Large Cross-Free Sets in Steiner Triple Systems

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Abstract: A *cross-free* set of size m in a Steiner triple system $(V, \mathcal{B})$ is three pairwise disjoint m-element subsets $X_1, X_2, X_3 \subset V$ such that no $B \in \mathcal{B}$ intersects all the three X_i-s. We conjecture that for every admissible n there is an STS(n) with a cross-free set of size $\lfloor \frac{n-3}{3} \rfloor$ which if true, is best possible. We prove this conjecture for the case $n = 18k + 3$, constructing an STS($18k + 3$) containing a cross-free set of size $6k$. We note that some of the 3-bichromatic STSs, constructed by Colbourn, Dinitz, and Rosa, have cross-free sets of size close to $6k$ (but cannot have size exactly $6k$). The constructed STS($18k + 3$) shows that equality is possible for $n = 18k + 3$ in the following result: in every 3-coloring of the blocks of any Steiner triple system STS(n) there is a monochromatic connected component of size at least $\lceil \frac{2n}{3} \rceil + 1$ (we conjecture that equality holds for every admissible n). The analog problem can be asked for r-colorings as well, if $r - 1 \equiv 1, 3 \pmod 6$ and $r - 1$ is a prime power, we show that the answer is the same as in case of complete graphs: in every r-coloring of the blocks of any STS(n), there is a monochromatic connected component with at least $\frac{n}{r-1}$ points, and this is sharp for infinitely many n. © 2014 Wiley Periodicals, Inc. J. Combin. Designs 00: 1–7, 2014

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1. INTRODUCTION

A hyperwalk in a hypergraph $H = (V, E)$ is a sequence $v_1, e_1, v_2, e_2, \ldots, v_{t-1}, e_{t-1}, v_t$ of vertices and edges such that for all $1 \leq i < t$ we have $v_i \in e_i$ and $v_{i+1} \in e_i$. We say that $v \sim w$, if there is a hyperwalk from v to w. The relation $\sim$ is an equivalence relation, and the subhypergraphs induced by its classes are called the *connected components* of H. A vertex v that is not covered by any edge forms a trivial component with one vertex v and no edge.

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The size of the largest monochromatic component in edge colorings of complete graphs and hypergraphs is well investigated, for a present survey see [5]. For example, in every 3-coloring of the edges of the complete graph K_n there is a monochromatic connected component of size at least $n/2$ and in every 3-coloring of the edges of K_n^3, the complete 3-uniform hypergraph, there is a monochromatic spanning component. What happens in between, when the blocks of a Steiner triple system $(V, \mathcal{B})$ are colored? For example, in every coloring of the blocks of STS(9) with three colors, there is a monochromatic connected component of size at least seven but in the 4-coloring of its blocks defined by the four parallel classes, every component in every color has only three points. Let $f(n)$ denote the largest m such that in every 3-coloring of the blocks of any STS(n) there is a monochromatic connected component on at least m points. It is easy to see that $f(7) = 6$, $f(9) = 7$. Our main result is the following.

Theorem 1. *$f(6k + 3) \geq 4k + 3$ with equality if k is divisible by 3. Moreover, $f(6k + 1) \geq 4k + 2$.*

In fact, the inequalities of Theorem 1 are probably always sharp (one can easily check the cases $k = 1, 2$):

Conjecture 2. *For every positive integer k, $f(6k + 1) = 4k + 2$, $f(6k + 3) = 4k + 3$.*

Three pairwise disjoint m-element sets of points, $X_1, X_2, X_3 \subset V$, is a *cross-free set of size m* in a Steiner triple system $(V, \mathcal{B})$ if no block $B \in \mathcal{B}$ intersects each X_i in exactly one point. To obtain the upper bound in Theorem 1, we need some STS with a cross-free set of size almost $n/3$. In Theorem 4, we construct an STS($6k + 3$) for the case $k \equiv 0 \pmod 3$ that contains a cross-free set of size $2k$ (and this is best possible).

