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HYPERGRAPH FAMILIES WITH BOUNDED EDGE COVER OR TRANSVERSAL NUMBER

A. GYÁRFÁS and J. LEHEL

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The transversal number, packing number, covering number and strong stability number of hypergraphs are denoted by τ, ν, ϱ and α, respectively. A hypergraph family $\mathscr{H}$ is called τ-bound (ϱ-bound) if there exists a "binding function" $f(x)$ such that $\tau(H) \leqq f(\nu(H))$ $(\varrho(H) \leqq f(\alpha(H)))$ for all $H \in \mathscr{H}$. Methods are presented to show that various hypergraph families are τ-bound and/or ϱ-bound. The results can be applied to families of geometrical nature like subforests of trees, boxes, boxes of polyominoes or to families defined by hypergraph theoretic terms like the family where every subhypergraph has the Helly-property.

1. Introduction

An essential part of hypergraph theory (we use the terminology of Berge [1]) concerns the connection between the transversal number, τ, and the packing number, ν, of hypergraphs.

$\tau(H)$: minimum number of vertices of hypergraph H representing all edges;

$\nu(H)$: maximum number of pairwise disjoint edges of hypergraph H.

So far many important hypergraph families satisfying the strongest connection $\nu = \tau$ were extensively investigated. In this paper we are dealing with the weakest possible connection, the functional dependence of these hypergraph numbers. Hypergraph families considered here exhibit the property that τ can be universally bounded by a function of ν. This concept leads to the following definition: a family $\mathscr{H}$ of finite hypergraphs is *τ-bound* if there exists a function $f(x)$ such that

$$\tau(H) \leqq f(\nu(H)) \quad \text{for all} \quad H \in \mathscr{H};$$

the function f is called a *binding function* for $\mathscr{H}$.

A τ-bound family obviously has a smallest binding function. To find this function or to determine its right order of magnitude usually leads to difficult extremal problems. The existence of binding functions is not known even for thoroughly studied hypergraph families. As an example, consider the family $\mathscr{H}$ of Helly-hypergraphs having line-graphs without induced holes and antiholes (C_{2k+1} and $\bar{C}_{2k+1}$ for $k \geqq 2$). It would be interesting to see that $\mathscr{H}$ is τ-bound, a rather

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reasonable claim justified by the strong perfect graph conjecture proposing the smallest imaginable binding function $f(x)=x$ for $\mathscr{H}$.

It is natural to look for general criteria implying a hypergraph family to be τ-bound. Our results show that the existence of a binding function in case of various τ-bound families is essentially based on the exclusion of octahedron graphs from the line-graphs.

Let us denote by Q^k the k-dimensional *octahedron graph* which is obtained from the complete graph K_{2k} by deleting k independent edges.

For our purposes the most important "octahedron-free" families are the tree hypergraphs, strong-Helly hypergraphs and the d-dimensional box hypergraphs which do not contain Q^2, Q^3 and Q^{d+1}, respectively in their line-graphs.

The dual notions of the hypergraph numbers τ and ν are the edge cover number, ϱ, and the strong stability number, α.

$\varrho(H)$: minimum number of edges of hypergraph H covering all vertices;

$\alpha(H)$: maximum number of vertices of hypergraph H containing at most one element from any edge.

Obviously, $\varrho(H)=\tau(H^*)$ and $\alpha(H)=\nu(H^*)$ where H^* denotes the usual hypergraph dual of H.

The notion of ϱ-binding of a family can be defined analogously to τ-binding: a family $\mathscr{H}$ of finite hypergraphs is *ϱ-bound* if there exists a binding function $f(x)$ such that

$$\varrho(H) \leqq f(\alpha(H)) \quad \text{for all} \quad H \in \mathscr{H}.$$

Clearly, $\mathscr{H}$ is ϱ-bound if and only if the family $\{H^* | H \in \mathscr{H}\}$ is τ-bound which means that τ-binding and ϱ-binding are equivalent notions. However, the dual family may appear as an artificial structure for concrete hypergraph families making the choice evident between the dual concepts of τ-binding and ϱ-binding.

