Red-blue clique partitions and (1-1)-transversals

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Abstract

Motivated by a problem of Gallai on (1-1)-transversals of 2-intervals, it was proved by the authors in 1969 that if the edges of a complete graph K are colored with red and blue (both colors can appear on an edge) so that there is no monochromatic induced C_4 and C_5 then the vertices of K can be partitioned into a red and a blue clique. Aharoni, Berger, Chudnovsky and Ziani recently strengthened this by showing that it is enough to assume that there is no induced monochromatic C_4 and there is no induced C_5 in *one of the colors*. Here we strengthen this result further, showing that it is enough to assume that there is no monochromatic induced C_4 and there is no K_5 on which both color classes induce a C_5. We also answer a question of Kaiser and Rabinovich, giving an example of six 2-convex sets in the plane such that any three intersect but there is no (1-1)-transversal for them.

1 Red-blue clique partitions of complete graphs

In 1968, thinking on a problem about piercing cycles in digraphs, Gallai arrived at the problem of piercing 2-intervals. He defined 2-intervals as subsets of the real line R having two interval components, one in $(-\infty, 0)$ and one in $(0, \infty)$ and asked: how many points are needed to pierce a family of pairwise intersecting 2-intervals? His question generated the paper [7] in which (as a special case of a general upper bound) we proved that two points always pierce pairwise intersecting 2-intervals and one of them can be selected from $(-\infty, 0)$ and the other from $(0, \infty)$. Let's call such a pair of points a (1-1)-transversal. This result can be extended to 2-trees, where a 2-tree is the union of two subtrees, one a subtree of T_1 the other a subtree of T_2, where T_1 and T_2 are vertex-disjoint trees. In [7] we proved a stronger result (see Theorem 1 below) using only properties of the intersection graph of subtrees of a tree. Consider 2-*colored complete graphs*, where edges are colored with red, blue, or both colors. Edges of one color only are called *pure* edges and can be

pure red or pure blue. Another view is to consider a complete graph (clique) as the union of a red and a blue graph on the same vertex set.

Theorem 1. (Gyárfás, Lehel [7], 1970) *Assume that G is a 2-colored complete graph containing no monochromatic induced C_4 and C_5. Then $V(G)$ can be partitioned into a red and a blue clique.*

Given a set of n pairwise intersecting 2-subtrees, one can represent their intersections by a 2-colored complete graph K_n. Then both colors determine *chordal graphs* i.e. graphs in which every cycle of length at least four has a chord. In particular, there is no monochromatic induced C_4 and C_5. Applying Theorem 1, the vertices of K_n can be partitioned into a red and a blue clique (empty sets or single vertices are accepted as cliques) and by the Helly-property of subtrees we have a (1-1)-transversal for the 2-subtrees. Thus Theorem 1 implies the following.

Corollary 2. (Gyárfás, Lehel [7], 1970*) Pairwise intersecting 2-subtrees have a (1-1)-transversal.*

Since 2-colorings of complete graphs with pure edges only can be considered as a graph and its complement, furthermore, the complement of C_4 is $2K_2$, we get another consequence of Theorem 1.

Corollary 3. (Földes, Hammer [6], 1977) *Assume that a graph G does not contain $C_4, 2K_2, C_5$ as an induced subgraph. Then G is a split graph, i.e. its vertices can be partitioned into a clique and an independent set.*

The seminal paper of Tardos [11] (1995) introduced topological methods, he proved that 2-intervals without $k+1$ pairwise disjoint members have $(k-k)$-transversals. Further results were obtained by Kaiser [8] (1997), Alon [2, 3] (1998, 2002), Matousek [10] (2001), Berger [5] (2005), and this list of references is very far from being complete. In this note we only consider the graph coloring approach. Very recently Theorem 1 was generalized as follows.

