On 3-uniform hypergraphs without linear cycles*

A. Gyárfás, [†]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
1053 Reáltanoda u 13-15, Budapest, Hungary
`gyarfas.andras@renyi.mta.hu`

E. Győri, [‡]
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
1053 Reáltanoda u 13-15, Budapest, Hungary
`gyori.ervin@renyi.mta.hu`

M. Simonovits,[§]

Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
1053 Reáltanoda u 13-15, Budapest, Hungary
`simonovits.miklos@renyi.mta.hu`

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Abstract

We explore properties of 3-uniform hypergraphs H without linear cycles. It is surprising that even the simplest facts about ensuring cycles in graphs can be fairly complicated to prove for hypergraphs. Our main results are that 3-uniform hypergraphs without linear cycles must contain a vertex of strong degree at most two and must have independent sets of size at least $\frac{2|V(H)|}{5}$.

1. Introduction

A subset S of vertices in a hypergraph H is *independent* if there are no edges of H inside S. The cardinality of a largest independent set of H is denoted by $\alpha(H)$. A *linear cycle* (often also called a loose cycle) in a hypergraph is a sequence of at least three edges where only the cyclically consecutive edges intersect and they intersect

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in exactly one vertex. Our original motivation was to prove the following conjecture that is still open.

Conjecture 1.1 (Gyárfás–G.N. Sárközy, [3])**.** *One can partition the vertex set of every* 3*-uniform hypergraph* H *into* $\alpha(H)$ *linear cycles, edges and subsets of hyperedges.*

Note that Conjecture 1.1 would extend Pósa theorem, see [5] from graphs to 3-uniform hypergraphs. Conjecture 1.1 in a weaker form (with weak cycles instead of linear cycles) has been proved in [3]. It is necessary to allow subsets of hyperedges in Conjecture 1.1, such an example is the complete hypergraph K_5^3. Let $\rho(H)$ denote the minimum number of edges (or subsets of edges) needed to partition $V(H)$ and let $\chi(H)$ denote the chromatic number of H, the minimum number of colors in a vertex coloring of H without monochromatic edges. The following result proves that Conjecture 1.1 is true if there are no linear cycles in H.

Theorem 1.2. *If* H *is a* 3*-uniform hypergraph without linear cycles, then* $\rho(H) \leq \alpha(H)$*. Moreover,* $\chi(H) \leq 3$*.*

We find the family of hypergraphs without linear cycles intriguing and the purpose of this paper is to prove further results about it. Let $H = (V, E)$ be a 3-uniform hypergraph, for $v \in V$ the link graph of v in H is the graph with vertex set V and edge set $\{(x, y) : (v, x, y,) \in E\}$. The *strong degree* $d^+(v)$ for $v \in V$ is the maximum number of independent edges (i.e. the size of a maximum matching) in the link graph of v. The *underlying* graph is the ordinary graph the edges of which are the pairs covered by the hyperedges of H. Our main results are motivated by the following trivial assertions: a graph of minimum degree 2 contains a cycle; if G_n has no cycles then $\alpha(G_n) \geq n/2$.

Theorem 1.3. *Suppose that* H *is a* 3*-uniform hypergraph with* $d^+(v) \geq 3$ *for all* $v \in V$*. Then* H *contains a linear cycle.*

Theorem 1.3 can be easily strengthened.

Theorem 1.4. *Suppose that* H *is a* 3*-uniform hypergraph with* $d^+(v) \geq 3$ *for all but at most one* $v \in V$*. Then* H *contains a linear cycle.*

Indeed, if a graph G is a counterexample with an exceptional vertex w to Theorem 1.4 then three copies of G can be joined together through cut vertex w to get a counterexample to Theorem 1.3 as well. Notice that Theorem 1.4 does not hold with *two* exceptional vertices: for an odd n consider the n triples $(i, i+1, i+2) \pmod{n}$ on $[n]$ together with two vertices x, y and with edges (x, y, i) for all $i \in [n]$. This hypergraph has no linear cycles and $d^+(i) = 3, d^+(x) = d^+(y) = 1$. It is worth mentioning that the condition $d^+(v) \geq 3$ cannot be weakened by requiring that the link graph of v cannot be pierced by at most two vertices. Indeed, K_5^3 or hypergraphs obtained by attaching further K_5^3's to it are examples. It is also interesting to note that the maximal number of edges in a 3-uniform hypergraph without linear cycles is $\binom{n-1}{2}$, the maximum number of edges without a linear triangle [1], [2].

