Matchings with few colors in colored complete graphs and hypergraphs

András Gyárfás*† Gábor N. Sárközy*‡§

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Abstract

The t-color Ramsey problem for hypergraph matchings was settled by the well-known result of Alon, Frankl and Lovász (answering a conjecture of Erdős). This result was the last step in a chain of special cases most notably Lovász's solution to Kneser's problem. We proposed an extension of the Erdős problem: for given $1 \leq s \leq t$, what is the maximum number of vertices that can be covered by a matching having at most s colors in every t-coloring of the edges of the complete graph K_n (or hypergraph K_n^r).

We revisit the first unknown case, $r = 2, s = 2, t = 4$, where we conjectured that in every 4-coloring of K_n there is a bicolored matching covering at least $\lfloor 3n/4 \rfloor$ vertices. We prove that this is true asymptotically by applying a recent twist of a standard application of the Regularity method: instead of lifting a (bicolored) matching of the reduced graph to regular cluster pairs, we lift a (bicolored) *basic 2-matching*, a subgraph whose connected components are edges and odd cycles. To find the bicolored basic 2-matching with at least $\lfloor 3n/4 \rfloor$ vertices in every 4-coloring of K_n we use Tutte's minimax formula.

1 Introduction

Let K_n^r denote the complete r-uniform hypergraph on n vertices, i.e. all r-sets of an n element ground set. A matching M in a hypergraph is a set of pairwise vertex disjoint edges, the size of M, $|M|$ is the number of edges in M.

The Ramsey number of matchings comes from the following well-known theorem of Alon, Frankl and Lovász [2].

*Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences, Budapest, P.O. Box 127, Budapest, Hungary, H-1364. gyarfas.andras@renyi.mta.hu, sarkozy.gabor@renyi.mta.hu

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‡Computer Science Department, Worcester Polytechnic Institute, Worcester, MA.

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Theorem 1.1. *([2]) Suppose $n = (t-1)(k-1) + kr$ and a coloring of the edges of K_n^r is given with t colors. Then there exists a monochromatic matching M such that $|M| \geq k$.*

Theorem 1.1 (conjectured by Erdős [7]) is at the crossroad of combinatorics and topology and were preceded by several notable special cases. The graph case ($r = 2$) is due to Cockayne and Lorrimer [5]. The 2-color case ($t = 2$) was solved in [1] and [10]. The case $k = 2$ is Kneser's conjecture, solved by Lovász [23] who introduced topological methods, then Bárány [4], Greene [9] and Matousek [25] gave new proofs.

Theorem 1.1 is sharp. To describe briefly certain t-colorings of K_n^r, consider *partition vectors* with t positive integer coordinates whose sum is equal to n. Assuming that $V(K_n^r) = \{1, 2, \ldots, n\}$, $[p_1, p_2, \ldots, p_t]$ represents the coloring obtained by partitioning $V(K_n^r)$ into parts A_i so that $|A_i| = p_i$ for $i = 1, 2, \ldots, t$ and the color of any edge e is the minimum j for which e has non-empty intersection with A_j. With this notation, the coloring

$$[k-1, k-1, \ldots, k-1, kr-1]$$

shows that Theorem 1.1 is sharp.

A possible extension of Theorem 1.1 was proposed in [17] (for hypergraphs in [11]) introducing an additional integer parameter s satisfying $1 \leq s \leq t$. A matching with edges colored by at most s distinct colors (out of t colors) is called an *s-colored matching.* When $s = 1, 2$ it is natural to use the terms *monochromatic* and *bicolored*. The problem is to find the smallest $n = n(r, k, s, t)$ such that in any t-coloring of the edges of K_n^r there is an s-colored matching of size k. For $s = 1$ Theorem 1.1 provides the answer. We do not risk a conjecture for $n(r, k, s, t)$ in general. However, the partition vector

$$[p, p, \ldots, p, pr, pr^2, \ldots, pr^{s-1}, pr^s], \tag{1}$$

(where the first $t - s$ coordinates are p-s), suggests that in certain special cases we have the following:

$$n(r, k, s, t) = kr + \left\lfloor \frac{(k-1)(t-s)}{1 + r + r^2 + \ldots r^{s-1}} \right\rfloor. \tag{2}$$

Indeed, (2) is trivial for $s = t$ and equivalent to Theorem 1.1 for $s = 1$. Furthermore, this has been conjectured [11] and proved in the following special cases:

- $s = 2, t = 3$. Proved for $r = 3$ by Terpai [27].
- $r = 2, s = t - 1$. Proved in [17].
- $s = t-1, n = \sum_{i=1}^{t-1} r^i$: In every t-coloring of K_n^r there is a perfect matching missing at least one color. (For $r = 2$ follows from the previous item.)

