Vertex Coverings by Monochromatic Paths and Cycles

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ABSTRACT

We survey some results on covering the vertices of 2-colored complete graphs by two paths or by two cycles of different color. We show the role of these results in determining path Ramsey numbers and in algorithms for finding long monochromatic paths or cycles in 2-colored complete graphs.

A graph is 2-colored if each edge is colored either red or blue. We consider results on covering the vertices of 2-colored complete (undirected, directed, or bipartite) graphs by monochromatic paths or cycles. Algorithmic proofs of these results are given, and these results are applied to the calculation of generalized Ramsey numbers.

A 2-colored path or cycle is called *simple* if it is monochromatic or it is the union of a red and a blue path. The following simple result was mentioned in a footnote in [2].

Theorem 1. A 2-colored complete graph K_n contains a simple Hamiltonian path.

Proof (Algorithm 1). Assume that $n \geqq 2$. We define simple paths P_i, $1 \leqq i \leqq n$, in a 2-colored complete graph K_n. Let $P_1 = x_1$ and $P_2 = x_1,x_2$, where x_1 and x_2 are arbitrary vertices of K_n. If $2 \leqq i < n$ and P_i is already defined, we choose an arbitrary vertex x_{i+1} from $K_n - P_i$. The symbol $c(x,y)$ denotes the color of the edge xy. The path P_{i+1} is defined as follows:

If $c(x_1,x_2) = c(x_{i-1},x_i)$ (P_i is monochromatic) or $c(x_{i-1},x_i) = c(x_i,x_{i+1})$ then $P_{i+1} = x_1,x_2, \ldots ,x_i,x_{i+1}$. Stop. If $c(x_{i+1},x_1) = c(x_1,x_2)$ then $P_{i+1} =$

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$x_{i+1}, x_1, \ldots, x_i$. Stop. If $c(x_i, x_1) = c(x_1, x_{i+1})$ then $P_{i+1} = x_2, \ldots, x_i, x_1, x_{i+1}$. Stop. If $c(x_{i+1}, x_i) = c(x_i, x_1)$ then $P_{i+1} = x_{i+1}, x_i, x_1, \ldots, x_{i-1}$. Stop.

It is easy to check that P_{i+1} is simple; therefore P_n is a simple Hamiltonian path. ■

Corollary 1 ([5]). A 2-colored complete graph K_n contains a simple Hamiltonian cycle if $n \geqq 3$.

A "vertex covering theorem," like Theorem 1 or Corollary 1, clearly provides a means to say something about the corresponding Ramsey numbers. Corollary 1 implies that a 2-colored K_{m+n-3} contains a red P_m or a blue P_n if $3 \leqq m \leqq n$, i.e., $R(P_m, P_n) \leqq m + n - 3$. This upper bound comes very easily compared to the proof of its exact value, $n + \lfloor m/2 \rfloor - 1$ ([2]). Another advantage is that Algorithm 1 is extremely simple and fast for finding a simple Hamiltonian path in a 2-colored K_n. It is easy to see that the number of elementary steps (comparisons, bookkeeping) to define P_{i+1} in Algorithm 1 does not depend on i. This observation gives the following result.

Proposition 1. Algorithm 1 finds a simple Hamiltonian path in a 2-colored K_n in $O(n)$ time. Consequently, it finds a monochromatic path of length at least $n/2$ in $O(n)$ time.

The next result deals with cycles covering a 2-colored K_n. It is convenient for our purpose to consider vertices and edges as cycles, a vertex is considered either as a red or a blue cycle, and an edge is considered as a cycle in its color.

Theorem 2. The vertices of a 2-colored K_n can be covered by the vertices of one red and one blue cycle, such that the two cycles have at most one common vertex.

Proof (Algorithm 2). The initial step is to construct a simple Hamiltonian cycle C in K_n using Algorithm 1 (we assume $n \geqq 3$). If C is monochromatic then we stop: C and any vertex of C give the required cycles. We can assume that C is the union of a red and a blue path: $C = R \cup B$, $R = x_1, x_n, x_{n-1}, \ldots, x_m$ and $B = x_1, x_2, \ldots, x_m$ for some $1 < m \leqq n$. Since the role of the edge $e = x_1 x_m$ is symmetric in C, we may assume that e is red.

