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THE STRUCTURE OF RECTANGLE FAMILIES DIVIDING THE PLANE INTO MAXIMUM NUMBER OF ATOMS

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Introduction

Let $\mathcal{A}=\{A_1,\ldots,A_n\}$ be a family of sets. The elements $x, y\in\bigcup_{i=1}^n A_i$ are called equivalent if for every i, $1\leq i\leq n$, $x\in A_i$ if and only if $y\in A_i$. The equivalence classes are called the *atoms* of the family $\mathcal{A}$. Rado asked in [4] the following question: what is the maximum number $f(n, d)$ of atoms, where the maximum is taken over families of n boxes in the d-dimensional Euclidean space. A *box* is a parallelopiped with sides parallel to the coordinate axes. A family of n boxes is *extremal* if it defines $f(n, d)$ atoms. Rado showed that $f(n, 1)=2n-1$. The authors of the present paper proved that $f(n, 2)=2n^2-6n+7$ if $n\geq 2$, determined $f(n, 3)$ asymptotically and gave upper and lower bounds for $f(n, d)$ (see [3]).

The present paper is devoted to the two-dimensional extremal families of boxes which we call *box diagrams*. Our main result, Theorem 3.1, is the characterization of box diagrams. It turns out that all box diagrams can be obtained by a slight modification (peripheral lifting) from a basic type: the caterpillar construction given in Section 2. Box diagrams defined by the caterpillar construction for $n=3$ and $n=4$ are shown in Figs. 6 and 9 in Section 2. We note that this characterization describes the structure of box diagrams completely.

It is remarkable that one-dimensional extremal families have no structural characterization. As proved in [3], these are interval families with connected overlap graphs for which only a non-structural characterization is known (cf. [2]).

We show two consequences of the main result. The first one concerns the enumeration of box diagrams: apart from axial symmetries, there are

$$\binom{2^{n-2}+3\cdot 2^{n-3}+1}{2}$$

combinatorially non-equivalent box diagrams for $n\geq 3$ (Theorem 3.2).

The second consequence of the main result is the characterization of simple box diagrams (Theorem 3.3). We call a box diagram simple if all atoms are connected regions of the plane. Since simple box diagrams for $2 \leq n \leq 4$ are Venn-diagrams (cf. [1]), a side-product of Theorem 3.3 is the catalogue of Venn-diagrams formed by two-dimensional boxes (see Figs. 10 and 11 in Section 3).

1. Preliminaries

A *box* is a closed rectangle with sides parallel to the perpendicular coordinate axes X and Y. Let X^+, X^-, Y^+, Y^- denote the positive and negative halves of X and Y, respectively. A *box system* is a finite set of boxes. We shall always assume that a box system B has the following properties:

(i) The boundary lines of the boxes of B are all different.

(ii) If B contains n boxes then the coordinates of all corners are integers whose absolute values are at most n.

(iii) B has non-empty intersection containing the origin in its interior.

We remark that properties (i) and (ii) are purely technical. Property (iii) is assumed because it is easy to prove that the boxes of a box diagram have non-empty intersection (see [3, Lemma 3.3]).

A box system B naturally defines four linear orders on the boxes of B. If $b_1, b_2 \in B$ then we define

$$\begin{aligned} b_1 >_L b_2 &\quad \text{if } b_1 \cap X^- \supset b_2 \cap X^-, \\ b_1 >_R b_2 &\quad \text{if } b_1 \cap X^+ \supset b_2 \cap X^+, \\ b_1 >_U b_2 &\quad \text{if } b_1 \cap Y^+ \supset b_2 \cap Y^+, \\ b_1 >_D b_2 &\quad \text{if } b_1 \cap Y^- \supset b_2 \cap Y^-. \end{aligned} \tag{1}$$

We refer these orders as L (left), R (right), U (up) and D (down) orders. On the other hand, any four linear orders L, R, U, D on the set $N = \{1, \ldots, n\}$ define a box system $B = \{b_1, \ldots, b_n\}$ as follows. For $i \in N$, let $L(i)$, $R(i)$, $U(i)$, $D(i)$ denote the position of i under L, R, U, D, respectively. For example, $L(i) = k$ means that $i \in N$ is the kth element of N under L. The box b_i is defined by the four lines

$$-x = L(i), \qquad x = R(i), \qquad y = U(i), \qquad -y = D(i).$$

On the basis of the above reasoning, a system of n boxes can be considered as four linear orders on a set of n elements. We shall use both the geometric and combinatorial views.

Two box systems of n boxes, B and B' are *equivalent* if there exists a one-to-one mapping between B and B' preserving the four orders L, R, D, U. If we think of B and B' as four linear orders on $N = \{1, \ldots, n\}$ then the equivalence of B and B' means that a suitable permutation of N maps L into L', R into R', U into U' and D into D'.

Two box systems are called *congruent* if they can be mapped into each other by applying reflections over the axes $x=0$, $y=0$ and the line $x+y=0$. Adopting the combinatorial view, a box system defined by the four linear orders L, R, U, D determines congruent box systems by applying (possibly repeatedly) some of the following three transformations: $L \leftrightarrow R$; $U \leftrightarrow D$; $L \leftrightarrow U$, $R \leftrightarrow D$.

It is obvious that equivalent or congruent box systems have the same number of atoms. As a consequence, box diagrams are closed under equivalence and congruence. Equivalent box diagrams are always considered identical. Congruent box diagrams are considered identical in enumerations and in figures where a catalogue of box diagrams is given.

Two intervals of a line *overlap* each other if they intersect but neither contains the other. The *overlap graph* of an interval system is defined by associating vertices to intervals and two vertices are connected if the corresponding intervals overlap each other. We shall use the following simple lemma established in [3].

Lemma 1.1 ([3]). *Let I be a system of n closed intervals without common endpoints. If I has a connected overlap graph then I defines $2n-1$ atoms.*

Let $B_n=\{b_1, \ldots, b_n\}$ be a box system. The boxes $b_i, b_j \in B_n$ *horizontally* (*vertically*) *overlap* each other if the intervals $X \cap b_i$ and $X \cap b_j$ ($Y \cap b_i$ and $Y \cap b_j$) overlap each other. The *horizontal* and *vertical overlap graphs* of B_n are defined as the overlap graphs of $\{X \cap b_1, \ldots, X \cap b_n\}$ and of $\{Y \cap b_1, \ldots, Y \cap b_n\}$, respectively. Using the linear orders defined in (1), the horizontal (vertical) overlap of two boxes means that they are compared oppositely under L and R (under U and D).

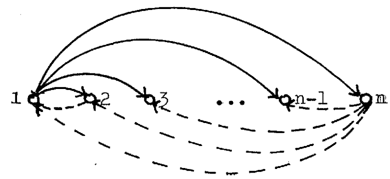
The number of atoms in a family B of boxes is denoted by $a(B)$.

2. The caterpillar construction and its peripheral liftings

Let $n, p, q, i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q$ be integers satisfying

$$\begin{aligned} &1=i_1<i_2<\cdots<i_p=n, \\ &n=j_1>j_2>\cdots>j_q=1. \end{aligned} \tag{2}$$

We define the caterpillar $C_v=C_v(n; i_1, \ldots, i_p)$ on the vertex set $N=\{1, \ldots, n\}$ with edges (i_m, k) for all k and m satisfying $i_m<k \leqslant i_{m+1}$, $1 \leqslant m \leqslant p-1$. The caterpillar $C_h=C_h(n; j_1, \ldots, j_q)$ is defined on the vertex set N with edges (j_m, k) for all k and m satisfying $j_{m+1} \leqslant k < j_m$, $1 \leqslant m \leqslant q-1$. We call C_v and C_h vertical and horizontal caterpillars. Let $C_v^+, C_v^-, C_h^+, C_h^-$ denote the directed graphs defined by the transitive orientations of C_v and C_h. (A tree has exactly two transitive orientations.) Now we define four linear orders, U, D, L, R on N as



boxes	U	D	L	R
1	1	n	n	n-1
2	2	1	n-1	n-2
3	3	2	n-2	n-3
⋮				
n-1	n-1	n-2	2	1
n	n	n-1	1	n