Due to the "forbidden subgraph" approach, our results concern τ-bound families (ϱ-bound families) containing every partial hypergraph (subhypergraph) of their members. This property, however, is not assumed a priori on the structures investigated here. For this reason we use the notion of partial hypergraph closure, $\mathrm{part}(\mathscr{H})$, and subhypergraph closure, $\mathrm{sub}(\mathscr{H})$, of a family $\mathscr{H}$.

In Section 2 we apply an earlier clique cover theorem of the first author ([7]) on colored graphs to derive conditions on the existence of binding functions for general hypergraph families. We show that

$\mathrm{sub}(\mathscr{H})$ is ϱ-bound if family $\mathscr{H}$ contains the union of conformal hypergraphs such that their 2-section graphs do not contain octahedron graphs of prescribed dimension (Theorem A);

$\mathrm{part}(\mathscr{H})$ is τ-bound if family $\mathscr{H}$ contains Helly-hypergraphs without Q^k in their line-graphs (Theorem C).

In Section 3 we prove that $\mathrm{part}(\mathscr{H})$ is τ-bound if family $\mathscr{H}$ is the join of Helly hypergraphs without $C_4=Q^2$ in their line-graphs (Theorem 1). As a special case, the family of forest hypergraphs is τ-bound.

We introduce the strong Helly hypergraphs in Section 4. Strong Helly hypergraphs form a self-dual family containing the family of balanced hypergraphs. We show that the family formed by unions of strong Helly-hypergraphs is ϱ-bound (Theorem 2). As a consequence, the family of strong Helly-hypergraphs is ϱ-bound and τ-bound.

Section 5 is devoted to box hypergraphs. The main result here is Theorem 3 stating that $5x^2$ is a ϱ-binding function for the subhypergraphs of 2-dimensional box hypergraphs. Some properties of polyomino hypergraphs are also established and it is proved that x^2 is a ϱ-binding function for subhypergraphs of polyomino hypergraphs.

2. The clique cover theorem and its hypergraph versions

The independence (or stability) number of a graph G is denoted by $\alpha(G)$ and the inclusion $G' \subset G$ is used to indicate that graph G' is an *induced subgraph* of G. The *t-coloring* of G is defined as an edge decomposition $E(G)=E_1 \cup E_2 \cup \dots \cup E_t$. The edges of G belonging to E_i are referred as edges of color i. Note that edges can have more than one color by definition. We say that $G' \subset G$ is *induced in color* i if $E(G') \subset E_i$ and $E(\bar{G}') \cap E_i = \emptyset$, where $\bar{G}'$ denotes the complement of G'.

The following clique cover theorem was proved on colored graphs in [7]:

CC. Theorem. *Let $t, k_1, k_2, \dots, k_t$ be positive integers. Then there exists a function $g(x; k_1, k_2, \dots, k_t)$ with the following property: for every graph G t-colored without induced Q^{k_i} in color i, $1 \leqq i \leqq t$, the vertex set of G can be covered by the vertices of at most $g(\alpha(G); k_1, k_2, \dots, k_t)$ monochromatic complete subgraphs.* ∎

1. It is worth noting that the existence of Ramsey-function $R(n_1, n_2, \dots, n_t)$ immediately follows from CC. Theorem.

2. CC. Theorem is sharp in the following sense: if $t \geqq 2$ and $G_1, G_2, \dots, G_t$ are forbidden induced subgraphs in color i then the function g does not exist whenever $G_i \neq Q^1$, $1 \leqq i \leqq t$, and, for some $1 \leqq j \leqq t$, G_j is not an induced subgraph of any octahedron graph (see [8]).

The clique cover theorem has natural interpretations for hypergraphs. We use the following notions: a hypergraph H' is a *partial hypergraph* of H if

$$E(H') \subset E(H) \quad \text{and} \quad V(H') = \cup \{e \mid e \in E(H')\};$$

H' is a *subhypergraph* of H if

$$V(H') \subset V(H) \quad \text{and} \quad E(H') = \{e \cup V(H') \mid e \in E(H)\}.$$

We denote by part$(\mathscr{H})$ and sub$(\mathscr{H})$ the *partial hypergraph closure* and the *subhypergraph closure* of a family $\mathscr{H}$, respectively: part$(\mathscr{H})$ (sub$(\mathscr{H})$) is the family containing every partial hypergraph (subhypergraph) of any $H \in \mathscr{H}$.

