Multicolor Ramsey numbers for triple systems

Maria Axenovich * András Gyárfás †
Hong Liu ‡ Dhruv Mubayi §

February 22, 2013

Abstract

Given an r-uniform hypergraph H, the multicolor Ramsey number $r_k(H)$ is the minimum n such that every k-coloring of the edges of the complete r-uniform hypergraph K_n^r yields a monochromatic copy of H. We investigate $r_k(H)$ when k grows and H is fixed. For nontrivial 3-uniform hypergraphs H, the function $r_k(H)$ ranges from $\sqrt{6k}(1+o(1))$ to double exponential in k.

We observe that $r_k(H)$ is polynomial in k when H is r-partite and at least single-exponential in k otherwise. Erdős, Hajnal and Rado gave bounds for large cliques K_s^r with $s \geq s_0(r)$, showing its correct exponential tower growth. We give a proof for cliques of all sizes, $s > r$, using a slight modification of the celebrated stepping-up lemma of Erdős and Hajnal.

For 3-uniform hypergraphs, we give an infinite family with sub-double-exponential upper bound and show connections between graph and hypergraph Ramsey numbers. Specifically, we prove that

$$r_k(K_3) \leq r_{4k}(K_4^3 - e) \leq r_{4k}(K_3) + 1,$$

where $K_4^3 - e$ is obtained from K_4^3 by deleting an edge.

We provide some other bounds, including single-exponential bounds for $F_5 = \{abe, abd, cde\}$ as well as asymptotic or exact values of $r_k(H)$ when H is the bow $\{abc, ade\}$, kite $\{abc, abd\}$, tight path $\{abc, bcd, cde\}$ or the windmill $\{abc, bde, cef, bce\}$. We also determine many new "small" Ramsey numbers and show their relations to designs. For example, the lower bound for $r_6(kite) = 8$ is demonstrated by decomposing the triples of [7] into six partial STS (two of them are Fano planes).

*Iowa State University, Ames, IA, USA and Karlsruher Institut für Technologie, Karlsruhe, Germany; maria.aksenovich@kit.edu, supported in part by NSF grant DMS-0901008.

†Rényi Institute of Mathematics, Budapest, Hungary; gyarfas@renyi.hu, supported in part by OTKA K104343.

‡University of Illinois at Urbana-Champaign, Urbana, IL, USA; hliu36@illinois.edu.

§University of Illinois at Chicago, Chicago IL, USA; mubayi@uic.edu, supported in part by NSF grant 0969092.

1 Introduction, results

An r-uniform hypergraph H is a pair (V, E) where V is a vertex set and $E \subseteq \binom{V}{r}$ is the set of edges. Let K_n^r be the complete r-uniform hypergraph containing all r-subsets of vertices as edges. For an edge $\{v_1, v_2, \ldots, v_r\}$ we often write $v_1v_2 \ldots v_r$. When $r = 2$, denote K_n^r by K_n. We shall also use the notation $\binom{[n]}{r}$ or $\binom{V}{r}$ for the edge set of K_n^r. An r-uniform hypergraph H is ℓ-partite if its vertex set can be partitioned into ℓ parts (called partite sets) such that each edge contains at most one vertex from each part; H is a complete r-partite hypergraph if each choice of r vertices from distinct partite sets forms an edge, and H is balanced if its partite sets differ in size by at most one. A matching is a hypergraph consisting of disjoint edges. A hypergraph $H = (V, E)$ is a subhypergraph of $F = (V', E')$ if $V \subseteq V'$ and $E \subseteq E'$. Denote by $\mathrm{ex}(n, H)$ the maximum number of edges in an n-vertex r-uniform hypergraph containing no copy of H as subhypergraph. The density of an r-uniform hypergraph $H = (V, E)$ on n vertices is $d(H) = |E|/\binom{n}{r}$.

The **multicolor Ramsey number** for an r-uniform hypergraph H, denoted by $r_k(H)$, is the minimum n such that no matter how the edges of K_n^r are colored with k colors, there is a monochromatic copy of H. While there is a number of results in the literature about $r_k(H)$ when k is a small fixed number (see [4]), the case when H is fixed and k grows appears not to have been extensively studied. The following three results are among the few results known in this area:

Theorem 1 (Lazebnik and Mubayi [25])**.** *Fix integers $r, s, t \geq 2$. Let $H^r(s,t)$ be the complete r-partite r-uniform hypergraph with $r-2$ parts of size 1, one part of size s and one part of size t. Then*

(i) $r_k(H^r(2, t+1)) = tk^2 + O(k)$;

(ii) $r_k(H^r(s,t)) = \Theta(k^s)$, for fixed $t, s \geq 2, t > (s-1)!$;

(iii) $r_k(H^r(3,3)) = (1 + o(1))k^3$.

Let M be a matching with two r-tuples. Notice that an edge-coloring of K_n^r without monochromatic copies of M corresponds to a proper vertex-coloring of Kneser graph $K(n, r)$, that is, the graph with vertex set $\binom{[n]}{r}$ and two r-sets are adjacent if and only if they are disjoint. Lovász proved that the chromatic number of $K(n, r)$ is equal to $n - 2r + 2$. Reformulating his result, we obtain the following.

