E(G))$ is a bipartite graph, $|X| = |Y| = n$ and $q_A(G) = n$. Is it true that $|E(G)| = o(n^2)$?

QUESTION 1.3 (B-variant). Assume that $G = (X \cup Y, E(G))$ is a bipartite graph, $|X| = |Y| = n$ and $q_B(G) = n$. Is it true that $|E(G)| = o(n^2)$?

Note that an affirmative answer to any of the two questions would provide an affirmative answer to Conjecture 1.1:

PROPOSITION 1.4. *A positive answer to Question 1.2 or Question 1.3 implies Conjecture 1.1.*

In fact, Question 1.2 was asked already by Erdős and Gyárfás ([12] Problem 1.), related to the minimum number of colors for a (not necessarily proper) edge coloring of K_n ensuring at least 9 colors within any 5 vertices. It was predicted that the answer is positive without noticing that it would imply Conjecture 1.1 (see Subsection 2.3 for more details). Dvořak, Mohar and Šámal raised the same problem from a different motivation ([10] Question 1) and introduced the term *star chromatic index*, $\chi'_s(G)$, which is exactly the same as what we denote here by $q_A(G)$. Many follow-up papers have addressed the problems raised in [10]; see the survey by Lei and Shi [26].

However, we could not find a reference to the rather natural Question 1.3. Combining the A-and B-variants, Conjecture 1.1 can be rephrased as follows.

CONJECTURE 1.5 (C-variant). *Assume that $G = (X \cup Y, E(G))$ is a bipartite graph, $|X| = |Y| = n$ and $q_C(G) = n$. Then $|E(G)| = o(n^2)$.*
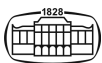
PROPOSITION 1.6. *Conjecture 1.5 and Conjecture 1.1 are equivalent.*

2. RESULTS

2.1. $(7,4)$-analogs of Ruzsa–Szemerédi graphs

A graph G is called an (r,t)-RS graph (*Ruzsa–Szemerédi graph*) if its edge set has a strong coloring with t color classes of size r. These graphs have been studied extensively. Many results focused on the case $r = cn$ (see e.g. Fox, Huang and Sudakov [16] and its references). In a similar vein we shall define (r,t)-A, (r,t)-B, and (r,t)-C balanced bipartite $G(n,n)$ graphs. A graph G is an (r,t)-A graph (similarly for B and C) if its edge set has an A-coloring with t color classes of size r. As in [16], we study that in different ranges for c, how large can t be if $r = cn$. Of course an affirmative answer to Question 1.2 would imply that r and t cannot both be linear in an (r,t)-A graph (similarly for B and C). It turns out that (r,t)-A and (r,t)-B graphs behave differently. First we study (r,t)-A graphs.

THEOREM 2.1. *Assume that $|X| = |Y| = n$ and $G = (X \cup Y, E(G))$ is a (cn,t)-A bipartite graph.*

   (i) If $c > 3/4$, then $t = 1$,
   (ii) If $c = 3/4$, then $t \leq 4$,
   (iii) If $c > 2/3$, then $t \leq \frac{1}{3(c-2/3)}$.

Thus for the range $c > 2/3$, t is still a constant, i.e. not only $rt = o(n^2)$, but actually it is linear. For $c = 3/4$, we can indeed have $t = 4$. See Figure 3 for an A-coloring of the cube with 4 color classes of size 3. Using disjoint copies of cubes, we have an example of a coloring with 4 color classes of size $\frac{3n}{4}$ for any n divisible by 8. By (ii) in the above theorem, this $t = 4$ is best possible.
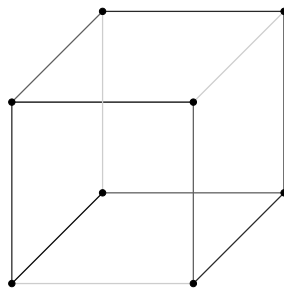

FIGURE 3. Decomposing the cube into 4 color classes of size 3.