A hypergraph has the Helly-property, or simply speaking H is a *Helly-hypergraph* if $\nu(H')=1$ implies $\tau(H')=1$ for every partial hypergraph $H' \subset H$. A hypergraph H is *conformal* if $\alpha(H')=1$ implies $\varrho(H')=1$ for every subhypergraph H' of H. The Helly-property and conformity are obviously dual notions.

The *line-graph* $L(H)$ of a hypergraph $H=(V, E)$ is defined as follows: the vertex set of $L(H)$ represents the edge set E and the pair $e_i, e_j \in E$ defines an edge of $L(H)$ if and only if $e_i \cap e_j \neq \emptyset$. The *2-section graph* $(H)_2$ is defined on the vertex set V and $v_i v_j$ is an edge of $(H)_2$ if and only if for some $e \in E$; $v_i, v_j \in e$. The line-graph and 2-section graph are obviously dual notions, $L(H)=(H^*)_2$.

The first possible interpretation of the CC. theorem is considering G as the 2-section graph of a hypergraph H. The meaning of a t-coloring of G is that H is the *edge union* of t partial hypergraphs, i.e., $H=H_1\cup H_2\cup\ldots\cup H_t$, where H_i are partial hypergraphs of H satisfying $E(H)=\bigcup_{i=1}^{t} E(H_i)$. If every H_i is conformal, $1\leq i\leq t$, then the monochromatic clique cover of $G=(H)_2$ becomes an edge cover of H. Then we obtain the following version of CC. Theorem:

Theorem A. *Let $t, k_1, k_2, \ldots, k_t$ be positive integers. The subhypergraph closure of family $\mathscr{H}$ is ϱ-bound with binding function $g(x; k_1, k_2, \ldots, k_t)$ if every $H\in\mathscr{H}$ has the form $H=H_1\cup H_2\cup\ldots\cup H_t$ where H_i is conformal hypergraph and $Q^{k_i}\not\subset(H_i)_2$, $1\leq i\leq t$.* ∎

It is worth formulating Theorem A in the special case of $t=1$:

Theorem B. *Let k be a positive integer. If $\mathscr{H}$ is a family such that for every $H\in\mathscr{H}$, H is conformal hypergraph and $Q^k\not\subset(H)_2$ then* sub $(\mathscr{H})$ *is ϱ-bound.* ∎

Another possible interpretation of the CC. theorem is considering G as the line-graph of a hypergraph H. In this way one can easily obtain the dual form of theorem A which gives a condition on the τ-binding of families. This result with different applications are considered in [7]. We mention here only the special case $t=1$, the dual version of theorem B:

Theorem C. *Let k be a positive integer. If $\mathscr{H}$ is a family such that for every $H\in\mathscr{H}$, H has the Helly-property and $Q^k\not\subset L(H)$ then part $(\mathscr{H})$ is τ-bound.* ∎

3. Helly-hypergraphs with C_4-free line-graph

A hypergraph H is a *tree-hypergraph* if one can give a tree T on the vertex set $V(H)$ with the property that every $e\in E(H)$ is a subtree of T. It is well-known that tree-hypergraphs are normal, moreover, they can be characterized as Helly-hypergraphs whose line-graphs do not contain an induced C_i for $i\geq 4$ ([5]).

It may be surprising at first glance that the family $\mathscr{D}$ of Helly-hypergraphs without induced C_i for $i\geq 5$ is not τ-bound. To see that, let G_k be a k-chromatic graph of girth at least 6, $k=1, 2, \ldots$. The existence of G_k follows from a well-known theorem of Erdős and Hajnal ([4]). Let H_k be the dual of the hypergraph defined by the maximal independent sets of G_k. It is easy to see that $H_k\in\mathscr{D}$, $\nu(H_k)=2$ and $\tau(H_k)=k$, $k=1, 2, \ldots$, showing that $\mathscr{D}$ is not τ-bound.

The situation changes if we consider the family of Helly-hypergraphs without $C_4=Q^2$ in their line-graphs: this family is τ-bound by theorem C. A more general statement involving the join operation will be proved in theorem 1 below.