It is worth noting that constructions of Colbourn, Dinitz, and Rosa in [2] provide STS(n)$_s$ with cross-free sets of size asymptotic to $n/3$. They construct 3-bichromatic STSs where all points are partitioned into X_1, X_2, X_3 so that every block intersects *precisely two* of the X_i-s and they can also control the sizes of the X_is. In particular, they provide 3-bichromatic STS(n)s where the sizes are nearly equal to $n/3$. However, it follows easily that in 3-bichromatic STS(n)s with $|X_1| \leq |X_2| \leq |X_3|$, $n/3 - |X_i|$ tends to infinity with n. Therefore, to achieve a cross-free set of size $2k$ in an STS($6k + 3$) the number of blocks inside the X_is tends to infinity with n.

To see the connection of cross-free sets to $f(n)$, let $G(n)$ be the size of the largest cross-free set present in **some** STS(n).

Lemma 3. *$f(n) \leq n - G(n)$.*

Proof. Suppose $|X_1| = |X_2| = |X_3| = G(n)$ for a cross-free set $X_1, X_2, X_3 \subset V$ in some STS(n). Then coloring any block B with the smallest i such that $B \subset V \setminus X_i$, we have a 3-coloring of the blocks with one nontrivial monochromatic connected component of size $n - G(n)$ in each color. □

The next result implies the equality $f(6k + 3) = 4k + 3$ for k divisible by 3 in Theorem 1.

Theorem 4. *For $n = 18k + 3$, $G(n) = 6k$.*

In fact, Theorem 4 probably can be extended, it would imply Conjecture 2.

Conjecture 5. $G(6k + 3) = 2k$, $G(6k + 1) = 2k - 1$.

It is easy to see that Conjecture 5 is sharp (if true). Indeed, a cross-free set of size $2k+1$ in an STS($6k+3$) would mean that there are at most $3\binom{2k+1}{2}$ blocks and that is less than $\binom{6k+3}{2}/3$. Similarly, a cross-free set of size $2k$ in an STS($6k+1$) would show that there are at most $3k + 3\binom{2k}{2}$ blocks, less than $\binom{6k+1}{2}/3$.

One can define $f_r(n)$ similarly for r-colorings of blocks. A lower bound on it can be easily derived from known results.

Proposition 6. $f_r(n) \geq \lceil \frac{n}{r-1} \rceil$.

Proof. Any r-coloring of the blocks of an STS(n) defines an r-coloring of the edges of K_n, by coloring the three pairs defined by a block with the color of the block. In this coloring there is a monochromatic, say red connected component C with at least $\lceil \frac{n}{r-1} \rceil$ vertices, proved first in [4], a more accessible account is the survey [5]. The blocks covering the red edges of C obviously span a red connected component on C. □

The lower bound of Proposition 6 is trivially sharp for $r = 2$ but also for certain other values of r, starting with $r = 4, 8, 10, 14, \ldots$

Proposition 7. *$f_r(n) = \frac{n}{r-1}$ for infinitely many n if $r-1$ is in the form 3^m, p^m, q^{2m} where $m \geq 1$, p, q are primes, $p \equiv 1$ (mod 6), $q \equiv -1$ (mod 6).*

Proof. $f_r(n) \geq \frac{n}{r-1}$ follows from Proposition 6. Suppose $r-1$ is a prime power in the form 3^m, p^m, q^{2m} where $m \geq 1$, p, q are primes, $p \equiv 1$ (mod 6), $q \equiv -1$ (mod 6). This implies that $r-1 \equiv 1$ (mod 6) or $r-1 \equiv 3$ (mod 6). Then there exists an affine plane P of order $r-1$ and we can define an STS($(r-1)^2$) by substituting each line of P by a copy of an STS($r-1$). Then the blocks of STS($(r-1)^2$) can be naturally colored with r colors according to the r parallel classes of P. In this coloring every component has size $r-1 = \frac{(r-1)^2}{r-1}$, providing an example with equality. To get infinitely many, we can apply the well-known direct product construction (see [1]) of STS($n_1 n_2$) from STS(n_1), STS(n_2). Assume we already know that for some $t \geq 0$ the blocks of STS($3^t(r-1)^2$) can be r-colored so that each color class has $r-1$ nontrivial components (of size $3^t(r-1)$) and consider the STS($3^{t+1}(r-1)^2$) defined as STS($3^t(r-1)^2$) $\times T$ where T is a single block on three points. Then each component C in each color class of STS($3^t(r-1)^2$) defines a component $C \times T$ in STS($3^{t+1}(r-1)^2$) whose blocks in $C \times T$ can be colored with the same color. This defines a natural r-coloring of the blocks of STS($3^{t+1}(r-1)^2$), preserving the property that each color class has $r-1$ nontrivial components. □