Theorem 1.5. *If* H_n *is a 3-uniform* n*-vertex hypergraph without linear cycles, then* $\alpha(H_n) \geq \frac{2}{5}n$*.*

The hypergraph consisting of n vertex disjoint copies of K_5^3 shows that equality can hold in Theorem 1.5. One may add further "transversal" copies of K_5^3's, to make the construction connected, if $n = 4k + 1$.

1.1. Skeletons, near-skeletons

A *linear tree* is a 3-uniform hypergraph that is obtained from a single edge by repeatedly adding edges that intersect the previous hypergraph in exactly one vertex. A single vertex is a *trivial tree.* A *linear path* is a linear tree built so that the next edge always intersects the previous edge in a vertex of degree one. A *linear cycle* is obtained from a linear path of at least two edges, by adding an edge that intersects the first and the last edges of the path in one of their degree one vertices. For brevity, we often just use the term tree for a linear tree. The *star* of a tree T at $v \in V(T)$ is the subtree of T containing the edges of T incident to v. For any $v \in V(T)$, the pairs (x, y) that are at equal distance from v in the underlying graph of T are called pairs *opposite* to v. Clearly, every edge of T has exactly one pair opposite to v.

A *skeleton* T in H is a non-trivial subtree which cannot be extended to a larger subtree by adding an edge $e \in E(H)$ for which $|e \cap V(T)| = 1$. A *near-skeleton* T with an exceptional vertex $v \in V(T)$ is a *non-trivial* subtree T with the following property: if $|e \cap V(T)| = 1$ for some $e \in E(H)$ then $e \cap V(T) = \{v\}$. Note that skeletons are not necessarily maximum subtrees, for example in the hypergraph with edge set $\{(a, b, c), (b, c, d), (c, d, e)\}$, $\{(b, c, d)\}$ and $\{(a, b, c), (c, d, e)\}$ are both skeletons. The following easy lemma is stated without proof.

Lemma 1.6. *Suppose H is a 3-uniform hypergraph having no linear cycle and T is a linear subtree in it. Let $v \in V(T)$ and $f = (v, a, b) \in E(H)$ be such that $\{a, b\}$ intersects $V(T)$ but does not intersect the star at $v \in V(T)$. Then $\{a, b\}$ is a pair opposite to v in T. Replacing the edge of T containing a, b by f is called a swap, it gives another linear tree on vertex set $V(T)$.*

The following is a useful corollary of Lemma 1.6.

Corollary 1.7. *Suppose T is a skeleton (near-skeleton) in a 3-uniform hypergraph H that has no linear cycle. Then any sequence of swaps with edges of $E(H[V(T)])$ results in a skeleton (near-skeleton) T' in H with $V(T') = V(T)$.*

1.2. Proof of Theorem 1.2

Consider a 3-uniform hypergraph H and choose a skeleton T_1 in it, then let T_2 be a skeleton in $H \setminus T_1$ and continue with $T_3, \ldots, T_m$ until an edgeless T_{m+1} remains. Let G_i be the underlying graph of T_i. Observe that $\alpha(G_i) = \theta(G_i)$ where $\theta(G)$ is the minimum number of complete subgraphs whose vertices cover $V(G)$. By the definition of skeletons, no edge of H intersects $V(T_i)$ in one vertex and intersects $V(H) \setminus (\cup_{j \le i} V(T_i))$ in two vertices. Suppose $S_i \subset V(G_i)$ is an independent set of G_i. Because H has no linear cycles, no edge of H is inside S_i and no edge of H contains two vertices of S_i and one vertex of $V(H) \setminus S_i$. Thus

$$\alpha(H) \geq \alpha(\cup_{i=1}^{m+1} G_i) = \sum_{i=1}^{m+1} \alpha(G_i) = \sum_{i=1}^{m+1} \theta(G_i) = \sum_{i=1}^{m+1} \rho(T_i) \geq \rho(H)$$

proving the first part of Theorem 1.2. The second part, $\chi(H) \leq 3$, follows from $\chi(G_i) = 3$ for $1 \leq i \leq m$ and $\chi(G_{m+1}) = 1$, using the remarks above, that union of independent sets of G_is are independent in H. In fact, one can also derive $\chi(H) \leq 3$ by induction, since Theorem 1.3 ensures that there is a vertex of H with strong degree at most two. □

2. Proof of Theorem 1.3

We shall prove Theorem 1.3 in the following slightly stronger form.