In the other direction, the lower bounds on t for (r,t)-RS-graphs are valid for (r,t)-A graphs as well. Thus, the constructions in [16] imply that for $c = 1/2$ we can have $t = \Omega(\log n)$.

The (r,t)-B graphs behave differently. The previous upper bounds on t for (r,t)-A graphs are not valid for (r,t)-B graphs. We can have even (n,t)-B graphs with large t because any proper
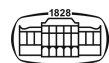
coloring of a C_4-free graph is a B-coloring. Indeed, note that a t-regular C_4-free bipartite graph $G(n, n)$ is an (n, t)-B graph and it is well known that we can have $t = \Omega(n^{\frac{1}{2}})$ (incidence graphs of finite planes). On the other hand, we have the following upper bound.

THEOREM 2.2. *Let $0 < \varepsilon \leq \frac{1}{3}$ and assume that $|X| = |Y| = n$ and $G = (X \cup Y, E(G))$ is a (cn, t)-B bipartite graph with $c = \frac{2}{3} + \varepsilon$. Then $t < (1 - \delta)n$ for sufficiently large n provided that $\frac{9\varepsilon}{6\varepsilon+4} > \delta$.*

Many questions remain, perhaps the most intriguing ones are the following.

- Where does the jump occur between $1/2$ and $2/3$ for the A-variant? More precisely, what is the largest $c \in [\frac{1}{2}, \frac{2}{3}]$ for which $t \to \infty$ when $n \to \infty$ for any sequence of (cn, t)-B bipartite graphs $G(n, n)$?
- Is $t = o(n)$ for any sequence of (n, t)-B bipartite graphs $G(n, n)$? (A very special case of Question 1.3.)
- Is $t < n$ for large enough even n in every $(\frac{n}{2}, t)$-C bipartite graph $G(n, n)$? (A very special case of Conjecture 1.5.)

2.2. Less strong chromatic indices of graphs

Fouguet and Jolivet [17] proposed to study $q(G)$ in connection to frequency assignment problems. Later, independently, Erdős and Nešetřil (at a seminar in Prague at the end of 1985) revitalized the subject, leading to problems and results, for example [4], [14], [27] (Section 10.4), [38] (survey).

The most natural problem is how the parameters q, q_A, q_B, q_C depend on the maximum degree $\Delta = \Delta(G)$. The natural upper bound on $q(G)$ ([14])

$$q(G) \leq 2\Delta^2 - 2\Delta + 1$$

is improved to $(2 - \varepsilon)\Delta^2$ by Molloy and Reed [27] and its conjectured best bound is asymptotic to $\frac{5\Delta^2}{4}$ [14]. On the other hand, as proved in [10], $q_A(G) \leq \Delta 2^{O(1)\sqrt{\log \Delta}}$, i.e. it is almost linear, but again the best bound is open. However, perhaps surprisingly, the best bound for $q_B(G)$ can be determined rather easily. In fact, the upper bound from the greedy coloring gives $q_B(G) \leq \Delta^2$ and this is sharp, shown by the complete bipartite graph.

The natural lower bound

$$q(G) \geq \max_{xy \in E(G)} \{d(x) + d(y) - 1\} \tag{2.1}$$

does not always hold for q_A, q_B (for example a result of Faudree et al. [15] is that for the d-dimensional cube $Q_d, d \geq 4, d \neq 5, q_B(Q_d) = d$). Assume that $|X| = |Y| = n, G = (X \cup Y, E(G))$ is a bipartite graph and let $\bar{d} = \bar{d}(G)$ denote the average degree $|E(G)|/n$. Then (2.1) implies

$$q(G) \geq 2\bar{d} - 1. \tag{2.2}$$

We start with a general lower bound for $q_A(G)$ that is not far off from (2.2).

THEOREM 2.3. *Assume that $|X| = |Y| = n$ and $G = (X \cup Y, E(G))$ is a bipartite graph with average degree $\overline{d} \geq 2$. Then we have*

$$q_A(G) \geq \frac{3\overline{d}}{2} - 3\sqrt{\frac{\overline{d}}{2}}.$$

It is worth noting that a weaker analogue of Theorem 2.3 with $4/3$ instead of $3/2$ was proved by Deng, Liu and Tian in [9]. A slightly better lower bound can be obtained from another assumption.