Our problem to determine $f(n)$ led to find $G(n)$, the size of the largest cross-free set present in **some** STS(n). It seems natural and interesting to find or estimate the size $g(n)$ of the largest cross-free set present in **any** STS(n). Obviously,

$$G(n) \geq g(n) \geq \frac{\alpha(n)}{3}$$

where $\alpha(n)$ is the largest independent set present in *any* STS(n). For the most recent result and history on $\alpha(n)$ see [3].

Problem 8. *Is $g(n)$ significantly smaller than $G(n)$?*

2. PROOF OF THEOREMS 1, 4

We prove first that $f(6k+3) \geq 4k+3$, $f(6k+1) \geq 4k+2$.

Suppose that the blocks of an STS $(V, \mathcal{B})$ with $|V| = n$ are 3-colored and consider the three components C_1, C_2, C_3 in colors 1, 2, 3 containing a point $v \in V$. There are some cases according to the number of C_is with points covered only by C_i, we call such points as "private parts" of C_i.

Case 1. No C_i has private part. In this case, the sets C_i doubly cover $V \setminus \{v\}$ and v is triply covered. This implies easily that $f(6k+3) \geq 4k+3$ and also $f(6k+1) \geq 4k+2$, unless if the C_is intersect in one point and all the three doubly covered sets have size $2k$. However, in this case we can have $3k$ blocks covering v and any other block must cover a pair of $(C_i \cap C_j) \setminus \{v\}$. Thus altogether we have at most $3k + 3\binom{2k}{2} < \frac{\binom{6k+1}{2}}{3}$ blocks in STS$(6k+1)$ and that is a contradiction.

Case 2. Only C_1 has a private part. Now there is no point $w \in V$ that belongs to $(C_2 \cap C_3) \setminus C_1$, otherwise no block can cover wx where x is from the private part of C_1. Thus in this case C_1 covers V.

Case 3. Two C_is, say C_1, C_2 have private parts. Now $(C_1 \cap C_3) \setminus C_2$ and $(C_1 \cap C_2) \setminus C_3$ are both empty and any pair of points x, y from the private parts of C_1, C_2, respectively, must be in a block colored with color 3. Thus the union of the private parts of C_1, C_2 is part of a component C of color 3. We can now apply the argument of Case 1 to the components C, C_1, C_2.

Case 4. All C_is have private parts. Now sets covered by precisely two of C_1, C_2, C_3 must be empty and the private parts $X_i \subset C_i$ together with $X_4 = C_1 \cap C_2 \cap C_3$, partition V. Pairs of points $x \in X_1, y \in X_2$ must be in a block of color 3, pairs of points $x \in X_1, y \in X_3$ must be in a block of color 2, pairs of points $x \in X_2, y \in X_3$ must be in a block of color 1, thus the union of any two X_is is covered by (in fact equal to) a monochromatic component. Observe that every block of our $(V, \mathcal{B})$ must contain a pair from some of the X_is, thus

$$s = \sum_{i=1}^{4} \binom{|X_i|}{2} \geq \frac{\binom{n}{2}}{3}. \tag{1}$$

First let $n = 6k+3$, assume w.l.o.g that

$$|X_1| \leq |X_2| \leq |X_3| \leq |X_4|.$$

If $|X_1| \geq 3k+1+t$ for some positive integer t then let X_j be the largest among X_2, X_3, X_4. Then

$$|X_1| + |X_j| \geq 3k+1+t + \frac{3k-t+2}{3} \geq 4k+3$$

proving what we need. However, if $|X_1| \leq 3k+1$ then the maximum of s (under the condition that each component has size at most $2k+2$) is obtained when $|X_1| = 3k+1$, $|X_2| = |X_3| = k+1$, $|X_4| = k$. But this contradicts to (1). Similar argument works if $n = 6k+1$, then

$$|X_1| = 3k+1, |X_2| = |X_3| = |X_4| = k$$

gives the largest s and the contradiction.