Starting from the red cycle $C_0 = R \cup \{e\}$, we construct red cycles C_i for $i = 1, 2, \ldots$ on the vertex set $x_{m-i}, x_{m-i+1}, \ldots, x_n, x_1, \ldots, x_i, x_{i+1}$, such that $x_{i+1} x_{m-i}$ is an edge of C_i. If C_i is already defined for some i and $|C_i| \geqq n - 2$ then we stop: C_i and $K_n - C_i$ (a vertex or a blue edge) give the required cycles. If $|C_i| \leqq n - 3$ then we stop when at least one of the following three conditions occurs:

(A) $x_{i+1}x_{m-i-1}$ is blue: C_i and $x_{i+1}, \ldots, x_{m-i-1}$ give the cycles.
(B) $x_{i+2}x_{m-i}$ is blue: C_i and $x_{i+2}, \ldots, x_{m-i}$ give the cycles.
(C) $x_{i+2}x_{m-i-1}$ is blue: C_i and $x_{i+2}, \ldots, x_{m-i-1}$ give the cycles.

The red and blue cycles are disjoint in case C; they have one common vertex in cases A and B.

If the conditions A, B, and C all fail then C_{i+1} is defined by removing the edge $x_{i+1}x_{m-i}$ from C_i and adding the red edges $x_{i+1}x_{m-i-1}$, $x_{i+2}x_{m-i}$ and $x_{i+2}x_{m-i-1}$. The construction eventually stops since $|C_{i+1}| = |C_i| + 2$. ■

It is not known whether a 2-colored K_n has a vertex cover by two *disjoint* (red and blue) cycles. We conjecture that the answer is yes and investigations of special cases in [1] seem to support this.

The analysis of Algorithm 2 gives the following.

Proposition 2. Algorithm 2 finds in $O(n)$ time a vertex cover of a 2-colored K_n by two (red and blue) cycles, having at most one common vertex. As a consequence, Algorithm 2 finds in $O(n)$ time a monochromatic cycle of length at least $n/2$ in a 2-colored K_n.

Propositions 1 and 2 show that the problem of finding a monochromatic path or cycle of length at least $n/2$ in a 2-colored K_n is of $O(n)$ complexity, a somewhat surprising fact since the problem input, a 2-colored K_n, contains $\binom{n}{2}$ bits of information. On the other hand, the algorithm coming from the proof of the path-path Ramsey number ([2]) finds in $O(n^2)$ time a monochromatic path of length at least $2n/3$ in a 2-colored K_n. These considerations lead us to a question.

Problem 1. Is there a real number $c > 1/2$ and an algorithm A of $O(n)$ time complexity, such that A finds a monochromatic path of length at least cn in any 2-colored K_n?

The next result (conjectured by Lehel and proved by Raynaud in [5]) generalizes Corollary 1 for complete symmetric directed graphs. (A directed graph G is *symmetric* if $xy \in E(G)$ implies $yx \in E(G)$.)

Theorem 3. A 2-colored complete symmetric directed graph with at least two vertices contains a simple directed Hamiltonian cycle.

Proof (Raynaud's proof with simplifications). Let k and m be natural numbers satisfying $1 \leq k \leq m$, where $k \neq m-1$. A loop $L = L(x_1, x_2, \ldots, x_m; k)$ is a 2-colored symmetric directed graph on vertices $x_1, x_2, \ldots, x_m$, with red edges $x_1x_2, x_2x_3, \ldots, x_{m-1}x_m$ and x_mx_k if $m \neq k$. The "reverse" edges $x_2x_1, x_3x_2, \ldots, x_mx_{m-1}$ and x_kx_m (if $m \neq k$) are blue. Note the special loops: the one vertex graph ($k = m = 1$), the path ($k = m$) and the cycle ($k = 1$,

$m \geqq 3$). The loops of a 2-colored complete symmetric directed graph have a natural partial ordering:

$L_1 \leqq L_2$ if $L_1 = L_1(x_1, \ldots, x_m; k_1), L_2 = L_2(x_1, \ldots, x_n; k_2)$ and $m < n$ or $m = n$ and $k_2 \leqq k_1$.