THEOREM 2.4. *Assume that $|X| = |Y| = n$ and $G = (X \cup Y, E(G))$ is a bipartite graph such that $d(v) = d$ for some $v \in X$ and $d(w) \geq d$ for all $w \in Y$ adjacent to v. Then we have*

$$q_A(G) \geq \left\lfloor \frac{3d}{2} \right\rfloor.$$

Note, that the lower bound of Theorem 2.4 is sharp for every tree satisfying the condition of the theorem and having maximum degree d (see [5] or [9], where it is proved that for each tree T with maximum degree d, we have $q_A(T) \leq \left\lfloor \frac{3d}{2} \right\rfloor$, see also [26]). In the next theorem we apply Theorem 2.4 for cubes, improving the earlier lower bound $\lceil 4d/3 \rceil$ (see [26]).

Next we consider the d-dimensional cube, Q_d, since $q_B(Q_d)$ have already been studied in [15]. We have $q(Q_d) = 2d$ ($d \geq 2$) [14]. For the A- and C-variants we have the following.

THEOREM 2.5. *For $d \geq 3$ we have $\lfloor \frac{3d}{2} \rfloor \leq q_A(Q_d) \leq q_C(Q_d) \leq 2d - 2$.*

The upper bound of $2d - 2$ on $q_A(Q_d)$ was shown earlier by Omoomi and Dastjerdi in [31] (see Corollary 3.2 on page 26 in [31]). Our lower bound on $q_A(Q_d)$ implies that $q_B(Q_d) < q_A(Q_d)$ for $d > 3, d \neq 5$ because a result of Faudree et al. [15] is that for $d \geq 4, d \neq 5$, $q_B(Q_d) = d$. It would be interesting to see how sharp are the inequalities in Theorem 2.5.

2.3. The cases K_n and $K_{n,n}$

It is obvious that $q(K_n) = q_B(K_n) = \binom{n}{2}$ and $q(K_{n,n}) = q_B(K_{n,n}) = n^2$ since no two edges can be colored with the same color in strong or in B-colorings of these graphs. However, to decide whether $q_A(K_n)$ or $q_A(K_{n,n})$ is linear in n seems difficult. In fact, it was predicted in [12] that $q_A(K_n)$ is super-linear and it remains so if K_n is replaced *by any graph with cn^2 edges*. This question is obviously difficult, since it would imply the positive answer to Question 1.2, consequently to Conjecture 1.1. The question about $q_A(K_n)$ is asked independently in [10].

The question about $q_A(K_n)$ emerged first from the following problem in [12]. Estimate $f(n, 5, 9)$, the minimum number of colors needed for a *not necessarily proper* edge-coloring ϕ of K_n such that at least 9 colors appear within every set of 5 vertices. Recalling the argument from [12], it is clear that an A-coloring of K_n must contain at least 9 colors in every K_5 (at most one color can contain two edges within 5 vertices). On the other hand, color classes of ϕ which do not form matchings must have just two intersecting edges and two such color classes cannot intersect. Thus, apart from at most $n/3$ exceptional ones, the color classes of ϕ are matchings. The exceptional ones can be recolored to give at most $2n/3$ one-edge matching. Thus

$$q_A(K_n) - n/3 \leq f(n, 5, 9) \leq q_A(K_n).$$
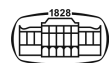
Axenovich [2] proved that

$$\frac{(1 + \sqrt{5})n}{2} - 3 \le q_A(K_n) \le 2n^{1+c/\sqrt{\log n}}.$$

The lower bound is improved to $(1 + o(1))2n$ in [10] and the proof method there also gives $(1 + o(1))2n \le q_A(K_{n,n})$. A further improvement is

$$(1 + o(1))3n \le q_A(K_n),$$

see [5], [28].

3. PROOFS

In the proofs we refer to color classes of proper colorings as *matchings*.

