The linear Turán number of small triple systems or why is the wicket interesting?

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Abstract

A linear *triple system* is a 3-uniform hypergraph $H = (V, E)$, where E is a set of three-element subsets of V such that any two edges intersect in at most one vertex. For linear triple systems H, F we say that H is *F-free* if H does not contain any subsystem isomorphic to F. We consider F fixed and call it a *configuration*. The (linear) Turán number $\mathrm{ex}_L(n, F)$ (or simply just $\mathrm{ex}(n, F)$) of a configuration F is the maximum number of edges in F-free linear triple systems with n vertices.

Here we call attention to some properties of the *wicket* W, formed by three rows and two columns of a 3×3 point matrix. On one hand we show that the problem whether $\mathrm{ex}(n, F) = o(n^2)$ can be decided for all configurations with at most five edges, except for $F = W$, which remains undecided. On the other hand we prove that $\mathrm{ex}(n, W) \le \frac{(1-c)n^2}{6}$ with some $c > 0$, separating it from the conjectured asymptotic of $\mathrm{ex}(n, G_{3\times 3})$, where $G_{3\times 3}$, the *grid*, formed by three rows and *three* columns of a 3×3 point matrix.

1 Introduction

A *linear triple system* is a 3-uniform hypergraph $H = (V(H), E(H))$, where $E(H)$ is a set of three-element subsets of $V(H)$ such that any two edges intersect in at most one vertex. When it is clear from the context, we just use V, E for the set of vertices and for the set of edges, respectively.

In this paper we just use the term triple system for linear triple systems. For triple systems H, F we say that H is *F-free* if H does not contain any subsystem isomorphic to F. We consider F fixed and call it a *configuration*. More generally, fixing a family $\mathcal{F}$ of configurations, we say that H is $\mathcal{F}$-free if H does not contain any member from $\mathcal{F}$.

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The most well-known triple systems are the Steiner triple systems, STS(n), they have n vertices and each pair of vertices is covered by an edge. They exist if and only if $n \equiv 1$ or $n \equiv 3 \pmod 6$, such values of n are called *admissible* (see [3]).

A configuration F is 3-*partite* if $V(F)$ can be partitioned into three sets V_1, V_2, V_3 so that all edges intersect each V_i in exactly one vertex.

The (linear) Turán number $\mathrm{ex}_L(n, \mathcal{F})$ (or simply just $\mathrm{ex}(n, \mathcal{F})$) of a family $\mathcal{F}$ of configurations is the maximum number of edges in $\mathcal{F}$-free triple systems with n vertices. For $\mathcal{F} = \{F\}$ we just write $\mathrm{ex}(n, F)$. Note that $\mathrm{ex}(n, F) = \frac{n(n-1)}{6}$ is equivalent to the statement that an F-free STS(n) exists, in this case F is called *avoidable*. Let $\mathcal{F}_{(k,\ell)}$ denote the family of all (linear) triple systems with ℓ triples on at most k vertices.

Let's call a configuration *small* if it has at most five edges. The complete catalogue of configurations with two, three and four edges and some of the ones with five edges are exhibited in the book of C. J. Colbourn and A. Rosa ([3], Figure 3.1 and Figure 13.4). There are $1 + 2 + 5 + 16 + 56$ small configurations with 1, 2, 3, 4 or 5 edges, respectively. Many of them obtained a name and one of them, the *wicket*, plays the main role in this note. Figures 1, 2 and 3 below show some important configurations among these, and one six-edge configuration, the grid. The edges of the configurations are represented as straight line segments. Note that all of them, apart from the sail are 3-partite.

