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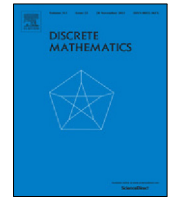
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## Discrete Mathematics

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## Q1 Chromatic Ramsey number of acyclic hypergraphs

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## ABSTRACT

This paper extends to hypergraphs a question discussed in Bialostocki and Gyárfás [4] and Garrison [8]: can one generalize a Ramsey type result from complete host graphs to graphs of sufficiently large chromatic number?

For an  $r$ -uniform tree  $T$  ( $r \geq 2$ ) we define the ( $t$ -color) chromatic Ramsey number  $\chi(T, t)$  as the smallest  $m$  with the following property: if the edges of any  $m$ -chromatic  $r$ -uniform hypergraph are colored with  $t$  colors in any manner, there is a monochromatic copy of  $T$ . The presence of a tree is not accidental:  $\chi(H, t)$  can be defined only for an acyclic hypergraph  $H$  since there are hypergraphs with arbitrary large chromatic number and girth. We prove that

$$\left\lceil \frac{R^t(T, t) - 1}{r - 1} \right\rceil + 1 \leq \chi(T, t) \leq |E(T)|^t + 1$$

where  $R^t(T, t)$  is the  $t$ -color Ramsey number of  $T$ . We give better upper bounds for  $\chi(T, t)$  when  $T$  is a matching or a star, proving that for  $r \geq 2, k \geq 1, t \geq 1, \chi(M_k^r, t) \leq (t - 1)(k - 1) + 2k$  and  $\chi(S_k^r, t) \leq t(k - 1) + 2$  where  $M_k^r$  and  $S_k^r$  are, respectively, the  $r$ -uniform matching and star with  $k$  edges.

The general upper bounds are improved for 3-uniform hypergraphs. We prove that  $\chi(M_k^3, 2) = 2k$ , extending a special case of the  $n$ -Frankl-Lovász theorem. We also prove that  $\chi(S_2^3, t) \leq t + 1$ , which is sharp for  $t = 2$ . This is a corollary of our main result which bounds the chromatic number  $\chi(H)$  of 3-uniform hypergraphs by the chromatic number of its 1-intersection graph  $H^{[1]}$ , whose vertices represent hyperedges and whose edges represent intersections of hyperedges in exactly one vertex. We prove that  $\chi(H) \leq \chi(H^{[1]})$  for any 3-uniform hypergraph  $H$  (assuming that  $H^{[1]}$  has at least one edge). The proof uses the list coloring version of Brooks' theorem. The more general question, whether  $\chi(H) \leq \chi(H^{[1]})$  holds for every  $r$ -uniform hypergraph ( $r > 3$ ) remains open. We could not decide either whether the above lower bound of  $\chi(T, t)$  is sharp for every  $r$ -uniform tree.

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## 1. Introduction, results

A hypergraph  $H = (V, E)$  is a set  $V$  of vertices together with a nonempty set  $E$  of subsets of  $V$ , which are called edges. In this paper, we will assume that for each  $e \in E$ ,  $|e| \geq 2$ . If  $|e| = r$  for each  $e \in E$ , then  $H$  is  $r$ -uniform; a 2-uniform  $H$  is a graph. A hypergraph  $H$  is *acyclic* if  $H$  contains no cycles (including 2-cycles which are two edges intersecting in at least two vertices). If  $H$  is a connected acyclic hypergraph, we say that  $H$  is a *tree*. In particular, a *star* is a tree in which exactly one vertex is common to every edge. A *matching* is a hypergraph consisting of pairwise disjoint edges, with every vertex belonging to some edge. We denote by  $S_k^r$  and  $M_k^r$  the  $r$ -uniform  $k$ -edge star and matching, respectively.