Proof of Proposition 1.4. Assume that we have a positive answer to Question 1.2 (Question 1.3 is similar). We will show that this implies Conjecture 1.1. Consider a 3-graph H on n vertices with $|E(H)| \ge cn^2$, where c is a constant and n is sufficiently large. We must show that H contains either a C_{14} or a C_{16} (see Figure 1). As it is discussed in the Introduction we may assume that H is a balanced, linear 3-partite 3-graph. Thus we may assume that $V(H) = X \cup Y \cup Z$ where X, Y, Z are pairwise disjoint, $|X| = |Y| = |Z|$ (and any two edges of H intersect in at most one vertex). Furthermore, also as in the Introduction we may define the projection $G = (X \cup Y, E(G))$ of H together with a proper coloring using $|Z|$ colors: for any $z \in Z$ define the color class (matching)

$$M_z = \{(x, y) : (x, y, z) \in E(H)\}.$$

Since we have a positive answer to Question 1.2, G contains a path or cycle with 4 edges using only 2 colors. Adding back the corresponding two vertices from Z, we get a C_{14} or C_{16} (see the last two figures in Figure 2), as desired.   □

Proof of Proposition 1.6. The direction that Conjecture 1.5 implies Conjecture 1.1 is similar to the proof of Proposition 1.4. For the other direction, assume that Conjecture 1.1 holds. We must show that Conjecture 1.5 holds as well. Assume that $|X| = |Y| = n$ and $G = (X \cup Y, E(G))$ is a bipartite graph with $|E(G)| \ge cn^2$, where c is a constant, n is sufficiently large and G has a proper edge-coloring with n colors. We must show that there is either a four-cycle with a repeated color or a four-path with only 2 colors. We will apply the fact that Conjecture 1.1 holds. First we define a balanced 3-partite 3-graph H with the *reverse* of the projection above: for each color class (matching) M_i we add a vertex $z_i \in Z$ and for each edge $(x, y) \in M_i$ we add the edge (x, y, z_i) to $E(H)$. By applying the fact that Conjecture 1.1 holds, we find either a C_{14} or a C_{16} in H. Taking the projection of this C_{14} or C_{16} back to $X \cup Y$ we get a four-cycle with a repeated color or a four-path with only 2 colors in G (see Figure 2, we get the first or the second figure from the C_{14} depending on whether we have 3 or 2 vertices in Z), as desired.   □

Proof of Theorem 2.1. Assume that $|X| = |Y| = n$ and $G = (X \cup Y, E(G))$ is a (cn, t)-A bipartite graph. Let us take an A-coloring of G with color classes (matchings) $M_1, \dots, M_t$ of size $r = cn$. Thus the union of any two color classes cannot contain paths or cycles with four edges by definition.

Proof of (*i*). Suppose $t = 2$, so we have two matchings M_1 and M_2 of size $r = cn$ with $c > 3/4$. Set $S = V(M_1) \cap V(M_2) \cap X$. Clearly $|S| > n/2$. Consider the set M_1' of those edges of M_1 that have an endpoint in S and the set M_2' of those edges of M_2 that have an endpoint in S. Then $|S| = |M_1'| = |M_2'| > n/2$ and thus $S' = V(M_1') \cap V(M_2') \cap Y \neq \emptyset$. This creates a four edge path or cycle in the union of M_1 and M_2, a contradiction. □

Proof of (*ii*). To avoid the contradiction of the previous proof, we must have

$$|V(M_i) \cap V(M_j) \cap X| = n/2$$

for any two matchings M_i, M_j. This means that the sets $S_i = X \setminus V(M_i)$ of size $n/4$ must be pairwise disjoint, thus $t \leq 4$. □

Proof of (*iii*). We will use the well-known Johnson bound (see [20]) from coding theory. If we have t sets of size cn on an n-element ground set such that the pairwise intersections have at most k elements, then

$$t \leq \frac{cn - k}{c^2 n - k}, \tag{3.1}$$

assuming that the denominator is positive.