Perhaps (2) is true in some other cases. However, it cannot be true in all cases. Indeed, for $s = 2, t = 6$, the partition vector $[p, p, p, p, p, 2p]$ with $p = \lfloor \frac{k-1}{2} \rfloor$ gives a lower bound on $n(2, k, 2, 6)$ asymptotic to $\frac{7k}{2}$ which is larger than the $\sim \frac{10k}{3}$ obtained from (2) (based on $[p, p, p, p, 2p, 4p]$).

In this paper we address the smallest unknown graph case, $s = 2, t = 4$ and show that (2) (provided by $[p, p, 2p, 4p]$) is asymptotically true in this case as well. In fact, we prove asymptotically the following conjecture [17].

Conjecture 1.2. *Every 4-coloring of K_n contains a bicolored matching covering at least $\lfloor 3n/4 \rfloor$ vertices.*

Our tool for the proof is the notion of a *basic 2-matching*: a collection of vertex-disjoint odd cycles and edges. Our main result is the proof of Conjecture 1.2 in a weaker form, for basic 2-matchings instead of (ordinary) matchings.

Theorem 1.3. *Every 4-coloring of K_n contains a basic bicolored 2-matching covering at least $\lfloor 3n/4 \rfloor$ vertices.*

Note that Theorem 1.3 is weaker than Conjecture 1.2 but still best possible. Its advantage is that a rather standard application of the Regularity method allows us to derive from Theorem 1.3 the following asymptotic version of Conjecture 1.2.

Theorem 1.4. *For every $\eta > 0$ there is an $n_0 = n_0(\eta)$ such that if $n \geq n_0$, then every 4-coloring of K_n contains a bicolored matching covering at least $\left(\frac{3}{4} - \eta\right) n$ vertices.*

A slight deviation from the standard applications, where matchings of the reduced graph are lifted to regular cluster-pairs of the original graph, is the presence of odd cycle components in the basic 2-matching appearing in Theorem 1.3. This method was used recently in [6] and in [22] and in our opinion it will find many future applications. The details of proving Theorem 1.4 from Theorem 1.3 are described in Section 3.

For the reader familiar with fractional matchings, we note that the size of a maximum fractional matching is equal to the size of a maximum 2-matching: the maximum sum of edge weights $0, 1, 2$ with sum of weights at most two at each vertex. Thus a 2-matching (in contrast to a basic 2-matching) may contain odd paths with weights one, and those in the reduced graph cannot be lifted with the method described. Fortunately, it follows from a result of Tutte [29] that there exists a basic 2-matching with as many vertices as the size of any maximum 2-matching (or maximum fractional matching).

We define the *2-deficiency* of G, $def_2(G)$, as the number of vertices uncovered by any maximum basic 2-matching of G. The formula for $def_2(G)$ is due to Tutte [29]. There are several different forms, see for example Problem 7.37 in [23], Theorem 2.2.6 in [28]. Let $q(G)$ denote *the number of isolated vertices of a graph.*

Lemma 1.5. *(Tutte)*
$def_2(G) = max\{q(V(G) \setminus X) - |X| : X \subset V(G)\}$.

A set X achieving the maximum in Lemma 1.5 is called a *Tutte set.*

2 Proof of Theorem 1.3

Let G be a 4-colored K_n with vertex set V, where the colors are $1, 2, 3$ and 4. Assume indirectly that the 2-deficiency of the graphs obtained as the union of edges in any two colors is at least $\lceil n/4 \rceil + 1 = p + 1$.

2.1 Partition by Tutte sets of complementary color pairs.

We apply Lemma 1.5 for the graphs G_{12}, G_{34} determined by the edges of G with colors $(1, 2)$ and $(3, 4)$, respectively, and denote the corresponding Tutte sets by X_1 for G_{12} and Y_1 for G_{34}. Let A_1, B denote the set of isolated vertices in $V \setminus X_1, V \setminus Y_1$, respectively. We may assume wlog that $|B| \geq |A_1|$.

Set

$$B_1 = B \setminus X_1, B_1' = V \setminus (A_1 \cup B_1 \cup X_1 \cup Y_1).$$

Moreover, set $X_1' = V \setminus (A_1 \cup B_1 \cup B_1')$. Note that $P_1 = \{A_1, B_1, B_1', X_1'\}$ forms a partition of V.