The *join* of hypergraphs $H_1, H_2, \ldots, H_c$ is defined as a hypergraph with vertex set $\bigcup_{i=1}^{c} V(H_i)$ and with edge set $\{e_1\cup e_2\cup\ldots\cup e_c | e_i\in E(H_i),\ 1\leq i\leq c\}$, ([1], p. 488). Let c be a positive integer and $\mathscr{H}$ be a hypergraph family. Consider every hypergraph which is the join of c identical copies of a $H\in\mathscr{H}$ and denote by $\mathscr{H}^{(c)}$ the partial hypergraph closure of this hypergraph family. As an example, if $\mathscr{T}$ denotes the family of tree-hypergraphs then $\mathscr{T}^{(c)}$ contains the hypergraphs defined by forests with c or less components of a tree, the so-called *c-forest hypergraphs.*

Theorem 1. *Let c be a positive integer. If $\mathscr{H}$ is a family such that for every $H\in\mathscr{H}$, H has the Helly-property and $C_4 \not\subset L(H)$ then $\mathscr{H}^{(c)}$ is τ-bound.*

Proof. Let hypergraph H be the join of c copies of $H_0=(V_0;\, e_1, e_2, \ldots, e_m)$ and $H'=(V;\, E_1, E_2, \ldots, E_p)$ be a partial hypergraph of H. Suppose that $c\geqq 2$ and for every $1\leqq i\leqq p$, $E_i=e_{i1}\cup e_{i2}\cup\ldots\cup e_{ic}$ with $1\leqq i1<i2<\ldots<ic\leqq m$. We say that e_{ij} is the j'th component of E_i and without the loss of generality one can suppose that every edge of H' has exactly c components.

For every $1\leqq j\leqq c$, let G_j be the line-graph of the j-th components, i.e., $V(G_j)=\{1, 2, \ldots, p\}$ and kl is an edge of G_j if and only if $e_{kj}\cap e_{lj}\neq\emptyset$. Since G_j is a subgraph of $L(H_0)$ and $H_0\in\mathscr{H}$, obviously $C_4\not\subset G_j$, $1\leqq j\leqq c$.

We show that the independence number of $G=G_1\cup G_2\cup\ldots\cup G_c$ depends only on the independence number of $L(H')$ and c. Supposing that $\alpha(G)\leqq h(\alpha(L(H'));\, c)$ is true, from the CC. theorem follows that $V(G)$ can be covered by at most $g(\alpha(G);\, c)$ complete subgraphs of G_j's $(1\leqq j\leqq c)$. Since H_0 is a Helly-hypergraph and G is a subgraph of $L(H')$, this clique cover of G corresponds to a transversal of H'. Thus we obtain that $f(x;\, c)=g(h(x;\, c), c)$ is a suitable τ-binding function for $\mathscr{H}^{(c)}$.

Let us colour the edges of $\bar{G}$, the complement of G, as follows: the edge kl $(1\leqq k<l\leqq p)$ is white if $E_k\cap E_l=\emptyset$; kl is blue if E_k and E_l have common components; kl is red otherwise.

The size of a white clique of $\bar{G}$ is obviously at most $\alpha(L(H'))$.

If a blue clique of $\bar{G}$ has q vertices, say $\{1, 2, \ldots q\}$, then every E_k $(1\leqq k\leqq q)$ has a component among $e_{11}, e_{12}, \ldots, e_{1c}$. On the other hand, any component e_{1j} $(1\leqq j\leqq c)$ belongs to at most c edges since $e_{1j}=e_{ki}=e_{li}$ implies $k=l$. Consequently, $q\leqq c^2$, that is, the size of a blue clique in $\bar{G}$ is at most c^2.

Suppose that $\{1, 2, \ldots, r\}$ spans a red clique of $\bar{G}$ and denote by G_0 the induced subgraph of $L(H_0)$ defined by the components of $E_1, E_2, \ldots, E_r$. By the definition of red edges, G_0 is a c-partite graph, i.e., its vertex set is the union of c pairwise disjoint r-element independent sets of $L(H_0)$, and G_0 contains at least $\binom{r}{2}$ edges. Then one can find two vertex classes of G_0 which spans a bipartite graph with at least $\varepsilon\cdot r^2$ edges, where $\varepsilon>0$ is a constant depending only on c. This bipartite graph contains a C_4 if r is large enough, according to a well known density theorem ([6], p. 95.). Since $C_4\not\subset L(H_0)$, r must be smaller than some constant $r(c)$.