We will apply the Johnson bound for the vertex sets of the matchings $V(M_i)$, so $|V(M_i)| = 2r = 2cn$ on a set with $2n$ elements. First we need a bound on the pairwise intersections. Let us consider two matchings, M_1 and M_2 (say with colors p and q), and put

$$S_X = V(M_1) \cap V(M_2) \cap X, \ S_Y = V(M_1) \cap V(M_2) \cap Y.$$

Then

$$V(M_1) \cap V(M_2) = S_X \cup S_Y.$$

Thus by definition from the vertices in S_X and S_Y there is both an edge in color p and an edge in color q (let us call this a pq-star). The pq-stars from S_X must all be pairwise disjoint and none of them can send two edges to S_Y otherwise we get a path or cycle with 4 edges in the union of the two color classes p and q, contradicting the fact that we have an A-coloring. Thus at least one edge of every pq-star must go from S_X to the symmetric difference of $Y \cap V(M_1)$ and $Y \cap V(M_2)$, implying

$$|S_X| \leq 2(cn - |S_Y|),$$

or

$$|S_X| + 2|S_Y| \leq 2cn. \tag{3.2}$$

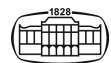Repeating the argument for the pq-stars from S_Y we get

$$|S_Y| + 2|S_X| \leq 2cn. \tag{3.3}$$

Adding (3.2) and (3.3) we get

$$3(|S_X| + |S_Y|) \leq 4cn,$$

or

$$|S_X| + |S_Y| \leq 4cn/3.$$

Then using $c > 2/3$, from the Johnson bound (3.1) with $k = 4cn/3$ we get

$$t \leq \frac{2cn - 4cn/3}{2c^2n - 4cn/3} = \frac{1}{3(c - 2/3)},$$

as desired.   □

                                                                                                                    □

Proof of Theorem 2.2. Let us take a B-coloring with matchings $M_1, \dots, M_t$ of size $r = cn$ with $c = \frac{2}{3} + \varepsilon$. Assume that $t \geq (1 - \delta)n$. We will show that there must be a 4-cycle with two edges from M_i for some i if n is large enough, giving a contradiction. From the assumptions

$$|E(G)| = cnt \geq c(1 - \delta)n^2. \tag{3.4}$$

We will use the following well-known lemma.

LEMMA 3.1 ([24, Lemma 9], see also [29]). *Assume that $|X| = |Y| = n$ and $G = (X \cup Y, E(G))$ is a bipartite graph with $|E(G)| \geq c'n^2$. Then G contains a double star with at least $2c'n$ vertices.*

Using this lemma with $c' = c(1-\delta)$ and (3.4) we get a double star S with center edge $(u, v) \in M_i$ and with order at least $(\frac{4}{3} + 2\varepsilon)(1 - \delta)n$. Therefore for $T = V(G) \setminus V(S)$ we get

$$\begin{aligned}
|T| &\leq 2n - \left(\frac{4}{3} + 2\varepsilon\right)(1 - \delta)n \\
&= \frac{2n}{3} - 2\varepsilon n + \delta\left(\frac{4}{3} + 2\varepsilon\right)n < \left(\frac{2}{3} + \varepsilon\right)n - 1,
\end{aligned} \tag{3.5}$$

provided that $\frac{9\varepsilon}{6\varepsilon+4} > \delta$ and $\frac{1}{n} < 3\varepsilon - \delta(\frac{4}{3} + 2\varepsilon)$.

Indeed, (3.5) is equivalent to

$$\delta\left(\frac{4}{3} + 2\varepsilon\right)n + 1 < 3\varepsilon n,$$

or

$$\frac{1}{n} < 3\varepsilon - \delta\left(\frac{4}{3} + 2\varepsilon\right),$$

(note that here the right hand side is positive if $\frac{9\varepsilon}{6\varepsilon+4} > \delta$).