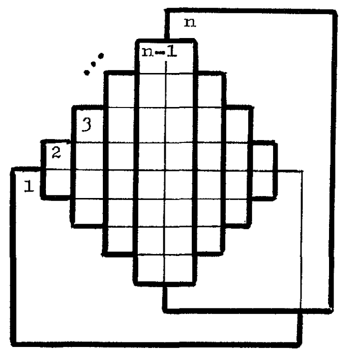
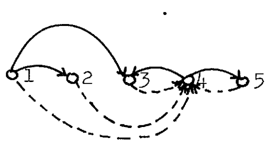


Fig. 1. The caterpillar construction with two stars.

follows:

$$
\begin{aligned}
i <_U j \quad & \text{if } i<j \text{ and } (i,j)\notin E(C_v) \text{ or } (i,j)\in E(C_v^+),\\
i <_D j \quad & \text{if } i<j \text{ and } (i,j)\notin E(C_v) \text{ or } (i,j)\in E(C_v^-),\\
i <_L j \quad & \text{if } i>j \text{ and } (i,j)\notin E(C_h) \text{ or } (i,j)\in E(C_h^+),\\
i <_R j \quad & \text{if } i>j \text{ and } (i,j)\notin E(C_h) \text{ or } (i,j)\in E(C_h^-).
\end{aligned}
\tag{3}
$$

It is easy to see that (3) defines four linear orders on $N=\{1,\ldots,n\}$ for each parameter set satisfying (2). These linear orders and the corresponding box system are referred as the *caterpillar construction* (C_v, C_h). Note that C_v and C_h are the vertical and horizontal overlap graphs of the box system (C_v, C_h). Two special cases of the caterpillar construction are displayed in Figs. 1 and 2. Catalogues of caterpillar constructions for $n=3, 4$ are given in Figs. 6 and 9, also in this section.



	U	D	L	R
1	1	3	4	5
2	2	1	3	4
3	4	2	2	3
4	3	5	5	1
5	5	4	1	2

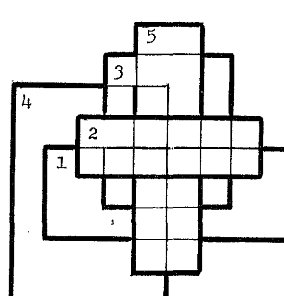


Fig. 2. A caterpillar construction for $n=5$.

Theorem 2.1. *Box systems defined by caterpillar constructions are box diagrams.*

Proof. We have to verify that a box system defined by the caterpillar construction with $C_v = C_v(n; i_1, \ldots, i_p)$ and $C_h = C_h(n; j_1, \ldots, j_q)$ has $2n^2 - 6n + 7$ atoms for $n \geq 2$. We apply induction on n. The case $n = 2$ is trivial (see Fig. 5).

Let us consider a caterpillar construction with $C_v = C_v(n+1; i_1, \ldots, i_p)$ and $C_h = C_h(n+1; j_1, \ldots, j_q)$. We may assume by obvious symmetry reasons that the edges of C_v^+ and C_h^+ are going out of vertex 1. The consequence of this assumption is that

$$U(1) = 1, \qquad D(2) = 1, \qquad L(1) = n, \qquad R(1) = n + 1.$$

We have to look at four similar cases.

Case (a) $i_2 = 2$, $j_{q-1} = 2$. Now $D(1) = 2$, $L(2) = n + 1$, $R(2) = n - 1$ (see Fig. 3(a)).

Case (b) $i_2 = 2$, $j_{q-1} > 2$. Now $D(1) = 2$, $L(2) = n - 1$, $R(2) = n$ (see Fig. 3(b)).

Case (c) $i_2 > 2$, $j_{q-1} = 2$. Now $U(2) = 2$, $L(2) = n + 1$, $R(2) = n - 1$, $L(3) = n - 1$ (see Fig. 3(c)).

Case (d) $i_2 > 2$, $j_{q-1} > 2$. Now $U(2) = 2$, $L(2) = n - 1$, $R(2) = n$ (see Fig. 3(d)).

The common feature of all four cases is that box b_1 (in Cases (a) and (b)) or box b_2 (in Cases (c) and (d)) is suitable to carry out the induction. We claim that b_1 (in Cases (a) and (b)) or b_2 (in Cases (c) and (d)) contains $4(n-1)$ atoms.

In Cases (a) and (b), the atoms in b_1 can be counted along the bottom line l_2 of b_2. The intersections of $b_2, b_3, \ldots, b_{n+1}$ with l_2 define n intervals with a connected overlap graph. By Lemma 1.1, there are $2n - 1$ atoms on l_2 and all but one support two box atoms in b_1. Thus we have $2(2n-2) = 4(n-1)$ atoms in b_1. In Cases (c) and (d), the atoms of b_2 can be counted along the top line l_1 of b_1. In Case (d), the intersections of $b_1, b_3, \ldots, b_{n+1}$ with l_1 define n intervals with connected overlap graph and the argument is the same as before. In Case (c), the intersections of $b_3, b_4, \ldots, b_{n+1}$ with l_1 have a connected overlap graph, therefore they define $2n - 3$ atoms. As $l_1 \cap b_1$ contains all these intervals, there are $2n - 2$ atoms defined by $l_1 \cap b_1, l_1 \cap b_3, \ldots, l_1 \cap b_{n+1}$ on l_1. Now every atom supports two box atoms in b_2, and our claim is proved.

In Cases (a) and (b), all atoms of $B_{n+1} - \{b_1\}$ having representative points inside b_1, intersect the boundary of b_1. The consequence is the equality $a(B_{n+1}) = a(B_{n+1} - \{b_1\}) + 4(n-1)$. Similarly we get $a(B_{n+1}) = a(B_{n+1} - \{b_2\}) + 4(n-1)$ in Cases (c) and (d). Since $B_{n+1} - \{b_1\}$ ($B_{n+1} - \{b_2\}$) is given by a caterpillar construction in Cases (a) and (b) (in Case (d)), the inductive hypothesis gives the theorem for the Cases (a), (b) and (d) because $2n^2 - 6n + 7 + 4(n-1) = 2(n+1)^2 - 6(n+1) + 7$. In Case (c), $B_{n+1} - \{b_2\} = B'$ is not a caterpillar construction; however, it is very close to it. In B' $L(1) = n$ and $L(3) = n - 1$. Since b_1 and b_3 vertically overlap each other, the exchange of the left sides of b_3 and b_1 does not change $a(B')$. This operation leads to a caterpillar construction with n boxes. □