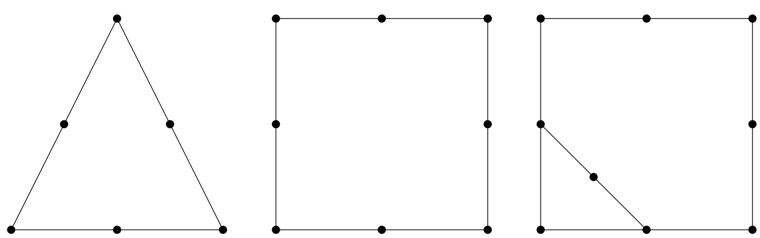

Figure 1: The triangle, 4-cycle and D_3

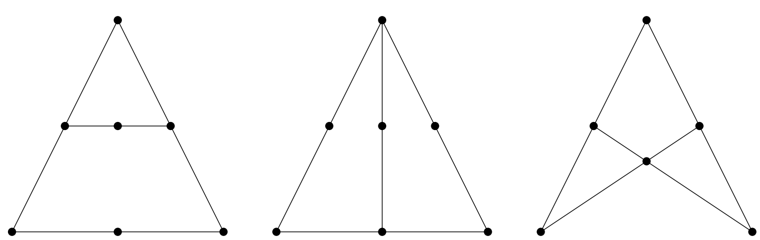

Figure 2: Configurations C_{14}, C_{15} (sail) and C_{16} (Pasch configuration)

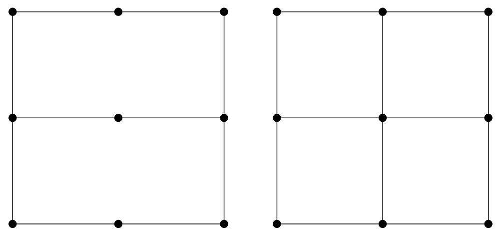

Figure 3: The wicket W and the grid $G_{3\times 3}$

Determining Turán numbers is one of the most fundamental problems in hypergraph theory. For example a celebrated conjecture of Brown, Erdős and T. Sós [1], [4] reduces to the following.

Conjecture 1.1 ([4]). *We have* $ex(n, \mathcal{F}_{(k+3,k)}) = o(n^2)$.

For $k = 3$ there is only one member in the $(6, 3)$-family, the triangle (see Figure 1). A celebrated theorem of Ruzsa and Szemerédi [13] proves Conjecture 1.1 for the triangle. However, already the cases $k = 4$ or 5 are wide open.

Motivated by Conjecture 1.1 and continuing our earlier studies in [9], here we address the problem whether $\mathrm{ex}(n, F) = o(n^2)$ for a *fixed* configuration F (as opposed to a family). Our starting point is the following.

Problem 1. *([9]) Is it true that* $ex(n, W) = o(n^2)$*?*

Our first result shows that the wicket is the only small configuration for which $\mathrm{ex}(n, F) = o(n^2)$ is in doubt.

Theorem 1.2. *For all small configurations F apart from W either $ex(n, F) = o(n^2)$ or $ex(n, F) \geq n^2/9$.*

In fact, the inequality in the second possibility is sharp, since $\mathrm{ex}(n, C_{15}) = n^2/9$ for $n \equiv 0 \pmod 3$ [5]. Using the following two general remarks, we can make Theorem 1.2 more explicit (Theorem 1.3).

Remark 1. *If F contains a subconfiguration F' such that there exists F'-free $STS(n)$ for every large enough n, (i.e. F' is avoidable), then $ex(n, F) = \frac{n(n-1)}{6}$.*

Remark 2. *If F contains a subconfiguration F' that is not 3-partite then $ex(n, F) \geq \frac{n^2}{9}$. Indeed, there exists 3-partite triple systems with n vertices and $n^2/9$ edges when $n \equiv 0 \pmod 3$ (transversal designs with three groups).*

These remarks (since C_{14}, C_{16} are avoidable and C_{15} is not 3-partite) allow the reduction of Theorem 1.2 to the following.

Theorem 1.3. *Every small configuration F satisfies at least one of the following.*

- *(i) $ex(n,F) = o(n^2)$,*
- *(ii) there is an $F' \subseteq F$ such that $F' \in \{C_{14}, C_{15}, C_{16}\}$,*
- *(iii) $F = W$.*

Our second result is a small step forward in Problem 1.