However, $|T| < (\frac{2}{3} + \varepsilon)n - 1$ implies that there exists an edge $(u', v') \in M_i$ not incident to T and different from (u, v) thus $(u, v), (u', v')$ are in a four-cycle, concluding the proof.   □

Proof of Theorem 2.3. Assume that $|X| = |Y| = n$, $G = (X \cup Y, E(G))$ is a bipartite graph with average degree $d = \bar{d}(G) \geq 2$ and we have an A-coloring with t matchings. We will show that we must have

$$t \geq \frac{3d}{2} - 3\sqrt{\frac{d}{2}}. \tag{3.6}$$

We call a matching *large* in this A-coloring if it has at least $cn = \left(\frac{2}{3} + \varepsilon\right)n$ edges, where $\varepsilon = \frac{1}{3}\sqrt{\frac{2}{d}}$. Note that $\varepsilon \leq \frac{1}{3}$. Otherwise, we call a matching *small*. By *(iii)* in Theorem 2.1 we can have at most

$\frac{1}{3(c-2/3)} = \frac{1}{3\varepsilon}$ large matchings. Thus the number of edges in the union of the small matchings is at least $nd - \frac{n}{3\varepsilon}$. This implies

$$t \geq \frac{nd - \frac{n}{3\varepsilon}}{\left(\frac{2}{3} + \varepsilon\right)n} = \frac{d - \frac{1}{3\varepsilon}}{\frac{2}{3} + \varepsilon} = \frac{d - \sqrt{\frac{d}{2}}}{\frac{2}{3} + \frac{1}{3}\sqrt{\frac{2}{d}}} \geq \frac{3d}{2} - 3\sqrt{\frac{d}{2}},$$

as desired (using the definition of ε). □

Proof of Theorem 2.4. Set $k = \lfloor \frac{d}{2} \rfloor$ for convenience. Let $W = \{w_1, w_2, \dots, w_d\}$ be the set of neighbors of v. Consider an A-coloring of G with colors $1, 2, \dots, d$ on the edges incident to v, say (v, w_i) is colored by i. Suppose indirectly that fewer than k additional colors are used in the A-coloring. Then, for any vertex w_i there are at least $d - 1$ edges going to $X \setminus \{v\}$ and at least

$$(d - 1) - (k - 1) = d - \left\lfloor \frac{d}{2} \right\rfloor = \left\lceil \frac{d}{2} \right\rceil$$

of them are colored with some color from $\{1, 2, \dots, d\}$. Thus the colors of $\{1, 2, \dots, d\}$ are used at least $d(\lceil \frac{d}{2} \rceil)$ times on the edges between W and $X' = X \setminus \{v\}$ implying that some of them, say 1 is used at least $\lceil \frac{d}{2} \rceil$ times. Since the coloring is proper, color 1 is used w.l.o.g. on edges (w_i, y_i) for $i = 2, \dots, \lceil \frac{d}{2} \rceil + 1$ where $y_i \in X'$. We claim that for every $j \in \{1, 2, \dots, \lceil \frac{d}{2} \rceil + 1\}$ no edge $(w_1, y), y \in X'$ can be colored with j. Indeed, for $j = 1$ this is obvious since the coloring is proper. Otherwise, assume that $j \neq 1$ and $(w_1, y), y \in X'$ is colored with j for some $j \in \{2, \dots, \lceil \frac{d}{2} \rceil + 1\}$. Then y, w_1, v, w_j, y_j is a path or a cycle colored by $j, 1, j, 1$ contradicting the definition of the A-coloring, proving the claim. We conclude that from w_1 to X' there are at most

$$d - (\lceil d/2 \rceil + 1) + \lfloor d/2 \rfloor - 1 < d - 1$$

colors, a contradiction proving Theorem 2.4. □

Proof of Theorem 2.5. The lower bound of $q_A(Q_d)$ follows from Theorem 2.4. For the upper bound of $q_C(Q_d)$ we construct inductively a C-coloring of Q_d with $s = 2d - 2$ matchings. Our starting point is the C-coloring of Q_3 with 4 matchings shown in Figure 3.