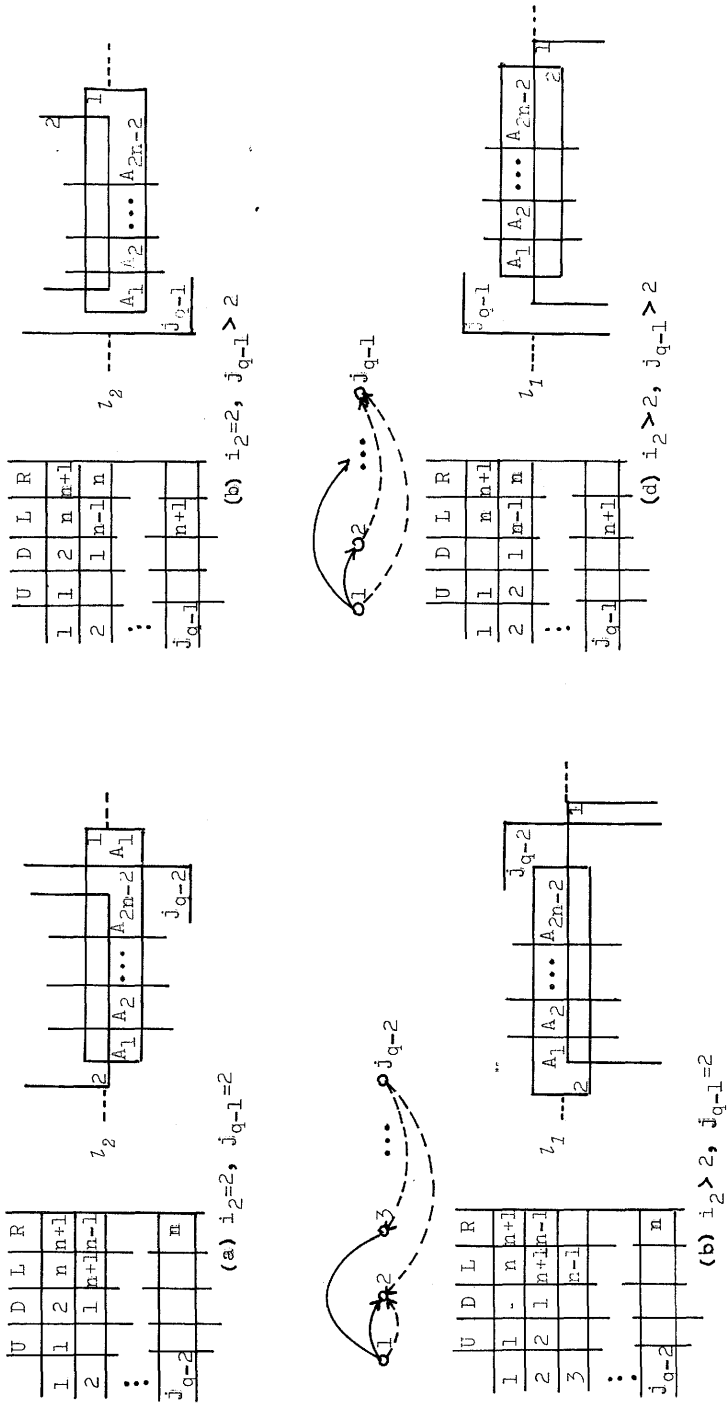


Fig. 3. Four cases in the proof of Theorem 2.1. (a) $i_2=2$, $j_{q-1}=2$; (b) $i_2=2$, $j_{q-1}>2$; (c) $i_2>2$, $j_{q-1}=2$; (d) $i_2>2$, $j_{q-1}>2$.

Now we introduce three minor modifications of the caterpillar construction $(C_{\mathrm{v}}, C_{\mathrm{h}})$. By symmetry reasons we assume $(i_{p-1}, n) \in E(C_{\mathrm{v}}^{+})$.

Augmentation. Assume that $i_{p-1} < n-1$ and let C^{+} denote the graph obtained by the addition of the edge $(n, n-1)$ to C_{v}^{+}. The box system defined by (3) with C^{+} and C^{-} in the role of C_{v}^{+} and C_{v}^{-} respectively, is called the augmentation of $(C_{\mathrm{v}}, C_{\mathrm{h}})$. The augmentation exchanges the order of b_{n-1} and b_n under U. Since b_{n-1} and b_n are horizontally overlapping, the augmentation does not change the number of atoms (see Fig. 4(a)).

One-point cut. Let C^{+} denote the graph obtained from C_{v}^{+} by removing the edge (i_{p-1}, n). The box system defined by (3) with C^{+} and C^{-} in the role of C_{v}^{+} and C_{v}^{-} respectively, is called the one-point cut of $(C_{\mathrm{v}}, C_{\mathrm{h}})$. The one-point cut exchanges the order of $b_{i_{p-1}}$ and b_n under U. Since either $b_n <_L b_{i_{p-1}}$ or $b_n <_R b_{i_{p-1}}$ holds, the one-point cut does not change the number of atoms (see Fig. 4(b)).

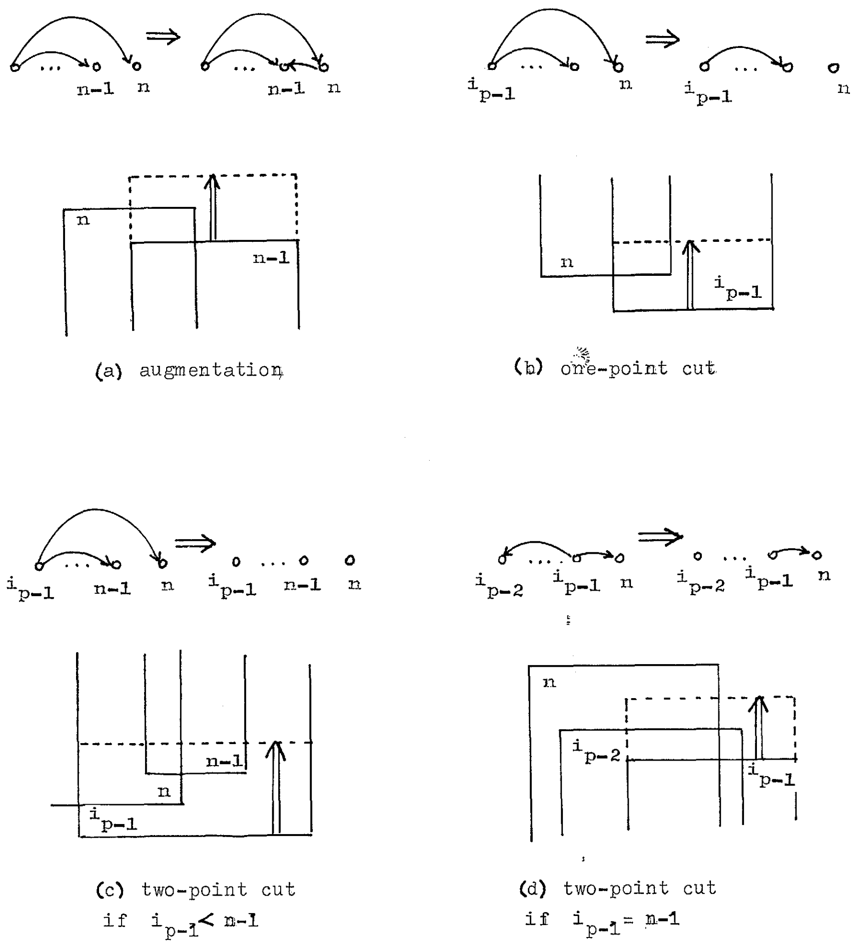


Fig. 4. The vertical transformations of caterpillar constructions. (a) augmentation; (b) one-point cut; (c) two-point cut if $i_{p-1} < n-1$; (d) two-point cut if $i_{p-1} = n-1$.

Two-point cut. Let C^+ denote the graph obtained from C_v^+ by removing the edges between the vertex sets $\{1, \ldots, n-2\}$ and $\{n-1, n\}$. The box system defined by (3) with C^+ and C^- in the role of C_v^+ and C_v^- respectively, is called the two-point cut of (C_v, C_h). By analysing the relation of the involved boxes under L and R, one can show easily that the two-point cut does not change the number of atoms. Its effect is illustrated in Figs. 4(c) and 4(d) corresponding to the cases $i_{p-1} < n-1$ and $i_{p-1} = n-1$.

The above-mentioned transformations which we call vertical transformations have horizontal analogons. It is easy to see that for $n \geq 4$ the vertical and horizontal transformations can be applied simultaneously to a caterpillar construction without changing the number of atoms. These transformations for $n = 3$ are shown in Figs. 7 and 8. Suggested by the geometric view, the box systems obtained from a caterpillar construction by vertical and/or horizontal transformations, are referred as the *peripheral liftings* of the caterpillar construction. The discussion above is summarized in the following theorem.

Theorem 2.2. *Peripheral liftings of caterpillar constructions are box diagrams if the number of boxes is at least four.*

The caterpillar constructions and their peripheral liftings for $n = 2$ and 3 are shown in Figs. 5, 6, 7 and 8. Note that the box systems of Fig. 8 are not box diagrams. For $n = 4$, see Fig. 9.

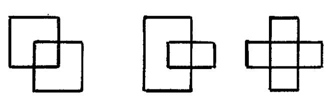


Fig. 5. The box diagrams for $n = 2$.

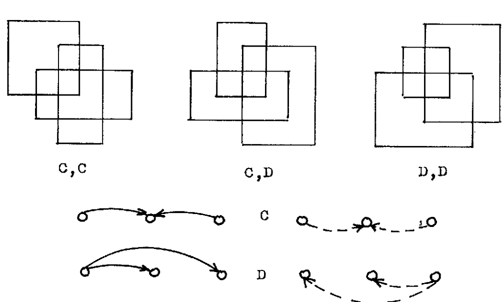


Fig. 6. The caterpillar constructions for $n = 3$.

