Order plus size of τ-critical graphs

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Abstract

Let $G = (V, E)$ be a τ-critical graph with $\tau(G) = t$. Erdős and Gallai proved that $|V| \leq 2t$ and the bound $|E| \leq \binom{t+1}{2}$ was obtained by Erdős, Hajnal and Moon. We give here the sharp combined bound $|E| + |V| \leq \binom{t+2}{2}$ and find all graphs with equality.

A set of vertices meeting every edge of a graph G is called a *transversal set* of G. The *transversal number* of G, $\tau(G)$, is defined to be the the minimum cardinality of a transversal set of G. A simple graph $G = (V, E)$ with no isolated vertex is called *τ-critical* if $\tau(G - e) = \tau(G) - 1$, for every $e \in E$ (where $G - e = (V, E \setminus \{e\})$). The primary sources for the properties of τ-critical graphs are Lovász and Plummer [5, Chapter 12.1], and Lovász [4, Chapter 8, Exercises 10–25].

The tight bounds for the number of edges and the number of vertices in a τ-critical graph $G = (V, E)$ with $\tau(G) = t$ are:

$$|V| \leq 2t \quad \text{and} \quad |E| \leq \binom{t+1}{2}. \tag{1}$$

The vertex bound is due to Erdős and Gallai [1], and the edge bound was obtained by Erdős, Hajnal, and Moon [2]. Here we derive the combined bound $|E|+|V| \leq \binom{t+2}{2}$ and determine all extremal graphs (Theorem 1). Note that the combined bound immediately gives the edge bound in (1) since $|V| \geq t+1$. The proof of the combined bound comes easily from the next degree bound.

Theorem A. [Hajnal [3]] *Let $G = (V, E)$ be a τ-critical graph of order n with $\tau(G) = t$. Then $deg_G(x) \leq 2t - n + 1$ for every $x \in V$.* □

Theorem 1. *If $G = (V, E)$ is a τ-critical graph of order n with $\tau(G) = t$, then*

$$|V| + |E| \leq \binom{t+2}{2}. \tag{2}$$

Furthermore, the bound is tight if and only if $G \cong K_n$, $n \geq 2$, or $G \cong 2K_2$ or $G \cong C_5$.

Proof. By Theorem A, we have

$$|V| + |E| \leq n + \frac{n(2t - n + 1)}{2}, \tag{3}$$

with equality if G is a $(2t - n + 1)$-regular graph. To prove (2), we show that the right hand side of (3) is at most $\binom{t+2}{2}$. This is equivalent to $0 \leq (n-t-1)(n-t-2)$ which is clearly true, since $n \geq t + 1$, with equality only for $n = t + 1$ or $n = t + 2$. Thus equality in (2) is possible only for $(n-1)$-regular and for $(n-3)$-regular graphs.

In the first case $G = K_n$. In the second case the candidates are the graphs whose complements are 2-regular (and have at least four vertices). Since these graphs are τ-critical with $t = n-2$, the deletion of any edge creates a set of three vertices inducing no edges; equivalently, including an edge in their complements produces a triangle. This implies that the complement of such a graph G must be a single cycle C, since otherwise, deletion of an edge between two cycles creates no triangle. In addition, C has at most five vertices because deletion of a long diagonal would not create a triangle. Thus C is a four-cycle (and then $G = 2K_2$), or C (and its complement G) is a five cycle. $\square$

References

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