Theorem 1.4. *There exist $c, n_0 > 0$ such that $ex(n, W) \leq \frac{(1-c)n^2}{6}$ for $n \geq n_0$.*

Theorem 1.4 relates to a conjecture of Füredi and Ruszinkó [6] who proved that

$$\Omega(n^{1.8}) \leq \mathrm{ex}(n, G_{3\times 3}) \leq \frac{n(n-1)}{6},$$

and they conjectured that the upper bound is the truth asymptotically. They formulated an even stronger conjecture: for every sufficiently large admissible n there is a grid-free STS(n), i.e. that the grid is avoidable. Gishboliner and Shapira [7] got close to the conjecture improving the lower bound of $\mathrm{ex}(n, G_{3\times 3})$ to $\Omega(n^2)$.

Note that Theorem 1.4 separates $\mathrm{ex}(n, W)$ from the conjectured value of $\mathrm{ex}(n, G_{3\times 3})$.

2 Projections, proof of Theorem 1.3

2.1 Projections

Assume that a configuration F has an *independent transversal* $S \subset V(F)$ which means that $|S \cap e| = 1$ for every $e \in E(H)$. Then we can represent F with a graph $G = G(F, S) = (V', E')$, the *projection of* F, as follows. Let α be a bijection from V to V' and for every $s \in S$ and every $\{s, v_1, v_2\} \in E(F)$ let $(\alpha(v_1), \alpha(v_2))$ be an edge of G. We consider G as a *properly edge-colored* graph where the color classes are defined by the elements of S. For example, Figure 4 shows how the wicket can be projected to the path *abcab*. (In these figures for simplicity we identify V and V' and projection points are always shown by capitalizing the letters of the corresponding patterns.) This is a natural and well-known technique going back to the induced matching lemma of Szemerédi, where the triangle was projected to the path *aba*, see [11]. Note that the projection of the grid is the *abcabca* cycle.

An *s-pattern*, defined in [9], is a graph forest with a proper s-edge-coloring where the color classes (matchings) $M_1, M_2, \ldots, M_s$ satisfy the following property: for any $1 \leq i \leq s$ every edge in M_i has a vertex that is not covered by any edge of any $M_j, i < j \leq s$. For example Figure 5 shows how D_3 can be projected to the 3-pattern path *abcba*. On Figure 1 the configurations have projections to 2- or 3- patterns, but the configurations on Figures 2 and 3 have no projections to any s-pattern. The connection of s-patterns with Turán numbers is shown by the main result of [9].

Theorem 2.1 ([9])**.** *If F has a projection to an s-pattern for some s, then $ex(n, F) = o(n^2)$.*

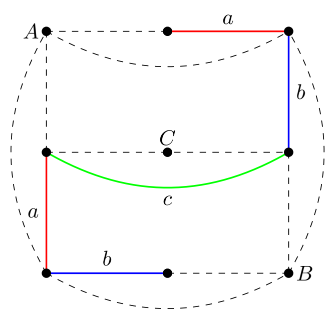

Figure 4: Projecting the wicket to the path *abcab*

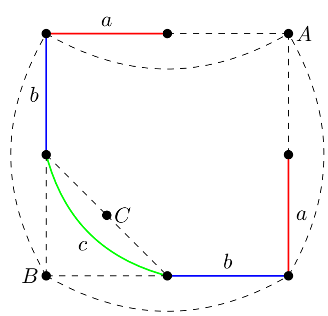

Figure 5: Projecting D_3 to the 3-pattern path *abcba*

2.2 Proof of Theorem 1.3

Theorem 2.1 allows us to replace (i) in Theorem 1.3 with the following (i').