Theorem 4. (Aharoni, Berger, Chudnovsky, Ziani [1], 2015) *Assume that G is a 2-colored complete graph such that there is no monochromatic induced C_4 and there is no red induced C_5. Then $V(G)$ can be partitioned into a red and a blue clique.*

We show that the proof of Theorem 1 in [7] yields an even stronger result. Let K_5^* denote the 2-colored K_5 where every edge is pure and both colors span a C_5.

Theorem 5. *Assume that G is a 2-colored complete graph such that there is no monochromatic induced C_4 and there is no K_5^*. Then $V(G)$ can be partitioned into a red and a blue clique.*

Proof. We prove the result by induction on $|V(G)|$. For $1 \leqslant |V(G)| \leqslant 3$ the theorem is obvious. Fixing any $p \in V(G)$, by the inductive hypothesis we have $V(G - p) = R \cup B$ where R and B are disjoint vertex sets spanning a red and a blue clique.

Set

$$R^* = \{r \in R : (p, r) \text{ is pure blue}\}, B^* = \{b \in B : (p, b) \text{ is pure red}\}.$$

Assume that among all choices of R, B satisfying $V(G-p) = R \cup B$, $|R^*| + |B^*|$ is as small as possible. We show that either R^* or B^* is empty and thus R or B can be extended with p, concluding the proof.

Suppose on the contrary that R^*, B^* are both nonempty. For any $q \in B^*$ there exists $r \in R$ such that (q, r) is pure blue, otherwise $R_1 = R \cup \{q\}$ and $B_1 = B \setminus \{q\}$ would be a red-blue clique partition of $V(G-p)$ with $|R_1^*| + |B_1^*| < |R^*| + |B^*|$, contradicting the assumption. In fact, we may assume that $r \in R^*$, otherwise, with any $s \in R^*$, consider the four-cycle $C = (p, q, s, r, p)$. If (q, s) would be red then C is a red cycle with pure blue diagonals $(q, r), (p, s)$, contradiction. Thus (q, s) is pure blue and we can choose $s \in R^*$ instead of r.

Applying the argument of the previous paragraph for any $s \in R^*$, there exists $q \in B^*$ such that (s, q) is pure red. So there exists a shortest even cycle $C = (s_1, q_1, s_2, \ldots, q_m, s_1)$ in the bipartite graph $[R^*, B^*]$ with edges alternating as pure red, pure blue, pure red $\ldots$. We claim that C is a four-cycle. Indeed, if $m > 2$, then from the minimality of m, all diagonals (s_i, q_j) must have both colors. In particular, $(s_1, q_2), (s_3, q_1)$ both have red colors. Now if (q_1, q_2) is pure blue then the red four-cycle (s_1, q_1, p, q_2, s_1) has pure blue diagonals. Otherwise, if (q_1, q_2) is red, then the red four-cycle $(q_1, q_2, s_2, s_3, q_1)$ has pure blue diagonals, contradicting the assumption that there is no induced monochromatic C_4. This proves the claim that $C = (s_1, q_1, s_2, q_2, s_1)$. Observe that (s_1, s_2) is pure red, otherwise $(s_1, s_2, q_1, q_2, s_1)$ is a blue four-cycle with pure red diagonals and (q_1, q_2) is pure blue, otherwise $(q_1, q_2, s_2, s_1, q_1)$ is a red four-cycle with pure blue diagonals, contradiction. Therefore $\{p, s_1, s_2, q_1, q_2\}$ spans a K_5^*, giving the final contradiction. $\square$

Remark 6. An anonymous referee pointed out the algorithmic aspect of our subject as follows. The decision problem whether there exists a red-blue clique partitioning of the 2-colored complete graph can be formulated as a 2-satisfiability (2SAT) problem which is described by a Boolean formula in conjunctive normal form using two literals per clauses (2-CNF). The feasibility of 2SAT is a well-studied polynomially solvable problem (see [4]). In our case the easy description of the obstructions denying feasibility may complete the view of the structural characterizations considered in this note.