Theorem 2.1. *Suppose that T is a near-skeleton in a 3-uniform hypergraph H and $d_H^+(v) \geq 3$ holds for every $v \in V(T)$. Then H contains a linear cycle.*

1.3.

Proof. Consider a minimum counterexample where $|V(H)|$ is as small as possible and within that $|V(T)|$ is as small as possible. The subhypergraph of H with vertex set $V(T)$ is denoted by $H(T)$. We may suppose that T has the longest linear path P among all near-skeletons T' of H with $V(T') = V(T)$. Set

$$P = \{e_1 = (y_0, x_1, y_1), e_2 = (y_1, x_2, y_2), \ldots, e_m = (y_{m-1}, x_m, y_m)\}.$$

(We can see P on Figure 1.) By the symmetry of y_0, x_1 in P we may assume that x_1 is not the exceptional vertex of T. For $1 \leq i < j \leq m$ an *upward path* B from e_i to e_j is a linear path in $H(T)$ whose *first edge* intersects e_i in $\{x_i\}$, its *last edge* intersects e_j in the pair $\{x_j, y_j\}$ and its other vertices (inner vertices) are not on P. It is possible that B is a one edge path $(x_i, x_j, y_j) \in E(H(T))$, in this case it is considered as the last edge (with no inner vertices). A set of upward paths are *internally disjoint* if their sets of inner vertices are pairwise disjoint.

Definition 2.2. *For $2 \leq j \leq m$ a* ladder *L_j is the subhypergraph of $H(T)$ containing the path $e_1, \ldots, e_j$ and a set of internally disjoint upward paths with the following property.*

- *For every $1 \leq i < j$ there exists an upward path from e_k to e_ℓ for some k, ℓ such that $1 \leq k \leq i < \ell \leq j$.*

Figure 2 shows a ladder with two upward paths. We shall use the ladder to ensure that for any vertex q not on the ladder the edge $(q, y_{i-1} y_i)$ can be continued to get a simple path from the edges of the ladder ending with a last edge of an upward path in the pair (x_j, y_j). Observe that by removing from L_j the last edges of its upward paths, we have a linear tree in $H(T)$ denoted by L_j^*. Ladders exist because $d^+(x_1) \geq 3$ implies that there is an edge $f = (x_1, a, b)$ in $H(T)$ for which $\{a, b\} \cap \{y_0, y_1\} = \emptyset$. The choice of x_1 and Lemma 1.6 imply that $\{a, b\} = \{x_j, y_j\}$ for some $2 \leq j \leq m$. Thus $P \cup f$ is a ladder L_j, see Figure 3.

PSfrag replacements

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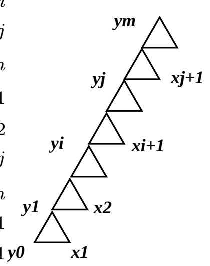

Figure 1: Path

Let L_j be a ladder such that j is as large as possible. Set $P' = \cup_{i>j} e_i$ and let M denote the linear tree $P' \cup L_j^*$. We extend M to a larger tree by adding a maximal linear subtree $F = F(x_j)$ of $H(T)$ with root x_j, so that its vertices (except its root) is in $V(T) \setminus V(M)$. Notice that from the construction, $U = V(M) \cup V(F) \subseteq V(T)$ and $M \cup F$ is a linear tree. (One can define F step by step using Corollary 1.7.) Let $q \in V(F)$ and suppose that there exists $h = (q, a, b) \in E(H)$ such that $\{a, b\} \cap V(F) = \emptyset$. The maximality of F implies that $\{a, b\} \cap M \neq \emptyset$. Applying Lemma 1.6 to the linear tree in $M \cup F$ at vertex q, we get that $\{a, b\}$ either intersects the star at q or it is a pair opposite to q. We have the following possibilities for $\{a, b\}$.