Assume that we already have a C-coloring of Q_{d-1} with $s - 2$ matchings. We consider Q_d as $K_2 \times Q_{d-1}$ i.e., two vertex-disjoint copies X_1, X_2 of Q_{d-1} where the corresponding vertices are connected with a perfect matching M. In X_1 take a C-coloring with $s - 2$ matchings, $M_1^1, \dots, M_{s-2}^1$ (ensured by the inductive hypothesis). Define the same C-coloring on X_2, denoting by $M_1^2, \dots, M_{s-2}^2$ the corresponding matchings.

To get a C-coloring of Q_d with s colors first we define $s - 2$ colors M_i' by extending M_i^1 with M_{i+1}^2: for $i \in [s - 2]$ set $M_i' = M_i^1 \cup M_{i+1}^2$ using (mod $s - 2$) index arithmetic. The last two colors are defined by partitioning M into two parts M', M'' using the well-known fact that Q_d is a bipartite graph with unique partite classes P, Q: let M' be the set of edges of M from the P-part of X_1 to the Q-part of X_2, similarly, let M'' be the set of edges of M from the Q-part of X_1 to the P-part of X_2. Now the $s - 2$ matchings M_i' together with M' and M'' give a proper edge-coloring π of the edges of Q_d into $s = 2d - 2$ color classes (matchings).
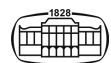
We claim that π is a C-coloring. Indeed, suppose indirectly that we have two edges e, f from the same matching of π such that $e \cup f$ is in a four cycle or in a 2-colored four edge path.

If e, f are in the same copy of Q_{d-1}, say in X_1 then the inductive hypothesis ensures this. Indeed, a four-cycle containing e, f is inside X_1 and this is true for an alternating four edge path containing e, f as well, since an edge leaving X_1 cannot be in the same matching as the edge connecting e, f. It is obviously impossible that one of $\{e, f\}$ is in X_i and the other is in $M' \cup M''$ because of the definition of M', M''.

Therefore we may assume that e, f are in X_1, X_2, respectively. The 4-cycle or the 2-colored 4-edge path containing e, f have to use an edge g from M' or from M'' w.l.o.g. $g \in M'$. Then we have a path with edges e, g, f where g connects a vertex of e in the P-part of X_1 to a vertex of f in the Q-part of X_2. This path cannot be extended to a 2-colored 4-edge path with an edge h. Indeed, since $g \in M'$, either $h \in M''$ or h is inside X_1 or inside X_2 and we can get only a 3-colored path with four edges. Thus the only possibility is that the path e, g, f is extended to a 4-cycle. Denote the vertices of this C_4 as $\{x_1, x_1', x_2', x_2\}$ in cyclic order where $e = (x_1, x_1'), f = (x_2, x_2')$ are edges in the copies of Q_{d-1} in X_1, X_2, respectively, joined by $(x_1, x_2), (x_1', x_2') \in M$. The edges $(x_1, x_2), (x_1', x_2') \in M$ belong to different matchings because one of them goes from the P-part of X_1 and the other from the Q-part of X_1. But then the edges $(x_1, x_1'), (x_2, x_2')$ must also belong to different matchings because $(x_1, x_1') \in M_i'$ implies (by the shifting) that $(x_2, x_2') \in M_{i-1}'$, a contradiction, proving Theorem 2.5. $\square$

4. CONCLUSION

We introduced two refinements of the chromatic index, $q(G)$, arising from the $(7, 4)$-conjecture. The first is $q_A(G)$, the smallest number of colors needed for a proper edge coloring of G such that the union of any two color classes does not contain a path or cycle with four edges, we called it an *A-coloring*. The second is $q_B(G)$, the smallest number of colors needed for a proper edge coloring of G such that all four-cycles are colored with four different colors, we called it a *B-coloring*. These notions led to two stronger and one equivalent form of the $(7, 4)$-conjecture in terms of $q_A(G), q_B(G)$ where G is a balanced bipartite graph. Since these are questions about graphs, perhaps they will be easier to handle than the original $(7, 4)$-conjecture. In order to understand the behavior of $q_A(G)$ and $q_B(G)$, we studied these parameters for some special graphs.

Acknowledgement
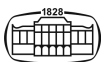
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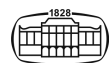
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