For a positive integer  $k$ , a function  $c : V \rightarrow \{1, \dots, k\}$  is called a  $k$ -coloring of  $H$ . A coloring  $c$  is *proper* if no edge of  $H$  is monochromatic under  $c$ . The chromatic number of  $H$ , denoted  $\chi(H)$ , is the least  $m \geq 1$  for which there exists a proper  $m$ -coloring of  $H$  and in this case, we say that  $H$  is  $m$ -chromatic. Given  $H = (V, E)$ , a partition  $\{E_1, \dots, E_t\}$  of  $E$  into  $t$  parts is called a  $t$ -edge-coloring of  $H$ . For  $r$ -uniform hypergraphs  $H_1, H_2, \dots, H_t$ , the  $(t$ -color) Ramsey number  $R^r(H_1, H_2, \dots, H_t)$  is the smallest integer  $n$  for which the following is true: under any  $t$ -edge-coloring of the complete  $r$ -uniform hypergraph  $K_n^r$ , there is a monochromatic copy of  $H_i$  in color  $i$  for some  $i \in \{1, 2, \dots, t\}$ . When all  $H_i = H$  we use the notation  $R^r(H, t)$ .

Bialostocki and the senior author of this paper extended two well-known results in Ramsey theory from the complete host graph  $K_n$  to arbitrary  $n$ -chromatic graphs [4]. One extends a remark of Erdős and Rado stating that in any 2-coloring of the edges of a complete graph  $K_n$  there is a monochromatic spanning tree. The other is the extension of the result of Cockayne and Lorimer [5] about the  $t$ -color Ramsey number of matchings. In [8], an acyclic graph  $H$  is defined as  $t$ -good if every  $t$ -edge coloring of any  $R^2(H, t)$ -chromatic graph contains a monochromatic copy of  $H$ . Matchings are  $t$ -good for every  $t$  [4] and in [8] it was proved that stars are  $t$ -good, as well as the path  $P_4$  (except possibly for  $t = 3$ ). Additionally,  $P_5, P_6, P_7$  are 2-good. In fact, as remarked in [4], there is no known example of an acyclic  $H$  that is not  $t$ -good.

In this paper, we explore a similar extension of Ramsey theory for hypergraphs, motivating the following definition.

**Definition 1.** Suppose that  $T$  is an acyclic  $r$ -uniform hypergraph. Let  $\chi(T, t)$  be the smallest  $m$  with the following property: under any  $t$ -edge-coloring of any  $m$ -chromatic  $r$ -uniform hypergraph, there is a monochromatic copy of  $T$ .

We call  $\chi(T, t)$  the *chromatic Ramsey number* of  $T$ . It follows from the existence of hypergraphs of large girth and chromatic number that the chromatic Ramsey number can be defined only for acyclic hypergraphs.

### 1.1. General trees and acyclic hypergraphs

To see that  $\chi(T, t)$  is indeed well defined, we use the following result.

**Lemma A** ([10,12]). If  $H$  is  $r$ -uniform with  $\chi(H) \geq k + 1$ , then  $H$  contains a copy of any  $r$ -uniform tree on  $k$  edges.

**Proposition 2.** For any  $r$ -uniform tree  $T$ ,  $\chi(T, t) \leq |E(T)|^t + 1$ .

**Proof.** Fix  $t \geq 1$ . Let  $T$  be an  $r$ -uniform tree with  $k$  edges and let  $H = (V, E)$  be a hypergraph with  $\chi(H) \geq k^t + 1$ . Let  $E = E_1 \cup \dots \cup E_t$  be a  $t$ -coloring of its edges and set  $H_i = (V, E_i)$ .

Then  $\chi(H_1 \cup \dots \cup H_t) = \chi(H) \geq k^t + 1$  holds and without loss of generality,  $\chi(H_1) \geq k + 1$ . By Lemma A,  $H_1$  contains a copy of  $T$ .  $\square$

The simple bound of Proposition 2 can be easily reduced for graphs.