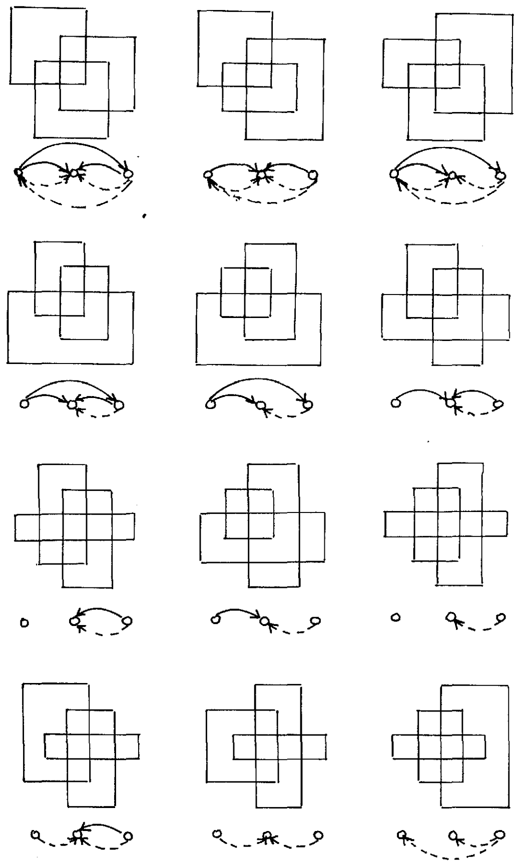


Fig. 7. The box diagrams obtained by peripheral liftings for $n = 3$.

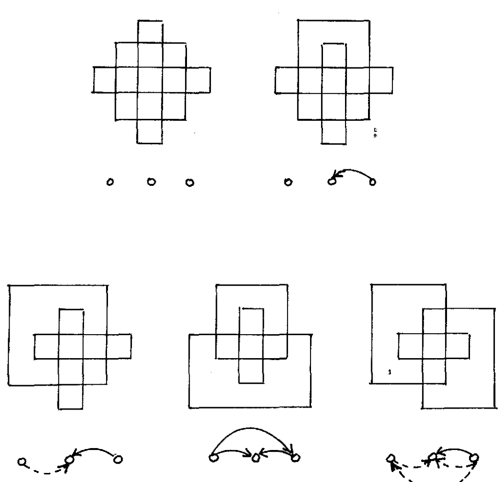


Fig. 8. Peripheral liftings, for $n = 3$, reducing the number of atoms.

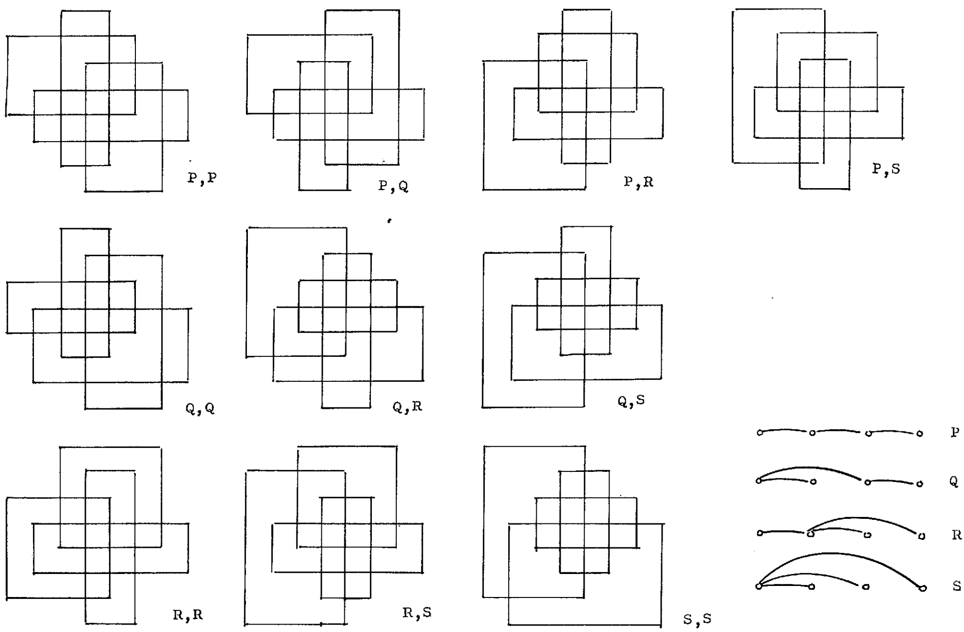


Fig. 9. The caterpillar constructions for $n=4$.

3. Characterization of box diagrams

Now we are ready to state the main result of the paper, the characterization of box diagrams.

Theorem 3.1. *All box diagrams can be obtained as caterpillar constructions and their peripheral liftings.*

The proof of Theorem 3.1 is given in Section 4; here we present some consequences. First we enumerate box diagrams. *Congruent box diagrams are considered identical.* By Theorems 3.1, 2.1 and 2.2, we have to enumerate caterpillar constructions and their peripheral liftings.

Let (C_v, C_h) be a caterpillar construction. The exchange of C_v^+ and C_v^- in (3) yields the axial symmetry $U \leftrightarrow D$. Similarly, the exchange of C_h^+ and C_h^- in (3) yields the axial symmetry $L \leftrightarrow R$. If $C_v = C_v(n; i_1, \ldots, i_p)$ and $C_h = C_h(n; j_1, \ldots, j_q)$ then let $p' = q$, $q' = p$, $i_1' = j_q, i_2' = j_{q-1}, \ldots, i_{p'}' = j_1$, $j_1' = i_p$, $j_2' = i_{p-1}, \ldots, j_{q'}' = i_1$. Now $C_v' = C_v'(n; i_1', \ldots, i_p')$ and $C_h' = C_h'(n; j_1', \ldots, j_{q'}')$ also define a caterpillar construction. The box diagrams belonging to (C_v, C_h) and (C_v', C_h') can be obtained from each other by the axial symmetry $U \leftrightarrow L$, $D \leftrightarrow R$.

Since the inequalities of (2) have 2^{n-2} integer solutions for fixed n, the number of caterpillar constructions (apart from congruence) is equal to

$$\binom{2^{n-2}+1}{2} \quad \text{for } n \geqslant 2.$$

Inspection shows that each of the three transformations (augmentation, one-point cut, two-point cut) define 2^{n-3} modified C_v^+'s (for $n \geq 4$). Repeating the previous argument, we obtain that the number of caterpillar constructions together with their peripheral liftings is equal to

$$\binom{2^{n-2}+3\cdot 2^{n-3}+1}{2} \quad \text{if } n \geq 4.$$

It is easy to see that these box diagrams are pairwise non-equivalent. So we obtain:

Theorem 3.2. *The number of non-equivalent box diagrams is equal to*

$$\binom{2^{n-2}+3\cdot 2^{n-3}+1}{2} \quad \textit{for } n \geq 4.$$

Remark. For $n=2$ there are 3 non-equivalent box diagrams (see Fig. 5). For $n=3$ there are 15 non-equivalent box diagrams (see Figs. 6 and 7). The formula of Theorem 3.2 is accidentally valid for $n=3$.

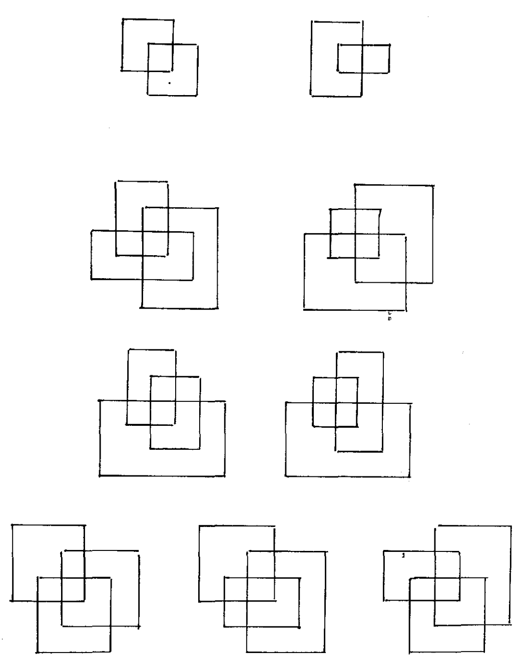


Fig. 10. Venn-diagrams by 2 and 3 boxes.