The proof proceeds by induction on the number of vertices of the 2-colored complete symmetric directed graph G. The case $|V(G)| = 2$ is trivial. For $|V(G)| > 2$, let $L = L(x_1, \ldots, x_m; k)$ be a maximal loop in the partial order defined above and let $H = G - \{x_k, \ldots, x_m\}$. If H is empty, then $k = 1$ and the red (or blue) edges of L form a Hamiltonian cycle of G. If H is nonempty, we apply the inductive hypothesis: let $C = \{y_1, y_2, \ldots, y_n\}$ be a simple Hamiltonian cycle of H. We may assume that y_n is the end-vertex of the red path of C (in a monochromatic C, y_n can be any vertex).

We observe that either $y_n x_m$ is a red edge or $x_m y_n$ is a blue edge in G by the maximality of L. If $y_n x_m$ is red, then $y_n x_m x_k \ldots x_{m-1} y_1 \ldots y_n$ is a simple Hamiltonian cycle of G. If $x_m y_n$ is blue, then $y_n y_1 \ldots y_{n-1} x_{m-1} \ldots x_k x_m y_n$ is a simple Hamiltonian cycle of G. ■.

Theorem 3 gives the value of a Ramsey number, $R(\vec{P}_m, \vec{P}_n)$, which is the smallest t such that a 2-colored complete symmetric directed graph $\vec{K}_t$ always contains either a red $\vec{P}_m$ or a blue $\vec{P}_n$. An obvious example shows that $R(\vec{P}_m, \vec{P}_n) \geqq m + n - 3$ for $m, n \geqq 3$. On the other hand Theorem 3 implies that $R(\vec{P}_m, \vec{P}_n) \leqq m + n - 3$ for $m, n \geqq 3$.

Corollary 2 ([4], [6]). $R(\vec{P}_m, \vec{P}_n) = m + n - 3$. for $m, n \geqq 3$.

It is easy to formulate the proof of Theorem 3 as an algorithm to find a simple Hamiltonian cycle in a 2-colored $\vec{K}_n$. Although the algorithm we obtain is still simple (compared to Algorithm 1 for the undirected case), its time complexity is $O(n^2)$. The reason for this behavior is that one needs $O(n^2)$ time to find a maximal loop in a 2-colored $\vec{K}_n$.

Path-path Ramsey numbers for complete bipartite graphs were independently established in [3] and [4]. The heart of the method in [4] can be stated as a vertex covering result (Theorem 4 below). An exceptional coloring of a complete bipartite graph with vertex classes A and B is a coloring, where $A = A_1 \cup A_2$, $B = B_1 \cup B_2$, $A_1 \cap A_2 = \emptyset$, $B_1 \cap B_2 = \emptyset$, moreover the edges between A_1 and B_1, A_2 and B_2 are all red, the edges between A_1 and B_2, A_2 and B_1 are all blue.

Theorem 4. A 2-colored complete bipartite graph $K(n,n)$ has either an exceptional coloring or contains a simple path P with the following properties: both the red and the blue path of P have an even number of vertices, P covers the vertices of $K(n,n)$ with one possible exception.

Theorem 4 is suitable to obtain the path-path Ramsey numbers for 2-colored complete bipartite graphs. Its simplest consequence is the following corollary.

Corollary 3. A 2-colored $K(m+n-1, m+n-1)$ contains a red P_{2m} or a blue P_{2n}.

The proof of Theorem 4 in [4] can be formulated as an $O(n^2)$ time algorithm to find the required simple path or the exceptional covering of a 2-colored $K(n,n)$. This algorithm is not presented here since it is longer and less illustrative than the algorithms shown in this paper.

We have seen that an $O(n)$ algorithm can find a "long" monochromatic path in a 2-colored K_n but only $O(n^2)$ algorithms are available for the same purpose if K_n is replaced by $\vec{K}_n$ or by $K(n,n)$. It would be interesting to know whether an $O(n)$ algorithm can have any power on $\vec{K}_n$ and $K(n,n)$. More precisely, we have the following problem.

Problem 2. Is there a positive real number c and an algorithm A_1 (A_2) of $O(n)$ time complexity such that A_1 (A_2) finds a monochromatic path of length at least cn in any 2-colored $\vec{K}_n$ ($K(n,n)$)?

References

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