The argument above shows that every monochromatic clique of $\bar{G}$ is bounded. From the Ramsey-theorem it follows that the size of the maximal clique of $\bar{G}$, therefore $\alpha(G)$, is not greater than some constant $h(\alpha(L(H'));\, c)$. ∎

Corollary. *The family of c-forest hypergraphs is τ-bound.*

Proof. It is easy to verify that every tree-hypergraph $H\in\mathscr{T}$ has the Helly-property and $C_4\not\subset L(H)$, thus the τ-binding of $\mathscr{T}^{(c)}$ follows from Theorem 1. ∎

Remark. We give an example which demonstrates that $C_4=Q^2$ can not be replaced by Q^3 in Theorem 1. For every $n=1, 2, \ldots,$ let us define a hypergraph $(V;\, e_1, e_2, \ldots$

$\ldots, e_n, f_1, f_2, \ldots, f_n)$ as follows: $V=\{(i,j) | 1 \leqq i, j \leqq n\}$ and $e_p=\{(p,j) | 1 \leqq j \leqq n\}$, $f_p=\{(i,p) | 1 \leqq i \leqq n\}$, $1 \leqq p \leqq n$.

Every hypergraph of this family $\mathscr{H}$ has the Helly-property and every line-graph is triangle-free, especially, contains no Q^3. The hypergraph $(V; e_1 \cup f_1, e_2 \cup f_2, \ldots \ldots, e_n \cup f_n)$ belongs to $\mathscr{H}^{(2)}$ for every $n=1, 2, \ldots$, furthermore, it has packing number 1 and transversal number equal to n. Consequently, $\mathscr{H}^{(2)}$ is not τ-bound.

4. Strong—Helly hypergraphs

A hypergraph H has the *strong-Helly property* if every subhypergraph of H is a Helly-hypergraph.

From the dual form of Gilmore's criterion (see in [1], p. 396) immediately follows:

Proposition. *A hypergraph H has the strong-Helly property if and only if any three edges* $e_1, e_2, e_3 \in E(H)$ *such that* $v_1 \in e_1 \cap e_2$, $v_2 \in e_2 \cap e_3$, $v_3 \in e_3 \cap e_1$ *satisfy* $e_1 \cap e_2 \cap e_3 \cap \{v_1, v_2, v_3\} \neq \emptyset$. ∎

One can observe that interval hypergraphs, more generally, balanced hypergraphs are strong-Helly hypergraphs.

Denote by $\mathscr{S}$ the family of all strong-Helly hypergraphs. The proposition shows that $\mathscr{S}$ is a *self-dual* family, i.e., if $H \in \mathscr{S}$ then $H^* \in \mathscr{S}$, especially, every $H \in \mathscr{S}$ is conformal.

Lemma. *If H is a strong-Helly hypergraph then* $Q^3 \not\subset (H)_2$.

Proof. If $Q^3 \subset (H)_2$ then by the conformity of H there exist three edges of H which meet Q^3 in three pairwise intersecting triangles without common vertex. This contradicts to the strong-Helly property. ∎

Theorem 2. Let t be a positive integer and $\mathscr{H}$ be the family of hypergraphs obtainable as the edge union of t strong Helly hypergraphs. Then sub $(\mathscr{H})$ is ϱ-bound.

Proof. The theorem follows from the lemma and theorem A. ∎

Since $\mathscr{S}$ is self-dual, Theorem 2 implies:

Corollary. *The family of strong-Helly hypergraphs is ϱ-bound and τ-bound.* ∎

Remark. The binding function we know for $\mathscr{S}$ is $x^4/4+O(x^3)$ which is valid for any Q^3-free conformal or Helly families. On the other hand, the best binding function is certainly not linear. Indeed, consider the subfamily of triangle-free graphs of $\mathscr{S}$ and denote by G_k, $k=2, 3, \ldots$, the triangle-free graph with $\alpha(G_k)=k$ and $|V(G_k)| = R(3, k+1)-1$ where R denotes the Ramsey-number. Since $R(3, k+1) \geqq c \cdot k^2/(\log k)^2$, (see in [3]) and $\varrho(G_k) \geqq |V(G_k)|/2$, for the ϱ-binding function of the family $\mathscr{S}$, $f(x) > c' \cdot x^2/(\log x)^2$ follows.