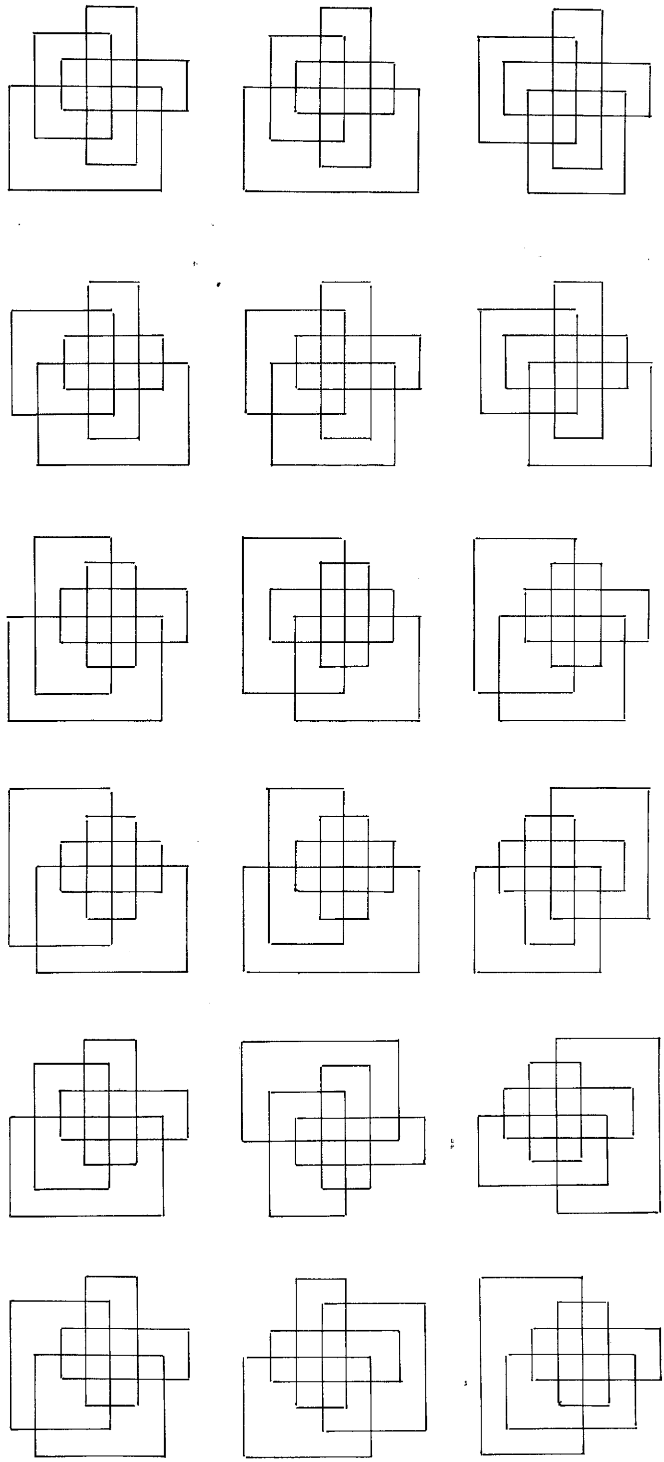


Fig. 11. Venn-diagrams by four boxes.

It may happen that a box diagram has disconnected atoms. The caterpillar construction (C, C) in Fig. 6 shows this possibility. We call a box diagram *simple* if its atoms are connected regions in the plane. Theorem 3.1 gives easily the following characterization of simple box diagrams.

Theorem 3.3. *A box diagram is simple if and only if it can be defined by one of the following constructions:*

(i) *a caterpillar construction* (C_v, C_h) *such that* C_v *and* C_h *have just one common edge;*

(ii) *vertical, horizontal or simultaneous augmentation of* (i);

(iii) *horizontal (vertical) one-point cut of a caterpillar construction* (C_v, C_h) *where* C_v *is a star* (C_h *is a star*).

Since simple box diagrams for $2 \leqslant n \leqslant 4$ are Venn-diagrams (cf. [1]), Theorem 3.3 gives all Venn-diagrams formed by boxes. The catalogue of these Venn-diagrams is shown in Figs. 10 and 11.

4. Properties of box diagrams and the proof of the main result

The aim of this section is to prove Theorem 3.1.

A family of boxes is called *connected* if both its vertical and horizontal overlap graphs are connected. Our first theorem shows that the problem of characterizing box diagrams can be reduced to characterizing connected box diagrams.

Theorem 4.1. *Let* $B = \{b_1, \ldots, b_n\}$ *be a box diagram which is not connected. Then* B *can be obtained by (vertical and/or horizontal) one-point or two-point cuts from a connected box diagram.*

Proof. Assume that the vertical overlap graph of B is not connected. It is easy to see that $B = B_1 \cup \cdots \cup B_k$ for some $k \geqslant 2$ where the vertical overlap graph of B_i is connected for $1 \leqslant i \leqslant k$; moreover, if $b \in B_i$, $b' \in B_j$ and $1 \leqslant i < j \leqslant k$ then $b <_U b'$ and $b <_D b'$.

We are going to show $|B_1| \geqslant n-2$ which implies our theorem immediately. Let b denote the largest box of B_1 under U. Let S denote the half-strip consisting of the points above the upper side of b and between the lines defined by the vertical sides of b. The atom A of B is called *separated* if it has a representative point in S and the boundary of A does not intersect the upper side of b. If B has at least one separated atom then we can modify B by lifting the upper side of b until it intersects a separated atom. This operation increases the number of atoms, contradicting the fact that B is a box diagram. We conclude that B has no separated atoms.

Let b_n, b_{n-1} and b_{n-2} be the three largest boxes of B in descending U-order. Assume indirectly that $|B_1| < n-2$, then $b_n, b_{n-1}, b_{n-2} \notin B_1$. Let b^* be a box among b_n, b_{n-1}, b_{n-2} such that the projection of b^* on the x-axis is covered by the union of the projections of the other two boxes. It is easy to see that $b^* \neq b_n$ (since B has no separated atoms) and the lifting of the upper side of b^* over b_n or b_{n-1} creates a separated atom while the number of atoms does not decrease—a contradiction. □

The remaining part of this section is devoted to connected box diagrams. The main tool for handling connected box diagrams is the notion of *overlay index*. Let B be a family of boxes, $b \in B$. If x is a corner of b then the overlay index of x, $\omega(x)$, is defined as $\max\{0, m-1\}$ where m denotes the number of boxes of B containing x and different from b. The overlay index of b, $\omega(b)$, is the sum of overlay indices of the four corners of b. The overlay index of a subfamily $B' \subseteq B$, $\omega(B')$, is defined as $\sum_{b \in B'} \omega(b)$. In particular, if $B' = B$, $\omega(B) = \sum_{b \in B} \omega(b)$.

Theorem 4.2. *Let $B_n = \{b_1, \ldots, b_n\}$ be a connected family of boxes. Let D denote the set of connected regions in the plane which belong to at least two boxes of B_n. Then*

$$|D| \leqslant 2^{n^2} - 6n + 5 - \omega(B_n).$$

Proof. Let $D = D_1 \cup D_2 \cup D_3 \cup D_4$ where $D_i \subset D$ denotes the subset of D having a non-empty intersection with the ith orthant, $i = 1, 2, 3, 4$. Let $d \in D_i$ and choose the point p of d with the largest distance from the origin. If p is a corner of a box b_i then we associate to d a pair (b_i, b_j) such that $j \neq i$ and $p \in b_j$. We have $\omega(p)+1$ possible choices for b_j. If p is not a corner of any box then we associate to d the uniquely determined pair of boxes (b_i, b_j) whose boundary lines intersect each other at p. This argument shows that $|D_i| \leqslant \binom{n}{2} - \sum \omega(x)$ where the summation is extended to all corners x of boxes in the ith orthant. Repeating this argument for the four orthants, we obtain

$$|D| \leqslant 4\binom{n}{2} - \omega(B_n). \tag{4}$$

The domains of D intersecting exactly two orthants were estimated twice in the right-hand side of (4); there are $2(2n-3)$ such domains. If we subtract $2(2n-3)$ from the right-hand side of (4) then the domains of D intersecting exactly three orthants were estimated three times and were subtracted twice. The domains of D intersecting four orthants and not twice connected were estimated four times and subtracted three times. The connectedness of the family B_n ensures that the only twice connected domain of D which intersects four orthants is $\bigcap_{i=1}^{n} b_i$; this domain was estimated four times and subtracted four times, thus 1 must be added to the right-hand side of (4) to get the correct estimation. This sieve argument leads to

$$|D| \leqslant 4\binom{n}{2} - \omega(B_n) - 2(2n-3) + 1 = 2n^2 - 6n + 5 - \omega(B_n). \quad \square$$

Corollary 4.3. *If $B_n = \{b_1, \ldots, b_n\}$ is a connected family of boxes then $a(B_n) \leqslant 2n^2 - 5n + 5 - \omega(B_n)$.*

Proof. The number of atoms covered by one box is at most n. $\square$

Let B_n be a family of n boxes. A *vertical order* on B_n is an indexing of the boxes of B_n by $1, \ldots, n$ such that for every i, $1 \leqslant i \leqslant n$, at least one of the following two properties holds:

(i) b_i is U-minimal in $\{b_i, b_{i+1}, \ldots, b_n\}$. If $i > 1$ then, for some j, $1 \leqslant j < i$, $b_j >_U b_i$.