**Proposition 3.** For any 2-uniform tree  $T$  with  $k$  edges,  $\chi(T, t) \leq 2kt + 1$ .

**Proof.** Let  $G$  be a  $t$ -colored graph with  $\chi(G) \geq 2kt + 1$  and  $T$  is a tree with  $k$  edges. Clearly  $G$  has a subgraph of minimum degree  $2kt$ , with a slight abuse of notation, we keep the name  $G$  for it. Then, for some color  $i$ ,  $\sum_{v \in V(G)} d_i(v) \geq 2k|V(G)|$ , where  $d_i(v)$  is the number of edges in color  $i$  incident to  $v$ . Thus the graph  $G_i$ , whose edges are the edges of  $G$  with color  $i$ , has average degree at least  $2k$  therefore  $G_i$  has a subgraph of minimum degree  $k$  which must contain every tree with  $k$  edges.  $\square$

**Question 4.** Let  $T$  be an  $r$ -uniform tree. Is there an upper bound for  $\chi(T, t)$  which is linear in both  $t, |E(T)|$ ?

Since any  $r$ -uniform acyclic hypergraph  $T$  may be found in some  $r$ -uniform tree  $T'$ ,  $\chi(T, t)$  is well-defined for any  $r$ -uniform acyclic hypergraph. Observe the following natural lower bound of  $\chi(T, t)$ . Let  $L(T, t, r) := \left\lceil \frac{R^r(T, t) - 1}{r - 1} \right\rceil + 1$ .

**Proposition 5.** For any acyclic  $r$ -uniform  $T$ ,  $L(T, t, r) \leq \chi(T, t)$ .

**Proof.** Let  $N := R^r(T, t) - 1$ . By the definition of the Ramsey number, there is a  $t$ -coloring of the edges of  $K_N^r$  without a monochromatic  $T$ . Since  $\chi(K_N^r) = \left\lceil \frac{N}{r-1} \right\rceil$ , the proposition follows.  $\square$

The notion of  $t$ -good graphs can be naturally extended to hypergraphs using [Proposition 5](#). An acyclic  $r$ -uniform hypergraph  $T$  is called  $t$ -good if every  $t$ -edge coloring of any  $L(T, t, r)$ -chromatic  $r$ -uniform hypergraph contains a monochromatic copy of  $T$ . In other words,  $T$  is  $t$ -good if  $L(T, t, r) = \chi(T, t)$ . Note that for  $r = 2$ , this gives the definition of good graphs. Although it is unlikely that all acyclic hypergraphs are  $t$ -good, we have no counterexamples.

For special families of  $r$ -uniform acyclic hypergraphs, namely for stars and matchings, we found better upper bounds for  $\chi(T, t)$ . Surprisingly, most of the bounds attained do not depend on  $r$ .

### 1.2. Stars

**Proposition 6.**  $\chi(S_k^r, t) \leq t(k - 1) + 2$ .

**Proof.** Fix  $t, k \geq 1$  and let  $p := t(k - 1) + 2$ . Suppose that  $H$  is  $r$ -uniform with  $\chi(H) \geq p$  and its edges are  $t$ -colored. By [Lemma A](#),  $\chi(S_{p-1}^r, 1) \leq p$ , so we can find a copy of  $S_{p-1}^r$  in  $H$ . By the pigeonhole principle,  $k$  of the edges of  $S_{p-1}^r$  have the same color, forming a monochromatic copy of  $S_k^r$ .  $\square$

How good is the estimate of [Proposition 6](#)? Notice first that for  $t = 1$  it is sharp.

**Proposition 7.**  $\chi(S_k^r, 1) = k + 1$ .

**Proof.** Consider the complete hypergraph  $K = K_{k(r-1)}^r$ . Clearly,  $\chi(K) = k$  and  $S_k^r$  is not a subgraph of  $K$ , as its vertex set is too large.  $\square$

If  $t = 2$ , [Proposition 6](#) gives  $\chi(S_k^r, 2) \leq 2k$ . For  $r = 2$  and odd  $k$ , this is a sharp estimate. For  $k = 1$ , this is trivial; for  $k \geq 3$ , the complete graph  $K_{2k-1}^2$  can be partitioned into two  $(k - 1)$ -regular subgraphs. However, for even  $k \geq 2$ ,  $\chi(S_k^2, 2) = 2k - 1$ .

An interesting problem arises when  $T = S_2^r$  with  $r \geq 3$ , when [Proposition 6](#) gives the upper bound  $t + 2$ . We can decrease this bound by introducing the notion of 1-intersection graphs of a hypergraph.