This finishes the proof of the two inequalities of Theorem 1. It is left to prove that $f(6k+3) = 4k+3$ if k is divisible by 3, i.e. to prove Theorem 4. In fact we need to prove only that $G(n) \geq 6k$, however $G(n) < 6k+1$ follows easily: a partition of V for a STS$(18k+3)$ into three sets of size $6k+1$ cannot be cross-free since then there are at most $t = 3\binom{6k+1}{2}$ blocks and t is less than the number of blocks required in an STS$(18k+3)$.

We construct an STS$(18k+3)$ with a cross-free set of size $6k$ as follows. Let H_k be the graph with $6k$ vertices and $4k$ edges, having $2k$ components, k of them a P_4, a path on four vertices, and k of them a single edge. We call the *middle* of H_k the union of the middle edges of the P_4 components in H_k. A *near factor* of a graph with $2m$ (or $2m-1$) vertices means $m-1$ pairwise disjoint edges.

Lemma 9. *Let T be the graph containing k vertex disjoint edges on $6k$ vertices. Then the edge set of $G_k = K_{6k} \setminus T$ can be partitioned into $2k$ factors $F_1, \ldots, F_{2k}$ and $4k$ near factors $E_1, \ldots, E_{4k}$ in such a way that the pairs uncovered by the near factors form a graph isomorphic to H_k and in the isomorphism the middle of H_k corresponds to the pairs of T.*

Based on the lemma, we define an STS$(18k+3)$ with a cross-free set of size $6k$. Take three disjoint copies of H_k on vertex sets X_0, X_1, X_2 and define $\mathcal{T}$ as a partial triple system PTS$(18k)$ on $\cup_{i=0}^{2} X_i$ as follows. Partition each X_i into k P_4 paths $a^i_{6j+1}, a^i_{6j+2}, a^i_{6j+3}, a^i_{6j+4}$ and k edges a^i_{6j+5}, a^i_{6j+6} for $j = 0, \ldots k-1$. This way each X_i spans a copy of H_k.

Now Lemma 9 can be applied with vertex set X_0 to obtain $2k$ factors and $4k$ near factors with the required properties (with respect to the copy of $H_k \subset X_0$). We can extend these factors and near-factors to blocks of $\mathcal{T}$, using vertices of X_1 as follows. Let a^1_{6j+4} define blocks with the pairs of the near factor E_{4j+1} with uncovered pair a^0_{6j+2}, a^0_{6j+3}, $j = 0, \ldots, k-1$. Then a^1_{6j+1} defines blocks with the pairs of the near factor E_{4j+2} with uncovered pair a^0_{6j+5}, a^0_{6j+6}, $j = 0, \ldots, k-1$; similarly a^1_{6j+6} defines blocks with the pairs of the near factor E_{4j+3} with uncovered pair a^0_{6j+1}, a^0_{6j+2}, $j = 0, \ldots, k-1$; and a^1_{6j+5} defines blocks with the pairs of the near factor E_{4j+4} with uncovered pair a^0_{6j+3}, a^0_{6j+4}, $j = 0, \ldots, k-1$. Finally, a^1_{6j+2}, a^1_{6j+3} define blocks with the pairs of the factors F_{2j+1}, F_{2j+2}, $j = 0, \ldots, k-1$.

The construction of the previous paragraph can be repeated cyclically, defining blocks with one vertex in X_2 and two in X_1, and a third time defining blocks with one vertex in X_0 and two in X_2. By Lemma 9, the partial STS $\mathcal{T}$ defined this way covers all pairs of $X_0 \cup X_1 \cup X_2$ except a 3-regular graph U of with the following edges: a^i_{6j+2}, a^i_{6j+3} for $i = 0, 1, 2$ and $j = 0, \ldots, k-1$ (formed by the middle of the three copies of H_k) and the $3 \times 8k$ edges between the pairs X_i, X_j that belong to the uncovered pairs of the $3 \times 4k$ near factors. It can be easily seen that the graph U can be factored into three 1-factors. In fact, these factors are

$$a^i_{6j+2}, a^i_{6j+3}, a^i_{6j+1}, a^{i-1}_{6j+6}, a^i_{6j+5}, a^{i-1}_{6j+4},$$
$$a^i_{6j+4}, a^{i-1}_{6j+2}, a^i_{6j+5}, a^{i-1}_{6j+3}, a^i_{6j+6}, a^{i-1}_{6j+1},$$
$$a^i_{6j+1}, a^{i-1}_{6j+5}, a^i_{6j+4}, a^{i-1}_{6j+3}, a^i_{6j+6}, a^{i-1}_{6j+2},$$

where $i = 0, 1, 2$ and $j = 0, \dots k-1$ with arithmetic on i, j-s are modulo $3, 6$, respectively.