Theorem 2.2. *Every small configuration F satisfies at least one of the following.*

- *(i') F has a projection to an s-pattern for some s,*
- *(ii) there is an $F' \subseteq F$ such that $F' \in \{C_{14}, C_{15}, C_{16}\}$,*
- *(iii) $F = W$.*

Proof of Theorem 2.2: Consider a counterexample small configuration F with as few edges as possible. By the assumption F does not fall into any of the three categories in Theorem 2.2.
Case I: There is an edge $e \in E(F)$ with at least two vertices of degree one. Remove e from $E(F)$ and denote the resulting configuration by F'. By the minimality assumption F' must fall into one of the three categories $(i'), (ii)$ and (iii), clearly this must be (i'), i.e. F' has a projection π to an s-pattern G for some s.

Clearly $|F' \cap e| \leq 1$ and if $F' \cap e$ is a projection point P of F' then π is also a projection of F to the s-pattern obtained from G by adding an isolated edge labeled with p. Otherwise one of the degree one vertices of e can be used as a new projection point Q extending the s-pattern G to an $(s+1)$-pattern with an isolated edge labeled with q. Thus we reach a contradiction, F has a projection to an s- or $(s+1)$-pattern.

Case II: There is no edge $e \in E(F)$ with at least two vertices of degree one. We can easily see (using that the last edge e of a longest path of F does not satisfy Case I) that F must contain a cycle.

Subcase II.1: The shortest cycle of F is a triangle T, say with edges $123, 345, 561$. Since T has a projection to a 2-pattern (aba) we must have further edges in F. However, a further edge f cannot intersect T in two or three vertices because that would result in one of the configurations F' in (ii). Thus further edges of F intersect T in at most one vertex. Moreover, to avoid Case I, the only possibility is to have two further edges f_1, f_2 forming a path connecting two vertices of T. There are four possibilities for these paths according to choosing corners or midpoints of T (corner to midpoint (2), midpoint to midpoint, corner to corner). All of these have projections to 2- or 3-patterns, leading to a contradiction (see Figure 6 for these patterns).

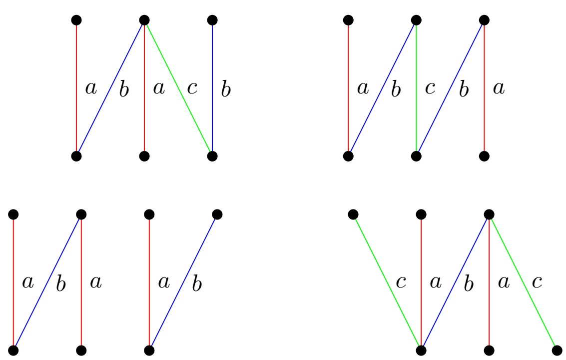

Figure 6: The 4 patterns in Subcase II.1

Subcase II.2: The shortest cycle of F is a four-cycle Q, say with edges $123, 345, 567, 781$. Since Q has a projection to a 2-pattern (two disjoint ab paths) we must have one further edge f in F. To avoid Case I and Subcase II.1, f must connect two opposite degree one vertex of Q, defining a wicket, a contradiction.

Subcase II.3: F is a five-cycle, it has a projection to a 3-pattern, a contradiction. □

3 Properties of the wicket and the proof of Theorem 1.4

3.1 Ramsey property

Although it is not known whether $\mathrm{ex}(n, W) = o(n^2)$ (see Problem 1), the wicket has a strong Ramsey-type property.

Theorem 3.1 ([9]). *For any fixed t there is an $n_0 = n_0(t)$ such that for any admissible $n > n_0$ in every t-coloring of the edges of any $STS(n)$ there is a monochromatic copy of W.*

For $t = 1$ we have $n_0(1) = 7$ which can be expressed as follows.

Proposition 3.2 ([9]). *Every $STS(n)$, except the Fano plane, contains W.*

Proposition 3.2 is a consequence of the following lemma, which will be applied in the proof of Theorem 1.4.