Using the indirect assumption and Lemma 1.5, we have

$$|A_1| \geq |X_1| + p + 1 \text{ and } |B| \geq |Y_1| + p + 1. \tag{3}$$

Note that the edges within A_1 and the edges in $[A_1, B_1], [A_1, B_1']$ are all in colors 3 and 4 and the edges within B_1 and the edges in $[B_1, B_1']$ are all in colors 1 and 2.

Lemma 2.1. *$A_1 \cap B = \emptyset$ and $A_1 \subseteq Y_1$.*

Proof. Note that $A_1 \cap B = A_1 \cap B_1$, since A_1 is disjoint from X_1. First we show that $|A_1 \cap B_1| \leq 1$. Indeed, otherwise suppose that $v, w \in A_1 \cap B_1$. Then the color of the edge (v, w) must be in $\{3, 4\} \cap \{1, 2\} = \emptyset$, a contradiction. Suppose next that $\{v\} = A_1 \cap B_1$. Using (3) and $|B| \geq |A_1|$, we have

$$|B \setminus (A_1 \cup X_1)| \geq |B| - 1 - |X_1| \geq |A_1| - 1 - |X_1| \geq p \geq 1,$$

thus there exists $w \in B \setminus (A_1 \cup X_1)$ so the color of the edge (v, w) must be in $\{3, 4\} \cap \{1, 2\} = \emptyset$, a contradiction. If there exists $v \in A_1 \setminus Y_1$ then for any $w \in B_1$, (v, w) cannot have a color, implying $A_1 \subseteq Y_1$. $\square$

From Lemma 2.1 we can write $Y_1 \setminus X_1 = A_1 \cup Z$ where $Z = Y_1 \setminus (A_1 \cup X_1)$. Then, using (3), we get

$$|B| \geq |Y_1| + p + 1 \geq |A_1| + |Z| + p + 1 \geq |X_1| + |Z| + 2p + 2, \tag{4}$$

implying

$$|B_1| \geq |Z| + 2p + 2. \tag{5}$$

Now (3), (5) and Lemma 2.1 imply

$$X_1' = X_1 \cup Z = V \setminus (A_1 \cup B_1 \cup B_1') \leq n - (|A_1| + |B_1|) \leq n - (|X_1| + |Z|) - (3p + 3),$$

from which we conclude that

$$|X_1'| \leq \frac{n-3p-3}{2}. \tag{6}$$

From the above we observe the following properties.

(i) A_1 spans a complete graph in colors 3 and 4.

(ii) B_1 spans a complete graph in colors 1 and 2.

(iii) $[A_1, B_1]$ is a complete bipartite graph in colors 3 and 4.

(iv) $[A_1, B_1']$ is a complete bipartite graph in colors 3 and 4.

(v) $[B_1, B_1']$ is a complete bipartite graph in colors 1 and 2.

(vi) $|X_1'| \leq \frac{n-3p-3}{2}$.

We can repeat the same argument for the graph pairs G_{13}, G_{24} and G_{14}, G_{23} to obtain partitions $P_2 = \{A_2, B_2, B_2', X_2'\}$ and $P_3 = \{A_3, B_3, B_3', X_3'\}$ of V and get the analogue properties (i)-(vi). We will show that this is impossible simultaneously.

Since the definition of A_j depends on which of $q(V \setminus X_j)$ and $q(V \setminus Y_j)$ is the smaller, we have to distinguish two possibilities. We say that three color pairs with a common color (as $(1,2)$, $(1,3)$ and $(1,4)$) are *star-type*, otherwise (as $(1,2)$, $(1,3)$ and $(2,3)$) they are *triangle-type*.

Let $X_i'' = X_i' \cup B_i'$. Then (3) and (5) imply

$$|X_i''| \leq n - 3p - 3. \tag{7}$$

2.2 Properties of AB atoms.

The three partitions P_i determine 4^3 *atoms*, sets obtained as intersections of three sets, one from each P_i. (We use the product notation for these intersections, for example $A_1B_2'X_3'$ is an atom.) First we consider the (eight) atoms with only the A_i, B_i sets and we call these AB-atoms. Some restrictions on these are described in the following claims. We have two cases.

Case 1: The coloring of the A_i-s is triangle-type, say wlog $(3,4)$ for A_1, $(1,3)$ for A_2 and $(1,4)$ for A_3 (and then the coloring of the B_i-s is star-type).