(ii) b_i is D-minimal in $\{b_i, b_{i+1}, \ldots, b_n\}$. If $i > 1$ then, for some j, $1 \leqslant j < i$, $b_j >_D b_i$.

A *horizontal order* on B_n is defined similarly by using L and R instead of U and D.

Proposition 4.4. *A connected family B_n of boxes has a vertical (horizontal) order.*

Proof. It is enough to show the existence of a vertical order. Let b_1 be the U-minimal box of B_n. Assume that for some k, $1 \leqslant k < n$, $b_1, \ldots, b_k$ are already defined so that (i) or (ii) is satisfied for $1 \leqslant i \leqslant k$. We show that either the U-minimal element b of $B_n - \{b_1, \ldots, b_k\}$ or the D-minimal element b' of $B_n - \{b_1, \ldots, b_k\}$ can be chosen as b_{k+1} to satisfy (i) or (ii). If b does not satisfy (i) and b' does not satisfy (ii) then $b_j <_U b$ and $b_j <_D b'$ for all $j \leqslant k$. The transitivity of U and D implies that $b_j <_U b^*$ and $b_j <_D b^*$ for all j, $1 \leqslant j \leqslant k$ and for all $b^* \in B_n - \{b_1, \ldots, b_k\}$. We have a contradiction to the assumption that B_n is vertically connected. $\square$

A box b_i is called *U-minimal, D-minimal* or *UD-minimal* in a vertical order if (i), (ii) or both hold for i. The box b_j, defined for all $i > 1$ in (i) or in (ii), is called the *overlap predecessor* of b_i in the vertical order (b_i and b_j vertically overlap each other). The *vertical overlap tree* is defined for a vertical order by taking the set of vertices $\{1, \ldots, n\}$ and defining an edge (j, i) if b_j is the overlap predecessor of b_i. The analogous notions can be obviously defined for a horizontal order.

From now on we assume that the families of boxes are *connected* and are *indexed in a vertical order.*

Let $B_n=\{b_1,\ldots,b_n\}$ and $1\leqslant i<j\leqslant n$. We say that $\{b_i, b_{i+1},\ldots,b_{j-1}\}=J(i,j)=J$ is an *L-block* if $b_i <_L b_j$ and $b_k >_L b_j$ for all k, $1\leqslant k\leqslant i-1$. The box b_i is called the *head* of the block. The definition of an *R-block* is similar, we must replace L by R. A *block* is either an L-block or an R-block. Note that $b_j\notin J(i,j)$ by definition.

Lemma 4.5. *Let J be a block in $B_n=\{b_1,\ldots,b_n\}$. Then*

$$\omega(J)\geqslant\begin{cases}|J| & \text{if the head of } J \text{ is not } b_1,\\ |J|-1 & \text{if the head of } J \text{ is } b_1.\end{cases} \tag{5}$$

Proof. Assume that $J=J(i,j)$ is an L-block. We divide the boxes of J, different from the head, into two disjoint sets X, Y as follows:

$$\begin{aligned} X&=\{b_k: i<k<j,\ b_i <_L b_k\},\\ Y&=\{b_k: i<k<j,\ b_i >_L b_k\}. \end{aligned} \tag{6}$$

We estimate $\omega(J)$ in two steps.

Step 1 (the overlay index of b_i). Suppose that b_i is U-minimal (D-minimal). Now the overlap predecessor of b_i, the boxes of X and b_j cover the upper left (lower left) corner of b_i. Then $\omega(b_i)\geqslant|X|+1$ if $i>1$ and $\omega(b_i)\geqslant|X|$ if $i=1$.

Step 2 (the overlay index of Y). We show that $\omega(Y)\geqslant|Y|$. We proceed by induction on $|Y|$. The case $|Y|=0$ is trivial. Let b_p be the box of Y with the largest index. Clearly $b_p <_L b_j$ by the definition of Y and by the transitivity of L. Assume that b_p is U-minimal (D-minimal) and let b_q denote the overlap predecessor of b_p. If $b_q >_L b_p$ then the upper left (lower left) corner of b_p is covered by b_q and b_j, i.e. $\omega(b_p)\geqslant 1$, and we are home by the inductive hypothesis on $Y-\{b_p\}$. If $b_q <_L b_p$ then $b_q\in Y$ and $b_q <_L b_j$ by transitivity. Now b_p covers the upper left (lower left) corner of b_q thus the overlay index of b_q in B_n is larger than in $B_n-\{b_p\}$, and we are home again.

Putting together the estimations of Step 1 and Step 2, we get the statement of the lemma, since $|X|+|Y|=|J|$. □

A block is called *extremal* if equality holds in (5).

Now we define a partition of $B_n-\{b_n\}$ into blocks, called the block partition of B_n. Let $j_1=n$ and let $J_1(i_1,j_1)$ be a block. If $J_1,\ldots,J_m$ are already defined and $J_1\cup\cdots\cup J_m$ does not cover $B_n-\{b_n\}$ then we continue by choosing a block $J_{m+1}(i_{m+1},j_{m+1})$ such that $i_{m+1}<i_m\leqslant j_{m+1}$. The connectivity of the horizontal overlap graph of B_n ensures that eventually $i_t=1$ for some block $J_t(i_t,j_t)$, i.e., we get a partition.

By applying Lemma 4.5 for the blocks of a block partition, we obtain immediately

Corollary 4.6. *If B_n is a connected family of boxes then*

$$\omega(B_n)\geqslant n-2. \quad \square$$

The facts established until this point allow to state some properties of connected box diagrams.

Theorem 4.7. *A connected box diagram B_n has the following properties:*

(i) $\omega(B_n) = n-2$;

(ii) $a(B_n) = 2n^2-6n+7$;

(iii) *the blocks of a block partition of B_n are extremal;*

(iv) *the atoms of B_n belonging to at least two boxes of B_n are connected regions.*

Proof. Corollaries 4.3 and 4.6 imply (i) and (ii) since the caterpillar construction defines $2n^2-6n+7$ atoms. Also, (iii) follows because the presence of a non-extremal block would violate (i). To prove (iv), let a_2 and d_2 denote the number of atoms and the number of connected regions belonging to at least two boxes. We have to show $a_2 = d_2$. From (i), (ii) and Theorem 4.2 we obtain $2n^2-6n+7 = a(B_n) \leqslant a_2+n \leqslant d_2+n \leqslant 2n^2-6n+5-\omega(B_n)+n = 2n^2-6n+7$ and $a_2 = d_2$ follows. □

For further analysis of connected box diagrams we have to study the structure of blocks. Let $J(i, j)$ be an L-block (R-block) of B_n. The box b_k for $i<k<j$ belongs to one of the following three types:

Type 1. $b_i <_L b_k, b_k <_L b_j$ $(b_i <_R b_k, b_k <_R b_j)$;

Type 2. $b_i <_L b_k, b_k >_L b_j$ $(b_i <_R b_k, b_k >_R b_j)$;

Type 3. $b_i >_L b_k, b_k <_L b_j$ $(b_i >_R b_k, b_k <_R b_j)$.