**Definition 8.** Let  $H = (V(H), E(H))$  be a hypergraph. The 1-intersection graph of  $H$  is denoted  $H^{[1]}$ , where  $V(H^{[1]}) = E(H)$  and

$$E(H^{[1]}) = \{(e, f) : e, f \in E(H) \text{ and } |e \cap f| = 1\}.$$

It is well-known that if  $H^{[1]}$  is trivial, i.e., no two edges of  $H$  intersect in exactly one vertex, then  $H$  is 2-colorable ([\[14\]](#), Exercise 13.33). Note that the stronger statement  $\chi(H) \leq \chi(H^{[1]}) + 1$  follows from applying the greedy coloring algorithm in any order of the vertices of  $H$ . Can we improve this?

**Question 9.** Let  $r \geq 3$ . Is it true that  $\chi(H) \leq \chi(H^{[1]})$  for any  $r$ -uniform hypergraph  $H$ , provided  $H^{[1]}$  has at least one edge?

Our main result is the positive answer to [Question 9](#) for the 3-uniform case and its corollary.

**Theorem 10.** If  $H$  is a 3-uniform hypergraph with at least one edge in  $H^{[1]}$  then  $\chi(H) \leq \chi(H^{[1]})$ .

**Corollary 11.** For  $t \geq 1$ ,  $\chi(S_2^3, t) \leq t + 1$ .

**Proof.** Suppose that we have a  $t$ -coloring  $c$  on the edges of a 3-uniform  $H$  with  $\chi(H) \geq t + 1$ . We claim that there are two edges of  $H$  with the same color that intersect in one vertex, defining a monochromatic  $S_2^3$ . Indeed, by [Theorem 10](#) we have  $t + 1 \leq \chi(H) \leq \chi(H^{[1]})$ . Then  $c$  defines a vertex  $t$ -coloring on the (at least)  $(t + 1)$ -chromatic graph  $H^{[1]}$ , this coloring cannot be proper: there are two vertices of the same color forming an edge in  $H^{[1]}$  and proving the claim.  $\square$

The case  $t = 2$  of [Corollary 11](#) was the initial aim of the research in this paper and it was proved first by Zoltán Füredi [\[7\]](#). Our proof of [Theorem 10](#) uses his observation ([Lemma 9](#)) and the list-coloring version of Brooks' theorem. [Corollary 11](#) is obviously sharp for  $t = 2$ ; it follows from [Proposition 5](#) that it is also sharp for  $t = 3$ , because  $R^3(S_2^3, 3) = 6$  [\[3\]](#). It would be interesting to see whether [Corollary 11](#) is true for any  $S_2^r$  (in particular for  $r = 4, t = 2$ ) as this is equivalent to the statement that  $r$ -uniform hypergraphs with bipartite 1-intersection graphs are 2-colorable.

### 1.3. Matchings

When  $T$  is a matching, the bounds of  $\chi(T, t)$  relate to the following well-known result of Alon, Frankl, and Lovász (originally conjectured by Erdős).

**Theorem B ([1]).** For  $r \geq 2, k \geq 1, t \geq 1$ ,

$$R^r(M_k^r, t) = (t - 1)(k - 1) + kr.$$

Note that special cases of [Theorem B](#) include  $r = 2$  [\[5\]](#),  $k = 2$  [\[13\]](#),  $t = 2$  [\[2,9\]](#).

We obtain the following linear upper bound for matchings using [Theorem B](#).

**Theorem 12.** For  $r \geq 2$ ,  $k \geq 1$ ,  $t \geq 1$ ,  $\chi(M_k^r, t) \leq (t-1)(k-1) + 2k$ . Equality holds for  $r = 2$ .

We tighten this bound, provided  $r \geq 3$  and  $t = 2$ .

**Theorem 13.** For  $r \geq 3$  and  $k \geq 1$ ,  $\chi(M_k^r, 2) \leq 2k$ .

We get two corollaries of [Theorem 13](#) when its upper bound coincides with the lower bound of [Proposition 5](#).