Claim 1. $|A_1A_2A_3| \leq 1$ *and if equality holds then we cannot have a non-empty atom with exactly two A-s.*

Proof. Indeed, otherwise if we have two vertices $v, w \in A_1A_2A_3$, then the edge (v, w) cannot have a color (it must have color $\{3,4\} \cap \{1,3\} \cap \{1,4\}$). The proof of the second half of the statement is similar. $\square$

Claim 2. *We cannot have two non-empty atoms with exactly two A-s, say $A_1A_2B_3$ and $A_1B_2A_3$.*

Proof. Indeed, otherwise if $v \in A_1A_2B_3$ and $w \in A_1B_2A_3$, then the edge (v, w) cannot have a color (it must have color $\{3,4\} \cap \{1,3\} \cap \{1,4\}$). $\square$

Claim 3. *For every non-empty atom with exactly two B-s, say $A_1B_2B_3$ we have $|A_1B_2B_3| \leq 1$ and if equality holds then $|B_1B_2B_3| = 0$.*

Proof. Indeed, otherwise if $v, w \in A_1B_2B_3$, then the edge (v, w) cannot have a color (it must have color $\{3,4\} \cap \{2,4\} \cap \{2,3\}$). The proof of the second half of the statement is similar. $\square$

These claims allow the following possibilities.

- **Case 1.a**: $|A_1A_2A_3| = 1$. Then $|B_1B_2B_3| = 0$. Claim 1 implies that there are no atoms with exactly two A-s. Claim 3 implies that $X_2'' \cup X_3''$ covers B_1, apart from at most two one-vertex atoms $(B_1B_2A_3, B_1A_2B_3)$.
- **Case 1.b**: $|A_1A_2A_3| = 0, |B_1B_2B_3| \neq 0$. Claim 3 implies that no atom has exactly two B-s. Claim 2 implies that at most one atom has exactly two A-s, say $B_1A_2A_3$.
- **Case 1.c**: $|A_1A_2A_3| = |B_1B_2B_3| = 0$ and there exists an atom with exactly two A-s, say $B_1A_2A_3$. From Claim 2 no other atoms have two A-s. From Claim 3 the atoms with exactly two B-s have at most one vertex (but $A_1B_2B_3$ cannot exist). Then $X_1'' \cup X_3''$ covers B_2, apart from at most one one-vertex atom $(B_1B_2A_3)$.
- **Case 1.d**: $|A_1A_2A_3| = |B_1B_2B_3| = 0$ and no atom has exactly two A-s. Now, apart from at most two vertices (from the possible one-vertex atoms $B_1A_2B_3, B_1B_2A_3$), B_1 is covered by $X_2'' \cup X_3''$. Note that here all atoms can be empty.

Case 2: The coloring of the A_i-s is star-type, say wlog $(3,4)$ for A_1, $(1,3)$ for A_2 and $(2,3)$ for A_3 (and then the coloring of the B_i-s is triangle-type). Similarly as in Case 1 we get the following claims, their proofs are left to the reader.

Claim 4. $|B_1B_2B_3| \leq 1$ *and if equality holds then we cannot have a non-empty atom with exactly two A-s.*

Claim 5. *We cannot have two non-empty atoms with exactly two B-s.*

Claim 6. *For every non-empty atom with exactly two A-s, say $A_1A_2B_3$, we have $|A_1A_2B_3| \leq 1$.*

These claims allow the following possibilities.

- **Case 2.a**: $|B_1B_2B_3| = 1$. Claim 4 implies that there are no atoms with exactly two A-s. Claim 5 implies that we have at most one atom with exactly two B-s, say $A_1B_2B_3$. Now apart from the vertex in $B_1B_2B_3$, $X_2'' \cup X_3''$ covers B_1.

- **Case 2.b**: $|B_1B_2B_3| = 0$. Claim 5 implies that at most one atom, say $A_1B_2B_3$ is non-empty among the atoms with exactly two B-s. By Claim 6 at most three one-vertex atom has exactly two A-s. The atom $A_1A_2A_3$ can exist but apart from one vertex (the atom $B_1A_2A_3$), $X_2'' \cup X_3''$ covers B_1. (As in Case 1.d, all atoms can be empty here.)

In Cases 1.a, 1.c, 1.d, 2.a and 2.b some B_i is covered by $\cup_{j \neq i} X_j''$ apart from at most two vertices, say $B_1 \subseteq X_2'' \cup X_3''$, apart from at most two vertices of B_1. Using (5) and (7) we get

$$|X_2''| + |X_3''| \leq 2n - 6p - 6 \text{ and } 2p - 1 \leq |B_1| - 2.$$

However, we claim that $2n-6p-6 < 2p-1$, contradicting the fact that $X_2'' \cup X_3''$ covers all but at most two vertices of B_1. Indeed, this is equivalent to

$$\frac{n}{4} - \frac{5}{8} < p = \left\lceil \frac{n}{4} \right\rceil.$$

Therefore from now on we may assume that Case 1.b holds, thus $|A_1A_2A_3| = 0, |B_1B_2B_3| \geq 1$ and at most one further nonempty AB-atom exists (with exactly two A-s, say $B_1A_2A_3$).