- Case 1. $\{a, b\} = \{x_k, y_k\}$ for some $k > j$
- Case 2. $\{a, b\} = \{y_{j-1}, y_j\}$
- Case 3. Either $\{a, b\} = \{y_{k-1}, x_k\}$ with some $1 \leq k < j$ or $\{a, b\}$ is on an upward path of L_j

Case 1 would contradict the choice of j since the path with first edge starting at x_j and last edge (q, a, b) would be an upward path extending the ladder L_j to a ladder L_k. Cases 2,3 for $q \neq x_j$ are also impossible since we could get a linear cycle from the definition of the ladder L_j. Indeed, in Case 2 one can start with $h = (q, a, b)$ and descend on P until an upward path leads back directly or through a jump on P to (x_j, y_j), closing a cycle at $\{y_{j-1}, y_j\}$. In Case 3 one can proceed similarly but upon reaching (x_j, y_j) get back to $q \in V(F)$, closing the cycle. We conclude that there is no $q \in V(F(x_j)) \setminus \{x_j\}$ and $h = (q, a, b) \in E(H(T))$ such that $\{a, b\} \cap V(F) = \emptyset$. Thus, if $F(x_j) \neq \{x_j\}$, $F(x_j)$ is a near-skeleton with exceptional vertex x_j, contradicting the assumption that $|V(T)|$ is as small as possible. If $F(x_j) = \{x_j\}$, the assumption $d^+(x_j) \geq 3$ allows to select $h = (x_j, a, b) \in E(H(T))$ such that $\{a, b\} \cap \{y_{j-1}, y_j\} = \emptyset$. Then $\{a, b\}$ must satisfy Case 3 and we get a linear cycle and a contradiction except when $h = (x_j, y_0, x_1)$ and L_j consists of only one upward path with one edge $f = (x_1, x_j, y_j)$ because in this case the cycle starting with edge (x_j, y_{k-1}, x_k) and ending with edge (x_1, x_j, y_j) degenerates. From here we assume that L_j is this simple ladder shown on Figure 3. In case of $j = 2$ the link graph of x_2 consists of the $\{a, b\}$ pairs that are either pairs of e_1 or intersect y_2 because if $\{a, b\} = \{u, y_1\}$ with $u \notin V(P)$ then $(u, y_1, x_2), e_1, f$ would form a linear triangle. Thus, from $d^+(x_2) \geq 3$, there is an edge of $H(T)$ on x_2 that is different from h and does not intersect $\{y_1, y_2\}$ and therefore would extend L_2 to a higher ladder. Thus we have $j \geq 3$.

For $2 \leq i \leq j$ define a maximal subtree $F(x_i)$ of $H(T)$ with root x_i, such that its vertices (except its root) are in $V(T) \setminus V(M)$.

Claim 2.3. *For* $2 \leq i \leq j$, $F(x_i) = \{x_i\}, g_i = (x_i, x_{i-1}, y_{i-1}) \in E(H)$ *and for* $3 \leq i \leq j$, $(x_{i-1}, x_1, y_0) \notin E(H)$.

Proof of Claim 2.3. For $i = j$, $F(x_j) = \{x_j\}$. Note that for $a \notin P$, $e = (a, y_{j-1}, x_j) \notin E(H)$ and $e' = (y_{j-1}, y_{j-2}, x_j) \notin E(H)$, otherwise $e, e_{j-1}, \ldots, e_1, f$

or $e', e_{j-2}, \ldots, e_1, f$ would be a linear cycle. Using this and $d^+(x_j) \geq 3$, it follows that $g_j \in E(H)$. Then $(x_{j-1}, x_1, y_0) \notin E(H)$, otherwise that edge with g_j, f would form a linear triangle. This proves the claim for $i = j$. Suppose that the claim is true for some $i \geq 3$, we show it remains true for $i-1$ as well.