2   Red-blue clique partition of complete 3-uniform hypergraphs

Extending 2-intervals Kaiser and Rabinovich [9] defined a planar 2-body as a union of two closed convex sets of the plane separated by a fixed line, say the y-axis. They asked whether the assumption 'any three 2-bodies intersect' implies that they have a (1-1)-transversal.

Following Theorem 1, where the (1-1)-transversal problem is translated into properties that imply a red-blue clique cover of a two-colored clique, this problem can be stated in

terms of red-blue colored 3-uniform complete hypergraphs. However, the obstructions for a red-blue clique cover of a hypergraph can be more complicated than those for graphs in Theorems 1, 4 and 5. In particular, as our next example shows, the answer is negative to the question above.

Example 7. We define six planar 2-bodies, $A_i \cup B_i$, $0 \leqslant i \leqslant 5$, as follows. On each side of the y-axis we are given 5 triangles formed by consecutive triples of vertices of a fixed regular pentagon, and the inner pentagon bordered by its diagonals is the sixth convex set. On the A-side the (clockwise) consecutive triangles are labeled A_1, A_2, A_3, A_4, A_5; on the B-side the labeling of the consecutive triangles is B_1, B_3, B_5, B_2, B_4; the inner pentagons are labeled A_0 and B_0 (see Figure 1).
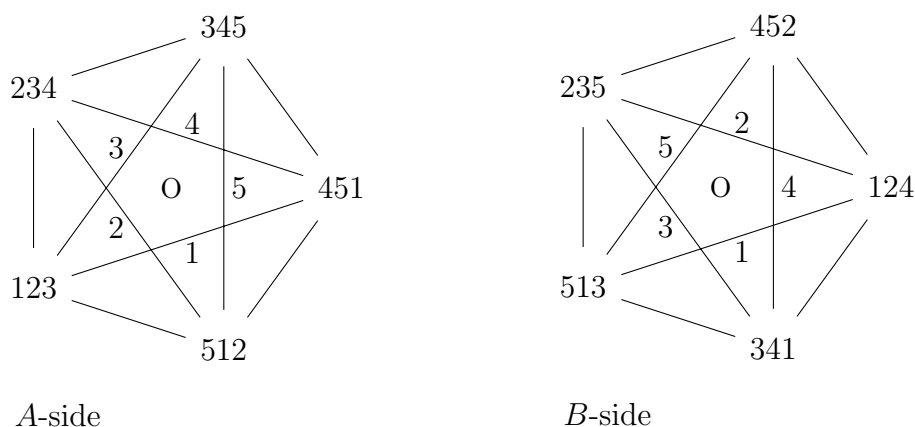

A-side   B-side

Figure 1: $A_i \cup B_i$, $0 \leqslant i \leqslant 5$, any three intersect without (1-1)-transversal

The 2-bodies of the example define a natural 2-coloring of the edges of $K_6^{(3)}$, the complete 3-uniform hypergraph on vertex set $V = \{0, 1, 2, 3, 4, 5\}$: if a triple of convex sets has non-empty intersection on the A-side (on the B-side), then the corresponding edge of the hypergraph is colored red (blue). It is easy to check that no four convex sets intersect on either side, furthermore, the 10 vertices and the 10 intersection points of diagonals are the intersections of the triples of the six 2-bodies. The red edges are the triples $\{(i, i+1, i+2) : 1 \leqslant i \leqslant 5\}$ (counting modulo 5) and their complements (with respect to V); the blue edges are the triples $\{(i, i+1, i+3) : 1 \leqslant i \leqslant 5\}$ and their complements. Thus V is covered by two triples of the same color, but it cannot be covered by a red and a blue triple.

Example 7 shows that the assumption 'any three 2-bodies intersect' does not imply that the 2-bodies have a (1-1)-transversal. However, Kaiser and Rabinovich [9] proved it from the condition that 'any four 2-bodies intersect'.

Theorem 8. (Kaiser, Rabinovich [9], 1999) *If S is a set of planar 2-bodies such that any four members of S have nonempty intersection, then S has a (1-1)-transversal.*   □

Acknowledgements

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