2.3 Properties of ABB' atoms.

The ABB' atoms are the ones in which A, B, B' parts can be selected from the partitions P_i. There are 3^3 ABB' atoms, we need two more claims about these atoms to finish the proof. The first one is an extension of Claim 2 and the second is similar to Claim 3.

Claim 7. *We cannot have two nonempty ABB' atoms with exactly two A-s (where the two A-s are different).*

Proof. Indeed, otherwise if wlog A_1, A_2 and A_1, A_3 are the two A-s, for any of the four choices (B_2 or B_2' from P_2, B_3 or B_3' from P_3), say $v \in A_1A_2B_3$ and $w \in A_1B_2'A_3$, the edge (v, w) cannot have a color: by (i) (applied to A_1) it must have a color from $\{3, 4\}$; by (iv) (applied to $[A_2, B_2']$) it must have a color from $\{1, 3\}$ and by (iii) (applied to $[A_3, B_3]$) it must have a color from $\{1, 4\}$). $\square$

Claim 8. *If $|B_1B_2B_3| > 0$ then we cannot have a non-empty ABB' atom with exactly one A.*

Proof. Let wlog A_1 be the only A. From the condition we have $v \in B_1B_2B_3$. Then, for any of the four choices (B_2 or B_2' from P_2, B_3 or B_3' from P_3), say $w \in A_1B_2'B_3$, the edge (v, w) cannot have a color: by (iii) (applied to $[A_1, B_1]$) it must have a color from $\{3, 4\}$; by (v) (applied to $[B_2, B_2']$) it must have a color from $\{2, 4\}$; by (ii) (applied to B_3) it must have a color from $\{2, 3\}$. $\square$

Thus, since we are in Case 1.b, the only non-empty atoms containing an A could be $B_1A_2A_3$ and perhaps $B_1'A_2A_3$. Then $X_2' \cup X_3'$ covers A_1, i.e. $A_1 \subset X_2' \cup X_3'$. Using property (vi) and (3) we get

$$|X_2'| + |X_3'| \leq n - 3p - 3 \text{ and } p + 1 \leq |A_1|.$$

However, we claim that $n-3p-3 < p+1$, contradicting $A_1 \subseteq X_2' \cup X_3'$. Indeed, this is equivalent to

$$\frac{n-4}{4} < p = \left\lceil \frac{n}{4} \right\rceil .$$

□

3 From basic 2-matchings to matchings via regularity

Here we outline how to prove Theorem 1.4 from Theorem 1.3 via the standard Regularity method based on the Regularity Lemma [26] and the Blow-up Lemma [19, 20]. The material of this section is fairly standard by now (see for example [8, 12, 13, 14, 15, 16, 18, 22, 24] for similar techniques) so we omit some of the details.

3.1 Perturbation

As in many applications of the Regularity Lemma, one has to handle irregular pairs. This translates to a small number of exceptional (or missing) edges in the reduced graph. Thus first we need an "*ε-perturbed*" version of Theorem 1.3.

Lemma 3.1. *For every $\eta > 0$ there exist an $\varepsilon > 0$ and an n_0 such that every 4-coloring of a graph G with $|V(G)| = n \geq n_0$ and $|E(G)| \geq (1-\varepsilon)\binom{n}{2}$ contains a bicolored basic 2-matching covering at least $\left(\frac{3}{4} - \eta\right) n$ vertices.*

Proof. Basically we have to follow the same proof as for complete graphs with minor modifications, so we omit some of the details.

We will need the following two standard, well-known tools about ε-perturbed graphs (see e.g. [14]). $\Delta(G)$ denotes the maximum degree of a graph G, $\overline{H}$ the complement of a graph H.

Lemma 3.2 (Lemma 9 in [14])**.** *Let G be a graph with $|V(G)| = n$ and $|E(G)| \geq (1-\varepsilon)\binom{n}{2}$. Then G has a subgraph H with at least $(1-\sqrt{\varepsilon})n$ vertices such that $\Delta(\overline{H}) < \sqrt{\varepsilon}n$.*

Lemma 3.3 (Lemma 10 in [14])**.** *Assume $\Delta(\overline{G}) < \sqrt{\varepsilon}n$ for a graph G and $H = [A, B]$ is a bipartite subgraph of G with $2\sqrt{\varepsilon}n < |A| \leq |B|$. Then H is a connected subgraph of G. Moreover, if only $2\sqrt{\varepsilon}n < |B|$ and $A \neq \emptyset$ are assumed then there is a subgraph H' which is connected and covers A and all but at most $\sqrt{\varepsilon}n$ vertices of B.*

To prove Lemma 3.1 we proceed similarly as in the proof of Theorem 1.3 with some straightforward modifications. Assume that we have a 4-coloring of a graph G with $|V(G)| = n \geq n_0$ and $|E(G)| \geq (1-\varepsilon)\binom{n}{2}$, where ε is sufficiently small compared to η.