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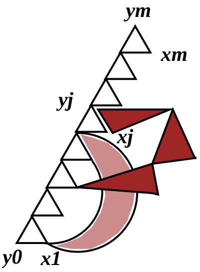

Figure 2: Ladder

Suppose $F = F(x_{i-1}) \neq \{x_{i-1}\}$. Then, as before, F is a near-skeleton with exceptional vertex x_{i-1}, a contradiction. Indeed, assuming that there exists $q \in V(F) \setminus \{x_{i-1}\}$ and $h = (q, a, b) \in E(H)$ such that $\{a, b\} \cap V(F) = \emptyset$ for some $q \in V(F)$, from Lemma 1.6 we get the following possibilities for $\{a, b\}$.

- Case A. $\{a, b\} = \{x_k, y_k\}$ for some $k > i-1$
- Case B. $\{a, b\} = \{y_{i-2}, y_{i-1}\}$
- Case C. $\{a, b\} = \{y_{k-1}, x_k\}$ with some $1 \leq k \leq i-1$

Case A would contradict the choice of j if $k > j$: the path with first edge starting at x_i and last edge (q, a, b) would be an upward path extending the ladder L_j to a ladder L_k. If $i \leq k \leq j$ then the linear cycle starting with the path $h, g_j, \ldots, g_i$ and ending with the linear path of F from x_{i-1} to q, a contradiction. Cases B,C are also impossible since we could get a linear cycle. Indeed, in both cases one can start with $h = (q, a, b)$ and go up on $g_i, \ldots, g_j$, return on h and close the cycle on $e_1, \ldots, e_{i-1}$. Thus F is a near-skeleton with exceptional vertex x_{i-1} leading to a contradiction. Therefore $F(x_{i-1}) = \{x_{i-1}\}$ as claimed. Moreover, $(x_{i-1}, x_1, y_0) \notin E(H)$ otherwise that edge with $g_i \ldots, g_j, f$ would form a linear cycle. Now we use $d^+(x_{i-1}) \geq 3$. Since $F_{i-1} = \{x_{i-1}\}$, every edge $(x_{i-1}, a, b) \in E(H)$ intersects $V(P)$ and from Lemma 1.6, we have Cases A,B,C plus those where the star at x_{i-1} intersects $\{a, b\}$, i.e. $\{a, b\} \cap \{y_{i-2}, y_{i-1}\} \neq \emptyset$. Notice that for $a \notin P$, $e = (a, y_{i-2}, x_{i-1}) \notin E(H)$ and $e' = (y_{i-2}, y_{i-3}, x_{i-1}) \notin E(H)$ otherwise $e, e_{i-2}, \ldots, e_1, f, g_j, \ldots, g_i$ or $e', e_{i-3}, \ldots, e_1, f, g_j, \ldots, g_i$ would be a linear cycle. One can easily check that only three cases left for which there is no linear cycle: $\{a, b\} \cap \{y_{i-1}\} \neq \emptyset$, or $\{a, b\} = \{x_i, y_i\}$, or $\{a, b\} = \{y_{i-2}, x_2\}$. From $d^+(x_{i-1}) \geq 3$, all of these possibilities must occur, in particular the third, so $g_{i-1} \in E(H)$ and this completes the proof of Claim 2.3. $\square$Observing that $g_j, \ldots, g_2, f$ is a linear cycle, the proof of Theorem 2.1 is completed. $\square$

3. Proof of Theorem 1.5

Let H be a 3-uniform hypergraph of n vertices not containing any linear cycle. We prove that $\alpha(H) \geq 2n/5$. To facilitate the constructive proof, a *mixed tree* is defined as an extension of linear 3-uniform trees where we allow 2-element edges as well. In particular, graph trees and 3-uniform (linear) trees are both mixed linear trees. A mixed forest is the vertex-disjoint union of mixed trees. *A path-ending* of a mixed forest T is a path with two edges g, h where h is a pendant edge (i.e. the vertices in $h \setminus g$ are of degree one in T) and the vertex $g \cap h$ has degree 2 in T. There are 4 types of path endings, depending whether g or h has 2 or 3 vertices. In fact, we define a

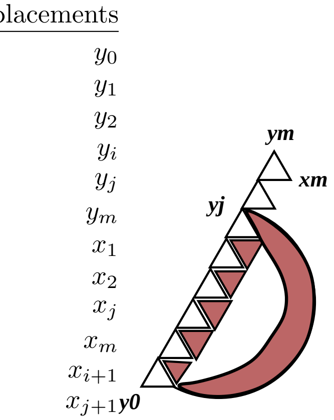

Figure 3: A simple ladder with edges forced by Claim 2.3

degenerate path-ending as well: in the one-edge tree the only edge is considered as a path-ending.