Lemma 3.3 ([9]). *Assume that $H = (V, E)$ is a triple system containing two disjoint edges $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}$ and containing all the nine edges covering the pairs x_i, y_j. Then H contains a wicket.*

The proof of this lemma amounts to checking that every proper edge coloring of $K_{3,3}$ contains a rainbow perfect matching.

3.2 Proof of Theorem 1.4

Assume that H is a wicket-free (linear) triple system. We will show that

$$|E(H)| \leq \frac{(1-c)n^2}{6}$$

for $c = 3/52$ and $n \geq n_0$. Let u denote the number of *uncovered* (unordered) pairs of vertices, i.e. pairs of vertices which are not contained in any edge of H. Then

$$3|E(H)| + u = \binom{n}{2},$$

hence

$$|E(H)| = \frac{1}{3}\left(\binom{n}{2} - u\right). \tag{1}$$

Next we will find a lower bound on u resulting in an upper bound on $|E(H)|$.

Let us take a maximal matching M in H, say with k edges for some $1 \leq k \leq n/3$. Using Lemma 3.3, between two edges in M there is at least one uncovered pair, so we get immediately at least $\binom{k}{2}$ uncovered pairs. We will add some more uncovered pairs of $V(H) \setminus V(M)$.

Note that all edges of H not covered by $V(M)$ intersect $V(M)$, since otherwise we could extend M. Let E_1 denote the set of edges of H which intersect

$V(M)$ in one vertex. Observe that if a vertex in an edge of M has degree at least three in E_1 then the other two vertices of this edge have degree zero in E_1, otherwise again we could extend M. Thus each edge in M have total degree at most

$$\max(6, \frac{n-3k}{2})$$

in E_1. Therefore at most $k \cdot \max(6, \frac{n-3k}{2})$ pairs of $V(H) \setminus V(M)$ are covered by $E(H)$. Then we get the following lower bound:

$$u \geq \binom{k}{2} + \binom{n-3k}{2} - k \cdot \max(6, \frac{n-3k}{2}). \tag{2}$$

We distinguish two cases.

Case 1: $\max(6, \frac{n-3k}{2}) = \frac{n-3k}{2}$.

Here from (1) and (2) we get

$$|E(H)| \leq \frac{1}{3}\left(\binom{n}{2} - \binom{k}{2} - \binom{n-3k}{2} + k\frac{n-3k}{2}\right) \leq$$
$$\leq \frac{1}{6}(7kn - 13k^2). \tag{3}$$

The maximum of the expression in (3) is attained at $k = 7n/26$, thus indeed

$$|E(H)| \leq \frac{(1-c)n^2}{6}$$

with $c = 3/52$.

Case 2: $\max(6, \frac{n-3k}{2}) = 6$.

In this case we must have

$$k \geq \frac{n}{3} - 4. \tag{4}$$

Here from (1) and (2) we get

$$|E(H)| \leq \frac{1}{3}\left(\binom{n}{2} - \binom{k}{2} - \binom{n-3k}{2} + 6k\right) \leq$$
$$\leq \frac{1}{6}(6kn - 10k^2 + 4k). \tag{5}$$

In the range of the k-s satisfying (4) the maximum of the expression in (5) is attained at $\frac{n}{3} - 4$ and thus we get

$$|E(H)| \leq \frac{1}{6}\left(\frac{8n^2}{9} + 4n\right) \leq \frac{(1-c)n^2}{6}$$

for $c = 3/52$ and $n \geq n_0$ ($n_0 = 75$ may be chosen). $\square$

4 Conclusion

First we showed that for all small configurations $F \neq W$ the problem whether $\mathrm{ex}(n, F) = o(n^2)$, can be answered. For the wicket the question remains open. We believe that $\mathrm{ex}(n, W) = o(n^2)$ and we have some evidence to back this up: we can derive it from a strong version of the $k = 5$ case of Conjecture 1.1.