**Corollary 14.**  $\chi(M_k^3, 2) = 2k$ .

**Corollary 15.** For  $r \geq 3$ ,  $\chi(M_2^r, 2) = 4$ .

[Corollary 14](#) extends [Theorem B](#) (for  $r = 3$ ,  $t = 2$ ) because  $\chi(K_{4k-1}^3) = 2k$ . However, [Corollary 15](#) does not extend [Theorem B](#) for  $r \geq 4$ . Indeed, for  $r = 4$ , the bound  $\lceil \frac{1+2r}{r-1} \rceil$  derived from [Theorem B](#) is 3.

## 2. Proofs of main results

### 2.1. Proof of [Theorem 10](#)

In this section, we use the phrase “triple system” for a 3-uniform hypergraph. The word “triple” will take the place of “edge” so that “edge” may be reserved for graphs. Our goal is to construct a proper  $t$ -coloring of  $H$  from a proper  $t$ -coloring of  $H^{[1]}$ . Note that a partition of  $E(H)$  into classes  $E_1, E_2, \dots, E_t$  such that for any  $i$ ,  $1 \leq i \leq t$ , no two edges of  $E_i$  1-intersect is precisely a proper  $t$ -coloring of  $H^{[1]}$ .

A triple system is connected if for every partition of its vertices into two nonempty parts, there is a triple intersecting both parts. Every triple system can be uniquely decomposed into pairwise vertex-disjoint connected parts, called components. Components with one vertex are called trivial components.

Let  $B_k$  denote the triple system with  $k$  edges intersecting pairwise in the vertices  $\{v, w\}$ , called the *base* of  $B_k$ . A  $B$ -component (also,  $B_k$ -component) is a triple system which is isomorphic to  $B_k$  for some  $k \geq 1$ . A  $K$ -component is either three or four distinct triples on four vertices.

**Lemma 16.** Let  $C$  be a nontrivial component in a triple system without 1-intersections. Then  $C$  is either a  $B$ -component or a  $K$ -component.

**Proof.** If  $C$  has at most four vertices then  $1 \leq |E(C)| \leq 4$  (where  $E(C)$  is here considered as a set, not a multiset) and by inspection,  $C$  is either  $B_1, B_2$ , or a  $K$ -component. Assume  $C$  has at least five vertices and select the maximum  $m$  such that  $e_1, e_2, \dots, e_m \in E(C)$  are distinct triples intersecting in a two-element set, say in  $\{x, y\}$ . Clearly,  $m \geq 2$ . Then  $A = \cup_{i=1}^m e_i$  must cover all vertices of  $C$ , as otherwise there is an uncovered vertex  $z$  and a triple  $f$  containing  $z$  and intersecting  $A$ , since  $C$  is a component. However, from  $m \geq 2$  and the intersection condition,  $f \cap A = \{x, y\}$  follows, contradicting the choice of  $m$ . Thus  $A = V(C)$  and from  $|V(C)| \geq 5$  we have  $m \geq 3$ . It is obvious that any triple of  $C$  different from the  $e_i$ 's would intersect some  $e_i$  in one vertex, violating the intersection condition. Thus  $C$  is isomorphic to  $B_m$ , concluding the proof.  $\square$

A multigraph  $G$  is called a *skeleton* of a triple system  $H$  if every triple contains at least one edge of  $G$ . We may assume that  $V(H) = V(G)$ . A *matching* in a multigraph is a set of pairwise disjoint edges. A *factorized complete graph* is a complete graph on  $2m$  vertices whose edge set is partitioned into  $2m-1$  matchings. The following lemma allows us to define a special skeleton of triple systems.

**Lemma 17.** Suppose that  $H$  is a triple system with  $\chi(H^{[1]}) = t \geq 2$  and  $H_1, H_2, \dots, H_t$  is a partition of  $H$  into triple systems (each  $H_i$  is considered on vertex set  $V(H)$ ) where each  $H_i$  has no 1-intersections. Then there exists a skeleton  $G$  of  $H$  with the following properties:

- $E(G) = \cup_{i=1}^t M_i$  where each  $M_i$  is a matching and a skeleton of  $H_i$ .
- For  $1 \leq i \leq t$ , edges of  $M_i$  are the bases of all  $B$ -components of  $H_i$  and two disjoint vertex pairs from all  $K$ -components of  $H_i$ .
- If  $K^* = K_{t+1} \subset G$  then  $K^*$  is a connected component of  $G$  factorized by the  $M_i$ 's and there is some  $e \in M_1 \cap E(K^*)$  such that  $e$  is from a  $B$ -component of  $H_1$ .