5. Boxes and polyominoes

Let C^d denote the set of cells in the infinite cube grid of the d-dimensional Euclidean space. A *d-dimensional box* is a parallelopiped defined by the union of some cells of C^d. A *box hypergraph* $H=(V, E)$ can be associated to a set of boxes $E=\{B_1, B_2, \dots, B_m\}$ where V denotes the set of cells in $\bigcup_{i=1}^{m} B_i$. The infinite family of d-dimensional box hypergraphs is denoted by $\mathscr{B}^d$.

It is well known that for every $H\in\mathscr{B}^d$, H has the Helly-property and $Q^{d+1} \not\subset L(H)$ ([10]). Then, by theorem C, $\mathscr{B}^d$ is τ-bound.

A subfamily of $\mathscr{B}^2$, the polyomino hypergraphs, has been studied recently by several authors ([2], [9]). A *polyomino* (or animal) is a finite set of cells of C^2. With each polyomino P, we may associate a *polyomino hypergraph* whose vertices are the cells of P and whose edges are the maximal boxes contained in P. We denote by $\mathscr{P}$ the family of polyomino hypergraphs. Since $\mathscr{B}^2$ is τ-bound and part $(\mathscr{P})\subset\mathscr{B}^2$, part$(\mathscr{P})$ is τ-bound.

Proposition 1. *Polyomino hypergraphs are conformal.*

Proof. Let $H\in\mathscr{P}$ defined by the polyomino P. Applying Gilmore's criterion, we have to show that for any three boxes $B_1, B_2, B_3\in E(H)$, the set $X=(B_1\cap B_2)\cup(B_2\cap B_3)\cup(B_1\cap B_3)$ is contained in some $B\in E(H)$. The assertion is true if $\{B_1, B_2, B_3\}$ contains two disjoint boxes, assume that $B_1\cap B_2\cap B_3\neq\emptyset$ and $B_i=[a_i, b_i]\times[c_i, d_i]$ $(i=1, 2, 3)$. Let a and c denote the next to smallest element of $\{a_1, a_2, a_3\}$ and $\{c_1, c_2, c_3\}$. Similarly b and d denote the next to largest element of $\{b_1, b_2, b_3\}$ and $\{d_1, d_2, d_3\}$. All cells of $B'=[a, b]\times[c, d]$ are in P, therefore $B'\subset B$ for some $B\in E(H)$. On the other hand, $X\subset B'$ implies $X\subset B$. ∎

Proposition 2. *If* $H\in\mathscr{P}$ *then* $Q^3\not\subset(H)_2$.

Proof. Assume that $Q^3\subset(H)_2$ for some $H\in\mathscr{P}$ defined by the polyomino P. Let $C=\{c_1, c_2, \dots, c_6\}$ be the set of cells corresponding to the vertices of Q^3 and (x_i, y_i) be an arbitrary point of c_i $(i=1, 2, \dots, 6)$. After suitable reindexing of the cells of C we can ensure that $x_1\leqq x_2\leqq x_3\leqq x_4$ and all pairs $c_i c_j$, $1\leqq i<j\leqq 4$, except $c_2 c_3$ have a box $B(i, j)$ in $E(H)$ containing c_i and c_j. (The pair $c_2 c_3$ is not contained in any box of $E(H)$.) Let B denote the box spanned by c_2 and c_3. It is easy to check that $B\subset B(1, 4)\cup B(1, 3)\cup B(2, 4)$ implying that B lies completely in P. Then B is covered by some edge of $E(H)$ contradicting the fact that $c_2 c_3$ is not contained together in any box of H. ∎

Propositions 1 and 2 show that Theorem B can be applied for the family of polyomino hypergraphs: sub$(\mathscr{P})$ is a ϱ-bound family. We proceed to show that the larger family, sub$(\mathscr{B}^2)$, is also ϱ-bound. It is worth mentioning that $(H)_2$ may contain Q^k for any k and H is not neccessarily conformal if $H\in\mathscr{B}^2$. The next lemma shows that $\mathscr{B}^2$ is not far from being conformal.