Lemma 4.8. *If $J(i, j)$ is an extremal L-block (R-block) then the following properties hold:*

(i) $b_p >_L b_q$ $(b_p >_R b_q)$ *for all p, q satisfying $i<p<j<q\leqslant n$;*

(ii) $b_p >_R b_q$ $(b_p >_L b_q)$ *for all p, q satisfying either* (a) $p<i<q\leqslant n$, *or* (b) $i\leqslant p<\min\{j, q\}$ *and* $(p, q) \neq (1, 2)$;

(iii) *Type 2 boxes precede the other boxes in the block, i.e., if $b_p, b_q \in J(i, j)$ and b_p is of Type 2 and b_q is not, then $p<q$;*

(iv) *if $b_p \in J(i, j)$ and b_p is of Type 1 or Type 3 then b_p is not UD-minimal in the vertical order;*

(v) *if $b_p \in J(i, j)$ and $1\leqslant q<r<p$ then no corner of b_p is covered by both b_q and b_r.*

Proof. We show that the falsity of any of the five properties allows to find a box with an 'extra overlay index', i.e., an overlay index which was not used in the estimation of $\omega(J(i, j))$ in Lemma 4.5. Let X and Y be the sets defined by (6).

If (i) does not hold then we have two cases. If $b_i <_L b_p$ then $b_i <_L b_q$, implying $\omega(b_i) \geqslant |X|+2$ for $i>1$ or $\omega(b_i) \geqslant |X|+1$ for $i=1$. We have an extra overlay index in Step 1. If $b_i >_L b_p$ then $b_p \in Y$ and b_q gives an extra overlay index on b_p in Step 2.

Condition (ii) follows from the fact that Step 1 and Step 2 used only left (right) corners for the overlay index estimation if $J(i, j)$ was an L-block (R-block).

Assume that (iii) does not hold. Let $b_p, b_q \in J(i, j)$, where b_q is of Type 2 and $p<q$. If b_p is of Type 1 then $b_p <_L b_q$, $b_p <_L b_j$. Since $b_p \in X$, neither Step 1 nor Step 2 defined overlay index on b_p, therefore b_q and b_j define an extra overlay index on b_p. If b_p is of Type 3 then $b_p <_L b_q$, $b_p <_L b_j$. Since $b_q \in X$, $b_p \in Y$, b_q increases the overlay index assigned to b_p in Step 2.

Condition (iv) follows from the fact that a box b_p of Type 1 or Type 3 must be in Y where the overlay index was assigned according to the U- or D-minimality of b_p.

Condition (v) follows from the observation that the index of at least one box defining an overlay index of b_p is larger than p. □

Lemma 4.9. *Let $J_1, \ldots, J_t$ be the blocks of the block partition of the connected box diagram B_n. If b_i is a Type 2 box in J_j then $j=t$ and $i=2$.*

Proof. Let $J_m = J_m(i, j)$ be a block of the block partition ($1 \leq m \leq t$). Assume that J_m is an L-block and let b_p be a Type 2 box in J_m with the largest index. Note that $b_{i+1}, \ldots, b_p$ are all of Type 2 by Lemma 4.8(iii).

Claim. For all q, r satisfying $i \neq r < p < q$, $b_r >_L b_q$ and $b_p >_R b_q$ hold.

Firstly, $b_r >_R b_q$ follows from Lemma 4.8(ii). For $q \geq j$, $r<i$, $b_r >_L b_q$ follows from the definition of the block partition. For $i<r\leq p$ and $q>j$, $b_r >_L b_q$ follows from Lemma 4.8(i). For $i<r<p$, $q=j$, $b_r >_L b_q$ follows from the fact that b_r is of Type 2. Finally, if $i<r<p<q<j$ then b_q is of Type 1 or Type 3, therefore $b_q <_L b_j$, $b_j <_L b_r$ which implies $b_q <_L b_r$ and the claim is proved.

We continue the proof by the indirect assumption that the lemma is not true. Assume that the head of J_m is U-minimal. If $m=t$ and there are at least two boxes of Type 2 in J_t then put $b'=b_2$, $b''=b_3$. If $m<t$ and J_t contains a box of Type 2 then let b' be such a box and let b'' denote the overlap predecessor of the head of J_t. The smaller of b' and b'' under U is denoted by b^*.

If $b^* >_U b_q$, for some $q>p$ then the transitivity of U implies $b' >_U b_q$, $b'' >_U b_q$. However, $b' >_L b_q$, $b'' >_L b_q$ by the previous claim which contradicts Lemma 4.8(v). We conclude that $b^* <_U b_q$ if $b_q \in \{b_{p+1}, \ldots, b_n\} = C$. Consider the following set of boxes:

$$D = \{b_s : 1 \leq s \leq p, b_s >_U b^*\}.$$

Obviously D is not empty (the larger of b' and b'' under U is in D) and $b_i \notin D$. The set $C \cup D$ contains all the boxes larger than b^* under U.

Let l be the upper horizontal side of b^*. Let X denote the union of the projections of the boxes of C into l, and let Y denote the intersection of the projections of the boxes of D into l. Our previous claim ensures that X is properly contained by Y, therefore the two intervals of $Y-X$ belong to the same atom A of B_n. The atom A belongs to at least two boxes (to b^* and to the boxes of D). Moreover, A is disconnected since $b_i <_U b^*$ and either $b_i >_D b^*$ (if b^* is the overlap predecessor of b_i) or $b_i >_L b^*$ (if b^* is a Type 2 box of J_m). We get a contradiction to Theorem 4.7(iv) □

Let b_i and b_j be two boxes vertically overlapping b_k (i, j, k are different). We say that b_i and b_j give a *UD-overlap* on b_k if either $b_i >_U b_k >_U b_j$ or $b_i >_D b_k >_D b_j$ holds. The definition of an *LR-overlap* is similar.

Lemma 4.10. *Let $B_n = \{b_1, \ldots, b_n\}$ be a connected box diagram indexed in vertical order. Then the vertical overlap graph of B_n is the vertical overlap tree with one possible additional edge (i, n). If the edge (i, n) is present then b_i and the overlap predecessor of b_n give an UD-overlap on b_n.*

Proof. Assume that b_i and b_j vertically overlap b_k for $1 \leq i < j < k \leq n$. We are going to show that in this case b_i and b_j give an UD-overlap on b_k and $k = n$ which clearly implies our lemma.

Case 1. Assume that b_i and b_j do not give an UD-overlap on b_k. We may assume (by symmetry) that $b_i >_U b_k$ and $b_j >_U b_k$. Consider the block J in the block partition of B_n which contains b_k or let $J = J_1$ if $k = n$, i.e. b_k is not contained in any block. Lemma 4.8(ii) shows that $b_i >_R b_k$, $b_j >_R b_k$ if J is an L-block, or $b_i >_L b_k$, $b_j >_L b_k$ if J is an R-block. In any case, we get a contradiction to Lemma 4.8(v).

Case 2. Assume that b_i and b_j give an UD-overlap on b_k. First we prove that b_k is UD-minimal in the vertical order. Assume that b_k is U-minimal but it is not D-minimal. Then there exists a $k' > k$ such that $b_k >_D b_{k'}$. Since $b_k <_U b_{k'}$, $b_{k'}$ and b_k vertically overlap each other. By transitivity we get that b_i and b_j vertically overlap $b_{k'}$ but they do not give an UD-overlap on $b_{k'}$. Now we get a contradiction through Case 1.

We know therefore that b_k is UD-minimal. If $k < n$ then b_k is in a block of the block partition of B_n. Since $1 \leq i < j < k$ implies $k \neq 2$, b_k is not of Type 2, by Lemma 4.9. If b_k is of Type 1 or Type 3 then b_k is not UD-minimal by Lemma 4.8(iv), a contradiction implying $k = n$. □

Theorem 4.11. *Let $B_n = \{b_1, \ldots, b_n\}$ be a connected box diagram. Then B_n can be obtained by a caterpillar construction or by vertical and/or horizontal augmentation of a caterpillar construction.*

Proof. Let $J_1, \ldots, J_t$ be the blocks of the block partition of B_n.

Step I. Assume that there is a Type 2 box b_k in some block. Lemma 4.9 implies that $k = 2$ and $b_2 \in J_t = J_t(1, j)$. By symmetry, assume that J_t is an L-block; now $b_1 <_L b_2$ by the definition of the Type 2 box.

