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PERFECT MATCHINGS IN SHADOW COLORINGS

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Happy birthday Endre!

Abstract. This birthday note gives a "non-asymptotic" version of our earlier result with G. N. Sárközy and Szemerédi [3], in which Endre had the lion's share.

A hypergraph H with vertex set V defines the *shadow graph* $G(H)$ whose vertex set is V and whose edge set is the set of pairs of V that are covered by some hyperedge of H. An edge coloring C of H defines a multicoloring, the *shadow coloring* C' on $G(H)$, by assigning all colors of C to an edge xy of $G(H)$ that appear on some edge of H containing $\{x, y\}$. A *matching* in a graph is a set of pairwise disjoint edges. A matching in a graph is *perfect* if it covers all vertices of the graph.

I show that in every $(r-1)$-coloring C of the complete r-uniform hypergraph K_n^r there is a monochromatic perfect matching in the shadow coloring C' (assuming $n \geq r \geq 2$ and n is even).

In many applications of the Regularity Lemma we start from a "non-asymptotic" statement and develop an "asymptotic" version of it. Here I turn this approach upside down and start from the following result, in which Endre had the lion's share.

Theorem 1 (Gyárfás, Sárközy, Szemerédi [3]). *In every $(r-1)$-coloring of the edges an almost complete r-uniform hypergraph there is an almost perfect monochromatic matching in the shadow coloring.*

Theorem 1 was conjectured by Gyárfás, Lehel, Sárközy and Schelp [1], where we proved that it implies for large enough n that in every $(r-1)$-coloring of the edges of the complete r-uniform hypergraph K_n^r there is a monochromatic Berge cycle with $(1 - o(n))n$ vertices. The stronger conjecture in [1] that for fixed r and sufficiently large n in every $(r-1)$-coloring

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of the edges of the complete r-uniform hypergraph K_n^r there is a monochromatic Hamiltonian Berge cycle, was proved eventually by Omidi [5].

I leave it to the imagination of the reader what "almost" means in Theorem 1, but state and prove here its "non-asymptotic" (and obviously sharp) version.

THEOREM 2. *Assume that n is even, $n \geq r \geq 2$ and C is an $(r-1)$-coloring of K_n^r. Then there is a monochromatic perfect matching in the shadow coloring C'.*

It is worth noting that Omidi's result [5] (with a quite involved proof relying also on [4]) implies Theorem 2 for large enough n ($n > 6r\binom{4r}{r-1}$). In contrast, in our simple proof we need that Theorem 2 is true for $n \leq \binom{r-1}{2}$. Luckily, this will follow from the following Ramsey-type result for matchings in colored complete graphs.

THEOREM 3 (Gyárfás, Sárközy, Selkow, [2, Corollary 2.4]). *In every t-coloring of the edges of K_m with even $m \leq 2^t - 2$, there is a perfect matching colored with at most $t-1$ colors.*

PROOF OF THEOREM 2. We assume that the colors of C are from $[r-1] = \{1, 2, \ldots, r-1\}$. Consider a minimal counterexample $K = K_n^r$ with an $(r-1)$-coloring C, first with minimal r then with minimal n. Note that $r \geq 3$, since for $r = 2$ the theorem is trivial for all n. We collect some properties of K_n, the shadow graph of K.

1. No edge xy of K_n receives all the $r-1$ colors in C'. Otherwise we remove x, y from the vertex set and (since the obtained hypergraph is not a counterexample) we find a monochromatic, say red perfect matching in the remaining K_{n-2} and this can be extended by xy to a perfect red matching of K_n contradicting the assumption that K is a counterexample.

2. No edge xy of K_n receives $i \leq r-3$ colors in C'. This is clear for $r = 3$, thus $r \geq 4$. In this case by the removal of x, y we consider $K' = K_{n-2}^{r-2}$ on which we define an i-coloring D by assigning the color of $e \cup \{x, y\}$ in C to any edge e of K_{n-2}^{r-2}. Since K' is not a counterexample, we find a monochromatic, say red perfect matching M in the coloring D'. There is at least one red hyperedge e in K_n^r covering $\{x, y\}$, thus the pair x, y extends M to a perfect red matching of K_n^r contradicting the assumption that K is a counterexample.

3. By 1 and 2 we may assume that every edge of K_n is colored in C' with *exactly* $(r-2)$ colors from the set $[r-1]$.

4. We may assume that $n > 2^{r-1} - 2$. Indeed, otherwise we can apply Theorem 3 with $t = r-1$ to the coloring of K_n obtained from C' by representing the color set $[r-1] \setminus i$ of an edge with a single color c_i. From Theorem 3 we have a perfect matching of K_n missing a color, say c_i, which means that the color sets of the edges of this matching all contain the color i

in the coloring C'. This contradicts the assumption that K is a counterexample.

For all $i \in [r-1]$ we define X_i as the set of vertices $v \in V(K_n)$ for which all edges incident to v contain color i in C'.

Claim 4. *Every vertex v of K_n belongs to at least two X_i-s.*

Proof. Suppose for contradiction that for all but at most one $i \in [r-1]$ there exists an edge vw_i colored by the colors of $[r-1] \setminus \{i\}$ in C'. If there is no exceptional i then the hyperedge $e = \{v, w_1, \dots, w_{r-1}\}$ of K_n^r cannot get a color in C, contradiction. If there is one exceptional i, say no edge incident to v has color set $[r-1] \setminus \{1\}$ then let xy be an edge of K_n with this exceptional color set (it exists otherwise every perfect matching has color 1). Clearly, one can select $r-2$ edges of K_n incident to v with color sets

$$[r-1] \setminus \{2\}, \quad \dots, \quad [r-1] \setminus \{r-1\}$$

that with xy span at most r (r or $r-1$) vertices. These vertices are either inside a hyperedge or form a hyperedge of K_n^r to which no color of C can be assigned, a contradiction again, proving the claim. $\square$

Let M_i be a maximum matching of $V(K_n) \setminus X_i$ with all edges containing color i in C' and set $Y_i = V(K_n) \setminus (X_i \cup V(M_i))$.

Claim 5. *$|X_i| < |Y_i|$ for all $i \in [r-1]$.*

Proof. Suppose for contradiction that for some $i \in [r-1]$ we have $|Y_i| \le |X_i|$. Then there is a matching M_i' in the complete bipartite graph $[X_i, Y_i]$ containing all vertices of Y_i. Define $Z_i = X_i \setminus V(M_i')$ and observe that $|Z_i|$ is even thus has a perfect matching M_i'' in Z_i. From the definition of X_i, M_i, all edges of the perfect matching $M_i^* = M_i \cup M_i' \cup M_i''$ of K_n contain color i in C'. This contradicts the assumption that K is a counterexample. $\square$

Using that $\binom{r-1}{2} \le 2^{r-1} - 2$, Property 4, Claims 4, 5 we get

$$n + \binom{r-1}{2} \le n + 2^{r-1} - 2 < n + n = 2n \le \sum_{i \in [r-1]} |X_i| < \sum_{i \in [r-1]} |Y_i|. \tag{1}$$

However, (1) implies that $n + \binom{r-1}{2} < \sum_{i \in [r-1]} |Y_i|$ thus there exist $i \ne j$ such that $|Y_i \cap Y_j| \ge 2$. From the choice of M_i, M_j the color set in C' assigned to an edge xy with $\{x, y\} \subset Y_i \cap Y_j$ can contain neither i nor j. This contradicts Property 3, finishing the proof. $\square$

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