**Proof.** From [Lemma 16](#) we can define  $M_i$  by selecting the base edges from every  $B$ -component of  $H_i$  and selecting two disjoint pairs from every  $K$ -component of  $H_i$ . The resulting multigraph is clearly a skeleton of  $H$  and satisfies properties 1 and 2. We will select the disjoint pairs from the  $K$ -components so that property 3 also holds. Notice that  $K^* = K_{t+1} \subset G$  must form a connected component in  $G$  because it is a  $t$ -regular subgraph of a graph of maximum degree  $t$ . Also,  $K_{t+1}$  is factorized by the  $M_i$ 's because the union of  $t$  matchings can cover at most  $\frac{t(t+1)}{2} = \binom{t+1}{2}$  edges of  $K_{t+1}$ , therefore every edge of  $K_{t+1}$  must be

covered exactly once by the  $M_i$ 's. Thus we have to ensure only that there is  $e \in M_1 \cap E(K^*)$  with  $e$  from a  $B$ -component of  $H_1$ . For convenience, we say that a  $K^* = K_{t+1}$  is a *bad component* if such  $e$  does not exist.

Select a skeleton  $S$  as described in the previous paragraph such that  $p$ , the number of bad components, is as small as possible. Suppose that  $(x, y) \in M_1$  is in a bad component  $U$ . In other words,  $(x, y)$  is in a  $K$ -component of  $H_1$ , where  $V(K) = \{x, y, u, v\}$  and  $(u, v) \in M_1$ . Now we replace these two pairs by the pairs  $(x, u)$ ,  $(y, v)$  to form a new  $M_1$ . After this switch,  $U$  is no longer a bad component. In fact, either  $U$  becomes a new component on the same vertex set (if  $(u, v)$  was in  $U$ ) or  $U$  melds with another component into a new component. In both cases, no new bad components are created and in the new skeleton there are fewer than  $p$  bad components. This contradiction shows that  $p = 0$  and proves the lemma.  $\square$

**Proof of Theorem 10.** Let  $H$  be a triple system with  $t := \chi(H^{[1]}) \geq 2$  and partition  $H$  into  $H_1, \dots, H_t$  so that each  $H_i$  is without 1-intersections. Let  $G$  be a skeleton of  $H$  with the properties ensured by Lemma 17.

Let  $G'$  be a connected component of  $G$ . By Brooks' Theorem, if  $G'$  is not the complete graph  $K_{t+1}$  or an odd cycle (if  $t = 2$ ),  $\chi(G') \leq \Delta(G') \leq t$ .

Suppose first that  $t$  is even. Now  $G' \neq K_{t+1}$  because that would contradict property 3 in Lemma 17:  $K_{t+1}$  cannot be factorized into matchings. Also, for  $t = 2$ ,  $G'$  cannot be an odd cycle since odd cycles are not the union of two matchings. Thus every connected component of  $G$  is at most  $t$ -chromatic, therefore  $\chi(G) \leq t$ . Since  $G$  is a skeleton of  $H$ , this implies  $\chi(H) \leq t$ , concluding the proof for the case when  $t$  is even.

Suppose that  $t$  is odd,  $t \geq 3$ . In this case the previous argument does not work when some connected component  $G' = K_{t+1} \subset G$ . However, from Lemma 17, every  $K_{t+1}$ -component  $C_i$  of  $G$  has an edge  $(x_i, y_i) \in M_1$  that is the base of a  $B$ -component in  $H_1$ . Define the vertex coloring  $c$  on  $X = \cup_{i=1}^m V(C_i)$  by  $c(x_i) = c(y_i) = 1$  and by coloring all the other vertices of all  $C_i$ 's with  $2, \dots, t$ .