An immediate consequence of the lower bound of [13] is that $\mathrm{ex}(n, F) = o(n^2)$ can happen "without an exponent".

Remark 3. *If F contains a triangle then for some constant $c > 0$*

$$2^{-c\sqrt{\log(n)}}n^2 \leq ex(n, F).$$

We suspect that "there is an exponent" if the triangle and the wicket are both forbidden: $\mathrm{ex}(n, \{W, C_3\}) \leq cn^{2-\varepsilon}$ holds for some $\varepsilon > 0$ (where C_3 is the triangle).

In Theorem 1.4 we merely wanted to separate the linear Turán number of the wicket from the conjectured linear Turán number of the grid. The constant 3/52 in Theorem 1.4 can certainly be increased but its significance depends on the outcome of Problem 1.

Concerning lower bounds on $\mathrm{ex}(n, W)$, we do not know of anything better than $\Omega(n^{3/2})$, which can be reached by a random construction (see [6]), or by lower bounds for C_4-free triple systems (see [2], [12]).

For certain configurations F stronger upper bounds are known on the linear Turán number. For example for C_t, the cycle with $t > 3$ edges, $\mathrm{ex}(n, C_t) \leq cn^{1+1/\lfloor t/2 \rfloor}$ [2]. For acyclic triple systems F, $\mathrm{ex}(n, F)$ is linear, the main problem in this case is determining the asymptotic which seems difficult even for some small configurations [10].

Among small configurations the wicket's Turán mystery can be paralleled by the sail's Ramsey mystery (see [8], [9]): does Theorem 3.1 hold for the sail?

References

[1] W.G. Brown, P. Erdős, V.T. Sós, Some extremal problems on r-graphs, in *New directions in the theory of graphs, Proc. 3rd Ann Arbor Conference on Graph Theory*, Academic Press, New York, 1973, pp. 55-63.

[2] C. Collier-Cartaino, N. Graber, T. Jiang, Linear Turán numbers of linear cycles and cycle-complete graph Ramsey numbers, *Combinatorics, Probability and Computing*, **27** (2018), 358-386.

[3] C. J. Colbourn, A. Rosa, Triple systems, *Oxford Mathematical Monographs*, Calendron Press, Oxford, 1999.

[4] P. Erdős, On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42.

[5] Z. Füredi, A. Gyárfás, An extension of Mantel's theorem to k-graphs, *American Mathematical Monthly* **127** (2020) 263-268.

[6] Z. Füredi, M. Ruszinkó, Uniform hypergraphs containing no grid, *Advances in Mathematics* **240** (2013), 302-324.

[7] L. Gishboliner, A. Shapira, Constructing dense grid-free linear 3-graphs, *arXiv:2010.144692v2*.

[8] E. Granath, A. Gyárfás, J. Hardee, T. Watson, X. Wu, Ramsey theory on Steiner triples, *Journal of Combinatorial Designs*, **26** (2018), pp. 5-11.

[9] A. Gyárfás, G. N. Sárközy, Turán and Ramsey numbers in linear triple systems, *Discrete Mathematics*, **344** (2021), 112258.

[10] A. Gyárfás, M. Ruszinkó, G.N. Sárközy, Linear Turán numbers of acyclic triple systems, submitted.

[11] J. Komlós and M. Simonovits, Szemerédi's Regularity Lemma and its applications in graph theory, in *Combinatorics, Paul Erdős is Eighty* (D. Miklós, V.T. Sós, and T. Szőnyi, Eds.), pp. 295-352, Bolyai Society Mathematical Studies, Vol. 2, Budapest, 1996.

[12] F. Lazebnik, J. Verstraëte, On hypergraphs of girth five, *Electronic Journal of Combinatorics* **10** (2004) R25.

[13] I. Z. Ruzsa, E. Szemerédi, Triple systems with no six points carrying three triangles, in *Combinatorics (Keszthely, 1976), Coll. Math. Soc. J. Bolyai 18, Volume II.* pp. 939-945.