A star-ending of a mixed forest is a set of at least two pendant edges with a common vertex. We state the following obvious lemma without proof.

Lemma 3.1. *Any mixed forest with at least one edge has either a path-ending or a star ending.*

Let T_1 be a maximum nontrivial skeleton, i.e. a linear tree in H such that $|V(T_1)|$ is maximum. We prove the theorem by constructing (step by step, details are in Subsection 3.1) an independent set S of H and a set $Z \subset V(H)$ such that

$$S \cap Z = \emptyset, \qquad \frac{|S|}{|S|+|Z|} \geq \frac{2}{5}, \text{ and } S \cup Z = V(H).$$

Initially set $S = Z = \emptyset$. First we cover $V(T_1)$ with $S \cup Z$ in several steps (see Subsection 3.1) so that $S \subset V(T_1)$ is an independent set in H. Then we iterate the process, taking a maximum skeleton T_2 in the subhypergraph of H induced by $X = V(H) \setminus (S \cup Z)$ and continue with $T_3, T_4, \dots, T_m$, etc. until the subhypergraph of H induced by $X = V(H) \setminus (S \cup Z)$ has no edges. At this point S is extended by X and the construction is completed.

Suppose that we have already defined $S \cup Z$ and T_i so that S is independent in H and $S \subset \cup_{j<i} V(T_j)$. We extend $S \cup Z$ by steps in Subsection 3.1, in each step using a mixed forest T in T_i, initially $T = T_i$. We choose a vertex set R (typically, but not always a subset of $V(T)$) such that a subset $R_0 \subseteq R \bigcap V(T)$ with $|R_0| \geq 2|R|/5$ vertices will be put into the independent set S and $R - R_0$ is placed in Z. We proceed in this way until all vertices of T_i are covered by $S \cup Z$. Note that the procedure defining $S \cup Z$ ensures the properties $\frac{|S|}{|S|+|Z|} \geq \frac{2}{5}, S \cup Z = V(H)$ because at each step $|R_0| \geq 2|R|/5$ and in the final step S is increased by $|X|$ but Z is left untouched. Thus we need to ensure only that the final S is independent. This will be done in Claim 3.2.

3.1. Construction of S and Z

Case 1. $T \subset T_i$ has a path-ending $Q = g \cup h$ with $|h| = 3$. Set $h = (a, b, c)$, $g = (b, d, e)$ (or $g = (b, d)$ if $|g| = 2$ or $g = \emptyset$ if T has one edge h).

Case 1.1. There is no edge (a, b, x) or (b, c, x) in H for which $x \notin Z \cup \{c, d, e\}$. Put a, b into S and c, d, e into Z (ratio at least 2/5). Replace T by the mixed forest obtained from T by deleting the vertices of Q. If Case 1.1 does not hold, we must have edges $(a, b, x_1), (b, c, x_2)$ in H such that $x_1, x_2 \notin Z \cup \{c, d, e\}$. However, $x_i \in V(T_i) \setminus Q$ would create a cycle in T_i, $x_i \in V(T_j)$ for $j < i$ would contradict the maximality of T_j. Thus x_1, x_2 are both in $X = V(H) \setminus (S \cup Z)$. If $x_1 \neq x_2$ then replacing (a, b, c) by (a, b, x_1) and (b, c, x_2), a skeleton larger than T_i could be defined, a contradiction.