Finally, $\mathcal{T}$ is extended to an STS$(18k+3)$ by extending each factor of U to a block with one of three new points A, B, C that also forms the last block. This finishes the proof of Theorem 4 and with it Theorem 1.

Proof of Lemma 9. The required partition is constructed from the *standard factorization* of K_{6k} on vertex set $\{1, 2, \dots, 6k-1\} \cup \infty$ where (for $i = 1, 2, \dots, 6k-1$) factor F_i contains (i, ∞) and $\{(i-j, i+j) : 1 \le j \le 3k-1\}$ with mod $6k-1$ arithmetic.

We shall keep $2k-1$ of the factors F_i and define the near factors $E_1, \dots, E_{4k}$ by deleting one edge from each of the other $4k$ factors so that the deleted edges form a graph isomorphic to H_k. The factor formed by the middle of H_k is left uncovered and all other edges of H_k form a new factor F^* that is added as the $2k$-th factor in the partition. To define the construction, it is enough to specify the set of $4k$ pairs (all from different F_i) that form a graph Z_k isomorphic to H_k. The construction is the simplest for $k \equiv 1 \pmod 2$ so we describe that first.

Suppose that $k \equiv 1 \pmod 2$ and set $W = \{(1,3), (2,4), (3,5), (5,\infty)\}$. Moreover, for $m \in \{6, 12, \dots, 6(k-2)\}$ let $L_m = A_m \cup B_m$ be the following set of eight pairs on 12 consecutive numbers:

$$A_m = \{(m, m+2), (m+2, m+4), (m+4, m+6), (m+1, m+3)\},$$
$$B_m = \{(m+5, m+7), (m+7, m+9), (m+9, m+11), (m+8, m+10)\}.$$

It is immediate to check that W, A_m, B_m are all define (6-vertex) graphs with a P_4 component and a K_2 component. Thus the graph Z_k defined by W (for $k=1$) and by $W \cup_{m=6}^{6(k-2)} L_m$ (for odd $k > 1$) is isomorphic to H_k. Moreover, since all edges (apart from $(5,\infty)$) of Z_k are in the form $(j, j+2)$ and $j \ne 4$, each edge of Z_k belongs to different F_i.

The case $k \equiv 0 \pmod 2$ is slightly more involved, we use another type of components C_m, D_m (beside W) to define Z_k.

$$C_m = \{(m, m+1), (m, m+2), (m+2, m+4), (m+3, m+5)\},$$
$$D_m = \{(m, m+2), (m+1, m+2), (m+1, m+3), (m+4, m+5)\}.$$

For $k = 2$ we use W followed by C_6 to define Z_2. For $k > 2$ start with W, then $\frac{k}{2}$ copies of C_m ($m = 6, 12, \dots, 3k$) then $\frac{k-2}{2}$ copies of D_m ($m = 3k+6, \dots, 6(k-1)$). To check here that each edge of Z_k belongs to different F_i, note that "jumping pairs" $(j, j+2)$ are obviously from different F_i (from F_{j+1}). The same is true for the "consecutive pairs" $(j, j+1)$. To check consecutive pairs against jumping pairs, notice that for $m = 6, 12, \dots, 3k$ the pair $(m, m+1)$ of C_m belongs to F_{3k+m}, a starting point of the D-block opposite to C_m thus it is not skipped by any jumping pair. Similarly, for $m = 3k+6, \dots, 6(k-1)$, the pairs $(m+1, m+2)$ and $(m+4, m+5)$ in D_m belong to F_{m+2-3k} and F_{m+5-3k}, respectively, and they are not skipped in their opposite C-blocks. □

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