First we apply the *trimming* lemma, Lemma 3.2, for G and we remove at most $\sqrt{\varepsilon}n$ vertices such that in the remaining graph G' on $n' \geq (1-\sqrt{\varepsilon})n$ vertices for every vertex the number of non-neighbors is less than $\sqrt{\varepsilon}n$. Then for G' we proceed as in the proof of Theorem 1.3. Again we assume indirectly that in G' for the union of any two colors the deficiency is at least

$$n' - \frac{3n}{4} + \eta n + 1 \geq (1-\sqrt{\varepsilon})n - \frac{3n}{4} + \eta n + 1 = \frac{n}{4} + (\eta - \sqrt{\varepsilon})n + 1 = p + 1. \quad (8)$$

Throughout the proof *complete graphs* have to be replaced by *almost complete graphs*, where the number of non-neighbors is less than $\sqrt{\varepsilon}n$. Instead of proving that a set S is *non-empty* we need that $|S| \geq \sqrt{\varepsilon}n$.
The perturbed proof of Theorem 1.3: We proceed for G' as in the proof of Theorem 1.3. Lemma 2.1 is identical but the proof is slightly different. First we show that $|A_1 \cap B_1| < \sqrt{\varepsilon}n$. Indeed, otherwise we can select $v, w \in A_1 \cap B_1$ such that (v, w) is an edge in G'. Then the color of this edge (v, w) must be in $\{3, 4\} \cap \{1, 2\} = \emptyset$, a contradiction. Suppose next that $v \in A_1 \cap B_1$. Using (3) and $|B| \geq |A_1|$, we have

$$|B \setminus (A_1 \cup X_1)| \geq |B| - \sqrt{\varepsilon}n - |X_1| \geq |A_1| - \sqrt{\varepsilon}n - |X_1| \geq p + 1 - \sqrt{\varepsilon}n \geq \sqrt{\varepsilon}n,$$

(using $\varepsilon \ll \eta$) and thus there exists $w \in B \setminus (A_1 \cup X_1)$ such that (v, w) is an edge in G'. Then the color of this edge (v, w) must be in $\{3, 4\} \cap \{1, 2\} = \emptyset$, a contradiction. If there exists $v \in A_1 \setminus Y_1$ then we select a $w \in B_1$ such that (v, w) is an edge in G'. Then this edge again cannot have a color, implying $A_1 \subseteq Y_1$.

Inequalities (4), (5), (6) remain valid and imply properties (i)-(v) with complete replaced by almost complete and property (vi) becomes

$$|X_1'| \leq \frac{n' - 3p - 3}{2} \leq \frac{n - 3p - 3}{2}.$$

Claims 1, 2, 3 are modified as follows.

Claim 9. *$|A_1A_2A_3| < \sqrt{\varepsilon}n$ and if it is > 0, then for every atom with exactly two A-s, say $B_1A_2A_3$, we have $|B_1A_2A_3| < \sqrt{\varepsilon}n$.*

Claim 10. *We cannot have two atoms with exactly two A-s, say $A_1A_2B_3$ and $A_1B_2A_3$, such that $|A_1A_2B_3| \geq \sqrt{\varepsilon}n$ and $|A_1B_2A_3| \geq \sqrt{\varepsilon}n$.*

Claim 11. *For every atom with exactly two B-s, say $A_1B_2B_3$, $|A_1B_2B_3| < \sqrt{\varepsilon}n$ and if it is > 0, then $|B_1B_2B_3| < \sqrt{\varepsilon}n$.*

Then we proceed similarly with the possible cases.

- **Case 1.a'**: $0 < |A_1A_2A_3| < \sqrt{\varepsilon}n$. Then $|B_1B_2B_3| < \sqrt{\varepsilon}n$. Claim 9 implies that for every atom with exactly two A-s, say $B_1A_2A_3$, we have $|B_1A_2A_3| < \sqrt{\varepsilon}n$. Claim 11 implies that $X_2'' \cup X_3''$ covers B_1, apart from at most $4\sqrt{\varepsilon}n$ vertices.