Thus $x_1 = x_2 = x$ and we have the following. **Case 1.2.** There is $x \in X$ such that (a, b, x) and (b, c, x) are edges in H. Put a, c into S and b, x into Z (ratio is $1/2$). Replace T by the mixed forest obtained from T by deleting $\{a, b, c\}$. **Case 2.** T has a path-ending $Q = g \cup h$ with $|h| = 2$, set $ab = h$. Put the degree one vertex of h into S and the other vertex of h into Z (ratio is $1/2$). Replace T by the mixed forest obtained from T by deleting the vertices of h. **Case 3.** T has a star-ending. Put one vertex of degree one from each edge of the star into S (there are at least two) and put its other vertices into Z. (Clearly at least $2/5$ of the vertices of the star go to S.) Replace T by the mixed forest obtained from T by deleting the vertices of the star-ending. **Case 4.** T has only isolated vertices. Place all vertices into S. The proof of Theorem 1.5 is complete with the following claim.

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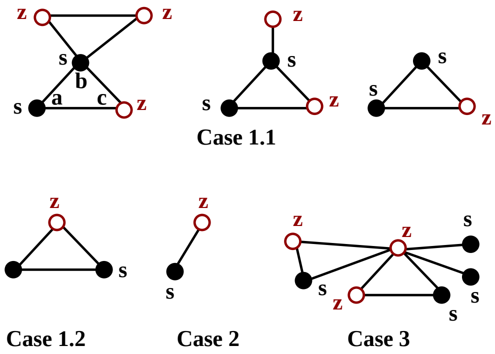

Figure 4: The cases

Claim 3.2. *S is an independent set in H.*

Proof. Suppose that $e = (s_1, s_2, s_3) \in E(H)$ is in S. Observing that the construction ensures $S \subset \cup_{j=1}^{m} V(T_j) \cup X_m$, the maximal choices of the T_j's imply that T_j with the smallest j for which $e \cap V(T_j) \neq \emptyset$ contains at least two vertices of e, say $s_1, s_2 \in T_j$ and $s_3 \in T_k$ for $j \leq k$ or $s_3 \in X_m$. If $s_3 \notin V(T_j)$ then s_3 was placed in S after s_1, s_2. We may assume that s_3 entered S not earlier than s_1, s_2 and s_2 entered S not earlier than s_1. The T-neighbors of a vertex $v \in V(T)$ are the vertices in the edges of T containing v. Notice that in Cases 2,3 the T-neighbors of the vertices placed in S are all placed in Z and in Cases 1.1 and 1.2 the T-neighbors of the pair placed in S are placed in Z (in Case 1.1 c, d, e, in Case 1.2 a, x) - we refer to this as the *neighbor rule.* If (s_1, s_2) or (s_2, s_3) were placed in S together as (a, b) in Case 1.1 then the definition of Case 1 excludes $e \in E(H)$. Suppose that (s_1, s_2) or (s_2, s_3) were placed in S together as (a, c) in Case 1.2, let y denote s_3 or s_1, depending on which pair is (a, c). Then $y \in V(H) - Z - \{a, b, c, x\}$. If $y \in X_j$ (in this case $(a, c) = (s_1, s_2), y = s_3$), replacing the triple (a, b, c) by (a, b, x) and (a, c, y) in T_j, we get a contradiction to the maximality of T_j. If $y \in S \cap V(T_i)$ and the skeleton path from y to $\{a, b, c\}$ reaches b first, then extending it with (a, b, x) and (a, c, y) we get a linear cycle, contradiction. If the skeleton path reaches a or c first, then extending it by (a, c, y), we get a linear cycle.

Thus we may assume that no pair of the vertices of e are placed into S through Cases 1.2 or 1.2, i.e. they entered S either in separate steps or some of them together in Case 3. Using the neighbor rule and Lemma 1.6 with $(v, a, b) = (s_1, s_2, s_3)$, e would create a cycle in H and this contradiction completes the proof of the claim and the proof of Theorem 1.5. $\square$

4. Concluding remarks

It would be desirable to understand better the structure of 3-uniform hypergraphs with no linear cycles. We conjectured that excluding K_5^3 from these hypergraphs essentially improves our results. Indeed, after the submission of this paper, [4] confirmed our conjecture that 3-uniform hypergraphs without linear cycles and without K_5^3 are 2-colorable, thus they contain independent sets of size at least half of their order. Naturally, it would be interesting to extend our results to r-uniform hypergraphs.

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