Lemma. *Let C be a finite set of cells in the plane and $\mathscr{B}$ be a set of boxes such that for any two cells $c_1, c_2\in C$ there is a box of $\mathscr{B}$ containing c_1 and c_2. Then C can be covered by at most 5 boxes of $\mathscr{B}$.*

Proof. Let B' denote the minimal box containing all cells of C. Each side of B' has a supporting cell from C. We choose four (not neccessarily distinct) cells c_1, c_2, c_3, c_4 of C supporting the four sides of B'. We may assume that c_1 and c_2 support opposite sides of B'. It is easy to see that the boxes of $\mathcal{B}$ containing the pairs $c_i c_j$ for $1 \leqq i < j \leqq 4$, $(i, j) \neq (1, 2)$ cover B'. ∎

We note that 5 can not be replaced by 4 in the lemma.

Theorem 3. *If* $H \in \mathrm{sub}(\mathcal{B}^2)$ *then* $\varrho(H) \leqq 5\alpha^2(H)$.

Proof. Let $H=(V, E)$ be a subhypergraph of a box hypergraph. Let $v_1, v_2 \in V$ and assume that $v_1 v_2$ is an edge of $G=\overline{(H)}_2$. Direct the edge $v_1 v_2$ from v_1 to v_2 if v_1 is placed left to v_2; otherwise $v_1 v_2$ is directed from v_2 to v_1. Assign colors to the (directed) edges of G: $v_1 v_2$ is red if v_1 is placed under v_2; otherwise $v_1 v_2$ is blue. Clearly the directed graphs G_r and G_b spanned by the red and blue edges of G have transitive orientations. Applying the well known fact that the size of the largest clique, ω, is equal to the chromatic number, χ, in comparability graphs, we get $\chi(G) \leqq \omega(G_r) \cdot \omega(G_b)$. Since $\omega(G_r)$, $\omega(G_b) \leqq \alpha(H)$, $(H)_2$ can be covered by at most $\alpha^2(H)$ cliques. The cells corresponding to a clique of $(H)_2$ can be covered by at most 5 edges of H by the lemma. Therefore $\varrho(H) \leqq 5 \cdot \alpha^2(H)$. ∎

Since polyomino hypergraphs are conformal, the proof of Theorem 3 yields the following result.

Proposition 3. *If* $H \in \mathrm{sub}(\mathcal{P})$ *then* $\varrho(H) \leqq \alpha^2(H)$. ∎

References

[1] C. Berge, *Graphs and Hypergraphs,* North-Holland 1973.
[2] C. Berge, C. C. Chen, V. Chvatal and C. S. Seow, Combinatorial properties of polyominoes, *Combinatorica* **1** (3) (1981), 217—224.
[3] P. Erdős, Graph theory and Probability II., *Canad. J. of Math.* **13** (1961), 346—352.
[4] P. Erdős and A. Hajnal, On chromatic number of graphs and set systems, *Acta Math. Acad. Sci. Hung.* **17** (1—2) (1966), 61—99.
[5] C. Flament, Hypergraphes arborés, *Discrete Math.* **21** (1978), 223—227.
[6] R. L. Graham, B. L. Rothschild and J. H. Spencer, *Ramsey Theory,* John Wiley, 1980.
[7] A. Gyárfás, Ramsey-type theorem and its application to relatives of Helly's theorem, *Periodica Math. Hung.* **3** (3—4) (1973) 261—270.
[8] A. Gyárfás, On Ramsey covering numbers, *Coll. Math. Soc. J. Bolyai* **10.**, *Infinite and finite sets,* (1973), 801—816.
[9] E. Győri, A minimax theorem on intervals, *to appear.*
[10] F. S. Roberts, On the boxicity and cubicity of a graph, *Recent Progress in Combinatorics, Proc. Third Waterloo Conf. on Combinatorics,* New York, 1969, 301—310.

A. Gyárfás and J. Lehel

Computer and Automation Institute of the
Hungarian Academy of Sciences
Kende u. 13—17, 1111—Budapest, Hungary