Let  $F$  be the subgraph of  $G$  spanned by  $V(G) \setminus X$  and define

$$Z := \{z \in V(F) : \{x_i, y_i, z\} \in E(H_1) \text{ for some } 1 \leq i \leq m\}.$$

Fix any  $z \in Z$ . Then there is a triple  $T = (x_i, y_i, z) \in H_1$  in a  $B$ -component of  $H_1$  where  $T$  has base  $(x_i, y_i)$ . If  $(z, u) \in M_1$  for some  $u \in V(G)$ , some triple  $S$  containing  $(z, u)$  would be in  $H_1$ . But then  $S, T$  would 1-intersect in  $z$ , contradicting to the definition of  $H_1$ . Thus  $d_G(z) \leq t - 1$  for all  $z \in Z$ . Note also that  $d_G(v) \leq t$  for all  $v \in V(F) \setminus Z$ .

We claim that with lists  $L(z) := \{2, \dots, t\}$  for  $z \in Z$  and  $L(v) := \{1, \dots, t\}$  for  $v \in V(F) \setminus Z$ ,  $F$  is  $L$ -choosable. We use the reduction argument present in many coloring proofs (see, for example, the very recent survey paper [6]).

Suppose  $F$  is not  $L$ -choosable and let  $F'$  be a minimal induced subgraph of  $F$  which fails to be  $L$ -choosable. We may assume that any  $z \in V(F') \cap Z$  has  $d_{F'}(z) = t - 1$  (otherwise we may  $L$ -choose  $F' - z$ , add  $z$  back and properly color it). Likewise we may assume  $d_{F'}(v) = t$  for all  $v \in V(F') \setminus Z$ . By the degree-choosability version of Brooks' theorem (see [11], Lemma 1 or [6], Theorem 11),  $F'$  is a Gallai tree: a graph whose blocks are complete graphs or odd cycles.

Let  $A$  be an endblock of  $F'$ . Then  $A \neq K_{t+1}$  because all  $K_{t+1}$ -components of  $G$  are in  $X$ . Since all vertex degrees in  $F'$  are  $t$  or  $t - 1$ ,  $A$  is either an odd cycle (if  $t = 3$ ) or  $A$  is a  $K_t$ .  $A$  must contain an edge  $e \in M_1$ . Otherwise  $M_2, \dots, M_t$  would cover the edges of  $A$ , a contradiction in either case. By the degree requirements, either

$$V(A) \cap (V(F) \setminus Z) = \{w\}$$

where  $w$  is the unique cut point of  $A$  or  $V(A) \subset Z$ . In both cases an endpoint of  $e$  must be in  $Z$ . Then there exists some triple  $\{x_i, y_i, z\} \in H_1$  which 1-intersects with the triple of  $H_1$  containing  $e$ , a contradiction to the definition of  $H_1$ , proving that  $F$  is  $L$ -choosable.

Let  $c' : V(F) \rightarrow \{1, \dots, t\}$  be an  $L$ -coloring of  $F$ . We extend  $c$  from  $X$  to  $V(H)$  by setting  $c(v) := c'(v)$  for all  $v \in V(F)$ . Observe that  $c$  properly colors all edges of  $G$  except for the edges of the form  $(x_i, y_i)$  which are monochromatic in color 1. Since  $G$  is a skeleton, every triple of  $H$  is properly colored except possibly the triples in the form  $(x_i, y_i, x)$ .

We claim that  $c(x) \neq 1$ . Suppose to the contrary that  $c(x) = 1$ . If  $x \in X$  then  $x \in \{x_j, y_j\}$  for some  $j \neq i$ , but this is impossible because the bases  $(x_i, y_i)$ ,  $(x_j, y_j)$  are from different  $B$ -components of  $H_1$ . If  $x \notin X$  then  $x \in Z$  from the definition of  $Z$ . However,  $1 \notin L(x)$  for  $x \in Z$  and this proves the claim.