We prove that $b_1 >_R b_2$. Assume in the contrary that $b_1 <_R b_2$. Now b_1 does not overlap b_2 horizontally. For any $q \geq 3$, $b_2 >_R b_q$ holds by Lemma 4.8(ii) (with condition (b)). For $q=j$, $b_2>_L b_q$ follows from the fact that b_2 is of Type 2. For any q satisfying $3\leq q<j$, $b_2>_L b_q$ follows from the fact that b_q is not a Type 2 box (because of $b_q<_L b_j<_L b_2$). Finally, $b_2>_L b_q$ for $q>j$ follows from Lemma 4.8(i). Therefore no box of B_n overlaps b_2 horizontally, contradicting the connectivity of B_n.

Now we exchange b_1 and b_2. In this way another vertical order is defined on B_n. The block partition belonging to this new order is $J'_i=J_i$ for $i<t$, $J'_t=J_t-\{b_2\}$, $J'_{t+1}=\{b_2\}$. It is obvious that there are no Type 2 boxes in this block partition.

Step II. Assume that there are no Type 2 boxes and $J_t=J_t(1,j)$, $j\geq 3$, i.e. $J_t\neq\{b_1\}$. Assume that J_t is an L-block. Now $b_1<_L b_j$ and $b_2<_L b_j$; moreover, $b_1>_R b_j$, $b_2>_R b_j$ by Lemma 4.8(ii). The exchange of b_1 and b_2 gives a new vertical order; the block partition relative to this new order is $J''_i=J_i$ for $i<t$, $J''_t=\{b_2, b_1, b_3, b_4, \ldots, b_{j-1}\}$. It is obvious that there are no Type 2 boxes in this block partition.

In the light of steps I and II, by the possible exchange of b_1 and b_2 in a vertical order, we can always obtain a vertical order $b_1, \ldots, b_n$ on B_n and a block partition $J_1, \ldots, J_t$ such that the following two properties hold:

there are no Type 2 boxes, (7)

if $J_t\neq\{b_1\}$ and J_t is an L-block (R-block)
then $b_1>_L b_2(b_1>_R b_2)$. (8)

Further on, (7) and (8) are assumed.

Claim 1. Let $J_m=J_m(i,j)$ be an L-block (R-block) where $m<t$. Then $J_{m+1}=J_{m+1}(i',j')$ is an R-block (L-block) and $j'=i$.

To prove the claim, assume that J_m is an L-block. If J_{m+1} is an L-block then $b_{i'}<_L b_{j'}$. Since $b_{j'}$ is not of Type 2 in J_m by (7), $b_{j'}<_L b_j$ and by transitivity $b_{i'}<_L b_j$ follows, contradicting the definition of the block J_m. Therefore J_{m+1} is an R-block. If $j'>i$ then applying Lemma 4.8(ii)(b) for J_m, one can see that $b_i>_R b_{j'}$. This implies that b_i is of Type 2 in J_{m+1}, contradicting (7); therefore $j'=i$ and the claim is proved.

Claim 2. If b' is the last element in a horizontal order of B_n then $b'=b_1$ or $b'=b_2$.

Assume in the contrary that $b'=b_m$, $m\geq 3$, and let J_t be an L-block. If $b_1, b_2\in J_t$ then $b_1>_R b_m, b_2>_R b_m$ by Lemma 4.8(ii). Now either $b_1>_L b_m$ or $b_2>_L b_m$ contradicts the definition of b_m since b_1 or b_2 is neither L- nor R-minimal in the horizontal order. If $b_1<_L b_m$ and $b_2<_L b_m$ then b_1 and b_2 are two overlap predecessors of b_m in the horizontal order which do not give an RL-overlap on b_m, contradicting the 'horizontal version' of Lemma 4.10. If $b_1\in J_t$, $b_2\in J_{t-1}=J_{t-1}(2, j_{t-1})$ then assume J_{t-1} to be an L-block. Now $b_1>_R b_m$, $b_2>_R b_m$ by Lemma 4.8(ii), and the contradiction follows as in the previous case.

Now we define the caterpillar C_h on the vertex set $N=\{1, \ldots, n\}$ with edges (k, j_m) where $1\leq m\leq t$, $j_{m+1}\leq k<j_m$. (Because of Claim 1, C_h is a caterpillar.) Let

C_h^+ be the transitive orientation of C_h in which (j_2, j_1) is a directed edge if J_1 is an L-block and (j_1, j_2) is a directed edge if J_1 is an R-block. Let C_h^- denote the 'reverse' orientation of C_h^+. Finally, let $C^+ = C_h^+ \cup (2,1)$, $C^- = C_h^- \cup (1,2)$ ((2, 1) and (1, 2) denote directed edges).

Claim 3. The L and R orders on B_n are defined by the caterpillars C_h^+, C_h^- or by the augmented caterpillars C^+, C^- according to the caterpillar construction, i.e., according to the third and fourth lines of (3) in Section 2.

To prove Claim 3, let L' and R' denote the linear orders on N according to the caterpillar construction. Let $p, q \in N$, $1 \leq p < q$. Since $p < n$, the box b_p is an element of some block, say $J_m = J_m(j_{m+1}, j_m)$ $(1 \leq m \leq t,\ j_{t+1} = 1)$. We shall prove that p and q are compared by L' and R' in the same way as b_p and b_q are compared by L and R. In the special case $(p, q) = (1, 2)$, $b_1 >_L b_2$, $b_1 <_R b_2$ are also acceptable since this is in accordance with the orders defined by C^+ and C^-. We distinguish some cases in the proof.

Case 1. $p = j_{m+1}$, $q = j_m$. If $(p, q) \neq (1, 2)$ then we should rely on Claim 1 and Lemma 4.8(ii)(b) (If $(p, q) = (1, 2)$ then condition (b) does not hold.) Assume that $(p, q) = (1, 2)$. Since $J_t = J_t(1, 2)$ in this case and J_t is an L-block, $b_1 <_L b_2$ follows. If $b_1 <_R b_2$ then $b_2 <_R b_{j_{t-1}}$ implies $b_1 <_R b_{j_{t-1}}$, contradicting the definition of the R-block J_{t-1}. Therefore $b_1 >_R b_2$ and we are home since L' and R' compare $1, 2 \in N$ in the same way by Claim 1.

Case 2. $j_{m+1} < p < j_m$, $q = j_m$. Now $p <_{L'} j_m$ and $p >_{R'} j_m$ since C_h^+ and C_h^- have transitive orientations. As b_p is of Type 1 or Type 3 in J_m by (7), $b_p <_L b_{j_m}$ follows. On the other hand, $b_p >_R b_{j_m}$ follows from Lemma 4.8(ii)(b).

Case 3. $p = 1, q = 2$. We may assume $j_m = j_t \geq 3$; otherwise the case was handled at Case 1. Now $b_1 >_L b_2$ holds by (8). If $b_1 >_R b_2$ then L' and R' compare $1, 2 \in N$ in the same way. If $b_1 <_R b_2$ then it is in accordance with the orders generated by C^+ and C^-.

Case 4. $j_{m+1} \leq p$, $q \neq j_m$, $p, q \neq (1, 2)$. Now (p, q) is not an edge of C_h; therefore $p >_{L'} q$, $p >_{R'} q$ by the definition (3) of the caterpillar construction. From Lemma 4.8(ii)(b), we get $b_p >_R b_q$. We have to show that $b_p >_L b_q$. The indirect assumption $b_p <_L b_q$ implies that b_p and b_q horizontally overlap each other, therefore the horizontal overlap graph of B_n contains a cycle. (We have proved in Cases 1 and 2 that pairs of boxes corresponding to the edges of C_h are overlapping.) By applying Lemma 4.10 for horizontal orders, one can see that the only excuse of having a cycle is that an RL-overlap is defined on the last box of the horizontal order. The only possibility is that the RL-overlap is defined on b_q in our case. However, $q \geq 3$ and we have a contradiction with Claim 2—so Claim 3 is proved.

It is easy to see that $b_n, b_{n-1}, \ldots, b_1$ is a horizontal order of B_n. Starting from this horizontal order, one can define U-blocks, D-blocks, horizontal block partitions etc. to state Lemmas 4.5, 4.8, 4.9 and 4.10 in a dual form (L and U, R and D exchange roles). Steps I, II, moreover, Claims 1, 2 and 3 can be dualized in the same spirit. Then Claim 3 in the dualized form gives the missing point of the proof of Theorem 4.11 □

Putting together Theorem 4.1 and Theorem 4.11, we obtain Theorem 3.1, the main result of the paper.

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