- **Case 1.b'**: $|A_1A_2A_3| = 0, |B_1B_2B_3| \geq \sqrt{\varepsilon}n$. Claim 11 implies that no atom has exactly two B-s. Claim 10 implies that there is at most one atom with exactly two A-s, say $B_1A_2A_3$, for which $|B_1A_2A_3| \geq \sqrt{\varepsilon}n$.

The other cases are modified similarly; the details are left to the reader. Then in all cases except Case 1.b', a B_i is covered by $\cup_{j\neq i} X''_j$ apart from at most $4\sqrt{\varepsilon}n$ vertices, say $B_1 \subseteq X''_2 \cup X''_3$, apart from at most $4\sqrt{\varepsilon}n$ vertices of B_1. Using (5) and (7) we get

$$|X''_2| + |X''_3| \leq 2n - 6p - 6 \text{ and } 2p - 4\sqrt{\varepsilon}n \leq |B_1| - 4\sqrt{\varepsilon}n.$$

However, we claim that $2n - 6p - 6 < 2p - 4\sqrt{\varepsilon}n$, contradicting the fact that $X''_2 \cup X''_3$ covers all but at most $4\sqrt{\varepsilon}n$ vertices of B_1. Indeed, this is equivalent to

$$\frac{n}{4} - \frac{3}{4} + \frac{\sqrt{\varepsilon}n}{2} < p = \frac{n}{4} + \eta n - \sqrt{\varepsilon}n$$

(using the fact that η is sufficiently large compared to ε).

Therefore we may assume again that we are in Case 1.b', thus $|B_1B_2B_3| \geq \sqrt{\varepsilon}n$ and perhaps for one of the atoms with exactly two A-s, say $B_1A_2A_3$, we have $|B_1A_2A_3| \geq \sqrt{\varepsilon}n$.

Next we consider again the ABB' atoms (say $A_1B_2B'_3$). Claims 7 and 8 are modified as follows.

Claim 12. *We cannot have two atoms with exactly two A-s where the two A-s are different, say $A_1A_2B_3$ and $A_1B'_2A_3$, such that $|A_1A_2B_3| \geq \sqrt{\varepsilon}n$ and $|A_1B'_2A_3| \geq \sqrt{\varepsilon}n$.*

Claim 13. *If $|B_1B_2B_3| \geq \sqrt{\varepsilon}n$, then we cannot have a non-empty atom with exactly two B-s, say $A_1B'_2B_3$.*

Since we are in Case 1.b', the only atoms containing an A and with size at least $\sqrt{\varepsilon}n$ could be $B_1A_2A_3$ and perhaps $B'_1A_2A_3$. Then $X'_2 \cup X'_3$ covers A_1 apart from at most $4\sqrt{\varepsilon}n$ vertices ($A_1B_2A_3$, $A_1B'_2A_3$, $A_1A_2B_3$ and $A_1A_2B'_3$). Using property (vi) and (3) we get

$$|X'_2| + |X'_3| \leq n - 3p - 3 \text{ and } p - 4\sqrt{\varepsilon}n \leq |A_1| - 4\sqrt{\varepsilon}n.$$

However, we claim that $n - 3p - 3 < p - 4\sqrt{\varepsilon}n$, a contradiction. Indeed, this is equivalent to

$$\frac{n-3}{4} + \sqrt{\varepsilon}n < p = \frac{n}{4} + \eta n - \sqrt{\varepsilon}n$$

(using the fact that η is sufficiently large compared to ε). This finishes the perturbed proof of Theorem 1.3 and thus Lemma 3.1. $\square$

3.2 Regularity

Next we give the standard tools from the Regularity method. Let $e(X, Y)$ denote the number of edges between X and Y in a graph G. For disjoint X, Y, we define the *density*

$$d(X, Y) = \frac{e(X, Y)}{|X| \cdot |Y|}.$$

For two disjoint subsets A, B of $V(G)$, the bipartite graph with vertex set $A \cup B$ which has all the edges of G with one endpoint in A and the other in B is called the pair (A, B).

A pair (A, B) is ε-*regular* if for every $X \subset A$ and $Y \subset B$ satisfying

$$|X| > \varepsilon|A| \text{ and } |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

A pair (A, B) is (ε, δ)-*super-regular* if it is ε-regular and furthermore,

$$deg(a) \geq \delta|B| \text{ for all } a \in A,$$

$$\text{and } deg(b) \geq \delta|A| \text{ for all } b \in B.$$

We need a 4-edge-colored version of the Szemerédi Regularity Lemma.[1]

Lemma 3.4. *For every integer m_0 and positive ε, there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For any n-vertex graph G, where $G = G_1 \cup G_2 \cup G_3 \cup G_4$ with $V(G_1) = V(G_2) = V(G_3) = V(G_4) = V$, there is a partition of V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$ such that*