Therefore  $c$  is a proper  $t$ -coloring of  $H$  and this completes the proof.  $\square$

## 2.2. Proof of Theorem 12

Let  $H = (V, E)$  be an  $r$ -uniform hypergraph with  $\chi(H) \geq p$  where

$$p = (t - 1)(k - 1) + 2k.$$

Consider any  $t$ -edge coloring  $\{E_1, \dots, E_t\}$  of  $H$  and any proper coloring  $c$  of  $H$  obtained by the greedy algorithm (under any ordering of its vertices). Clearly  $c$  uses at least  $p$  colors and for any  $1 \leq i < j \leq p$  there is an edge  $e_{ij}$  in  $H$  whose vertices are colored with color  $i$  apart from a single vertex which is colored with  $j$ . Let  $\{F_1, \dots, F_t\}$  be a  $t$ -edge-coloring of  $K_p^2$  defined so that  $F_s := \{\{i, j\} : 1 \leq i < j \leq p, e_{ij} \in E_s\}$  for each  $s$ ,  $1 \leq s \leq t$ . From the definition of  $p$ , Theorem B (in fact the Cockayne-Lorimer Theorem suffices) implies that there is a monochromatic  $M_k^2$  in  $K_p^2$ . Observe that

$$\{e_{ij} : \{i, j\} \in M_k^2\}$$



is a set of  $k$  pairwise disjoint edges in  $H$  in the same partition class of  $\{E_1, \dots, E_t\}$ . This completes the proof that  $\chi(M_k^t, t) \leq (t-1)(k-1) + 2k$ . The lower bound  $R^2(M_k^2, t) \leq \chi(M_k^2, t)$  implies equality in the  $r = 2$  case.  $\square$

### 2.3. Proof of Theorem 13

We fix  $r \geq 3$  and proceed by induction on  $k$ . Suppose  $k = 1$  and let  $H$  be some  $r$ -uniform hypergraph with  $\chi(H) \geq 2$ . Then any 2-edge-coloring of  $H$  contains a single monochromatic edge since  $H$  has at least one edge. Now suppose the theorem is true for  $k-1 \geq 1$  and let  $H = (V, E)$  be  $r$ -uniform with  $\chi(H) \geq 2k$ . Without loss of generality,  $H$  is connected. Fix some 2-edge-coloring  $\{E_1, E_2\}$  of  $H$ , calling the edges of  $E_1$  “red” and the edges of  $E_2$  “blue”. If  $E_1$  or  $E_2$  is empty, then Theorem 12 with  $t = 1$  implies the desired bound.

So we may assume otherwise, and there exist edges  $e, f \in E$  with  $e$  red and  $f$  blue. Let  $s := |e \cap f|$  and  $A := e \cup f$ . If  $H[A]$  is 2-colorable, then  $\chi(H - A) \geq \chi(H) - 2 \geq 2(k-1)$  so by induction we find a monochromatic  $M_{k-1}^r$  matching in  $H - A$ . Without loss of generality,  $M_{k-1}^r$  is red and  $M_{k-1}^r + e$  is a red  $M_k^r$  in  $H$ .

If  $s > 1$ , then  $|A| = 2r - s \leq 2r - 2$  thus  $H[A]$  is certainly 2-colorable and the induction works. If  $s = 1$  and  $H[A]$  is not 2-colorable then  $H[A]$  is  $K_{2r-1}^r$ . Writing  $e = \{w, u_1, \dots, u_{r-1}\}$  and  $f = \{w, v_1, \dots, v_{r-1}\}$ , the edge  $g = \{w\} \cup \{u_1, u_3, \dots\} \cup \{v_2, v_4, \dots\} \in E(H)$ . Without loss of generality,  $g$  is red and  $|g \cap f| = 1 + \lfloor (r-1)/2 \rfloor \geq 2$  since  $r \geq 3$ . So the previous case applies to the red edge  $g$  and blue edge  $f$ . Finally, if  $s = 0$  and  $H[A]$  is not 2-colorable there must be  $g \in H[A]$  that intersects both  $e$  and  $f$ . Then either  $e, g$  or  $f, g$  is a pair of edges of different colors that intersect, and a previous case can be applied again.  $\square$

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