- *$m_0 \leq \ell \leq M_0$, $|V_1| = |V_2| = \cdots = |V_\ell|$, $|V_0| < \varepsilon n$,*
- *apart from at most $\varepsilon\binom{\ell}{2}$ exceptional pairs, all pairs $G_s|_{V_i \times V_j}$ are ε-regular, where $1 \leq i < j \leq \ell$ and $1 \leq s \leq 4$.*

Our other main tool is the Blow-up Lemma (see [19, 20]). It basically says that super-regular pairs behave like complete bipartite graphs from the point of view of bounded degree subgraphs. Actually we will need the following consequence of the Blow-up Lemma.

Lemma 3.5. *For every $\delta > 0$ there exist an $\varepsilon > 0$ and n_1 such that the following holds. Let G be a bipartite graph with bipartition $V(G) = V_1 \cup V_2$ such that $|V_1| = |V_2| = n \geq n_1$, and let the pair (V_1, V_2) be (ε, δ)-super-regular. Then G contains a perfect matching.*

Note that to prove this lemma directly is much easier than the Blow-up Lemma. Furthermore, an easier approximate version of this lemma would suffice as well, but for simplicity we use this lemma.

[1] For background, this variant and other variants of the Regularity Lemma see [21].

Proof of Theorem 1.4: With these preparations now we are ready to prove Theorem 1.4 from Lemma 3.1. Let $\eta > 0$ be given as in Theorem 1.4. Let

$$\varepsilon \ll \eta \tag{9}$$

(where $\alpha \ll \beta$ means that α is sufficiently small compared to β), let m_0 be sufficiently large compared to $1/\varepsilon$ and n_0 (from Lemma 3.1). Then M_0 is obtained from Lemma 3.4. Suppose we have a 4-coloring of a complete graph with vertex set V, $|V| = n$, where we assume that this is a sufficiently large integer compared to M_0 and n_1 (from Lemma 3.5). We apply Lemma 3.4 with ε. We obtain a partition of V, that is $V = \cup_{0 \le i \le \ell} V_i$. We define the following *reduced graph* G^R: The vertices of G^R are $p_1, \dots, p_\ell$, and there is an edge between vertices p_i and p_j if the pair (V_i, V_j) is ε-regular in all 4 colors. The edge $p_i p_j$ is colored with a majority color in $K(V_i, V_j)$. Thus G^R is a $(1-\varepsilon)$-dense 4-colored graph on ℓ vertices. Applying Lemma 3.1 with $\eta/2$ to G^R, we find a 2-colored (say in colors 1 and 2) basic 2-matching M (its connected components are odd cycles and single edges) in G^R covering at least $\left(\frac{3}{4} - \frac{\eta}{2}\right)\ell$ vertices.

We "lift" M back to the original graph and we remove some vertices from each cluster to achieve super-regularity in the color (1 or 2) of the cluster pair corresponding to M for each edge of M. Then we remove some more vertices from each cluster to achieve that we have exactly the same number of vertices left in each cluster corresponding to M. We may assume that this number is even by removing one more vertex from each cluster if necessary. From ε-regularity and (9) the total number of vertices left in the clusters corresponding to vertices of M is still at least $\left(\frac{3}{4} - \eta\right) n$.

Next we show that there is a perfect matching in colors 1 and 2 in the original graph on the remaining vertices in the clusters corresponding to M and thus finishing the proof of Theorem 1.4. For a single edge in M, Lemma 3.5 immediately implies that we can span the two corresponding clusters with a matching in the color of the edge (1 or 2). Finally, let the clusters $V^1, V^2, \dots, V^{2t+1}, t \ge 1$ correspond to an odd cycle in M. For each cluster V^i we find a random bipartition $V^i = V^i_1 \cup V^i_2$ with $|V^i_1| = |V^i_2|$, $1 \le i \le 2t+1$. Then (V^i_1, V^{i+1}_2) is still super-regular (with slightly weaker parameters) with high probability in the color (1 or 2) of the (V^i, V^{i+1}) edge for each $1 \le i \le 2t+1$, where $2t+2 = 1$. Indeed, regularity follows from the standard Slicing Lemma (Fact 1.5 in [21]), while the minimum degree condition follows from the Chernoff bound (see [3]). Using Lemma 3.5 we find a spanning matching in (V^i_1, V^{i+1}_2) in the color (1 or 2) of the (V^i, V^{i+1}) edge for each $1 \le i \le 2t+1$, where $2t+2 = 1$. This spans all the vertices of the odd cycle finishing the proof. $\square$

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