Theorem 2 (Lovász [28])**.** *If M is a matching with two r-tuples, then $r_k(M) = k + 2r - 1$.*

Gyárfás and Raeisi observed that results of Csákány and Kahn [6] and the standard coloring of the Kneser graph imply the following.

Proposition 3 (Gyárfás and Raeisi [14])**.** *If C_3^3 is the hypergraph with edge set $\{abc, cde, efa\}$, then $k + 5 \leq r_k(C_3^3) \leq 3k + 1$.*

In this paper, we start a systematic investigation on the growth rate of $r_k(H)$ for some fixed H as k grows. Our first result shows that $r_k(H)$ is polynomial in k if and only if H is r-partite.

Proposition 4. *Let $r \geq 2$ be fixed and H be a connected r-uniform hypergraph. Then $r_k(H)$ is polynomial in k if and only if H is r-partite. In particular, there are positive constants c and c', such that*

(i) If H is r-partite, then $r_k(H) = O(k^c)$

(ii) If H is not r-partite, then $r_k(H) \geq 2^{c'k}$.

Determining the growth rate of $r_k(H)$ in general is known to be a very hard problem. For example, the best known bounds even for the smallest nontrivial graph case are $c^k < r_k(K_3) < c'k!$ for some positive constants c and c' (see Chung [5] and Erdős, Szekeres [11]). Define the tower function as follows: $t_1(n) = n$ and $t_{i+1}(n) = 2^{t_i(n)}$ for all $i \geq 1$. Erdős, Hajnal and Rado gave an upper bound for all cliques and a lower bound for only large cliques.

Theorem 5 (Erdős and Rado [10], Erdős et al. [9])**.** *Let $s > r \geq 2$. There are positive integers $c = c(s,r) \leq 3(s-r)$, $s_0(r)$, and $c' = c'(s,r)$ such that*

$$t_r(c'k) < r_k(K_s^r) < t_r(ck \log k)$$

where the lower bound holds for $s \geq s_0(r)$.

It is worth noting that the lower bound in [9] was stated for the case when the number of colors, k, is fixed while r grows and the bound was only for large cliques. But the proof in [9] applies naturally to our case as well, when k grows and the other parameters are fixed. Recently, an improved stepping-up lemma was proved by Conlon et al [3]. Their main result implies a lower bound for cliques of smaller sizes, but still only for $s \geq 2r - 1$. Duffus, Lefmann and Rödl [7] took another approach, using shift graphs, and proved a lower bound for cliques of all sizes $s > r$, but require k being fixed and $r \gg k$. Our next result gives a proof for cliques of all sizes using a slight modification of the stepping-up lemma, due to Erdős and Hajnal (see Chapter 4.7 in [13]).

Theorem 6. *For any $s > r \geq 2$ and $k > r2^r$ we have*

$$r_k(K_s^r) > t_r\left(\frac{k}{2^r}\right).$$

Our remaining results are all for 3-uniform hypergraphs and we will address the question of determining $r_k(H)$ for most interesting H's with 6 or fewer vertices. Let $K_4^3 - e$ be a hypergraph obtained from K_4^3 by removing one edge. Our next theorem gives bounds on $r_k(K_4^3 - e)$ in terms of $r_k(K_3)$, showing that compared to the double-exponential bounds for K_4^3 from Theorems 5 and 6, the correct order of magnitude for $r_k(K_4^3 - e)$ is single-exponential.

Theorem 7. *For any $k \geq 2$,*

$$r_k(K_3) \leq r_{4k}(K_4^3 - e) \quad \textit{and} \quad r_k(K_4^3 - e) \leq r_k(K_3) + 1.$$

Moreover $r_2(K_4^3 - e) = r_2(K_3) + 1 = 7$.

Denote by F_5 the hypergraph with edges $\{abc, abd, cde\}$. We show that $r_k(F_5)$ behaves similarly to $r_k(K_3)$.

Theorem 8. *There is a positive constant c such that, for $k \geq 4$,*

$$2^{ck} \leq r_k(F_5) \leq k!,$$

and $r_2(F_5) = 6, r_3(F_5) = 7$.

The simplest non-trivial triple systems have just two edges. The **kite** is a 3-uniform hypergraph with two edges sharing two vertices. The **bow** is a 3-uniform hypergraph with two edges sharing a single vertex.

Theorem 9. *Let $r_k = r_k(bow)$. Then*

$$r_k = (1 + o(1))\sqrt{6k}.$$

If $k = \frac{\binom{n}{3}}{n}$ and $n \equiv 4, 8 \pmod{12}$, then $r_k = n + 1$. Moreover, $r_2 = 5, r_3 = r_4 = r_5 = 6$, $r_6 = 7$, $r_7 = r_8 = r_9 = r_{10} = 9$, $9 \leq r_{11} \leq r_{12} \leq r_{13} \leq r_{14} \leq 10, r_{15} = 11$.

Remark. Note that $r_k(bow)$ is the smallest multicolor Ramsey number among nontrivial 3-uniform hypergraphs since $r_k(H) \geq \min\{r_k(bow), r_k(kite), r_k(M)\}$, where M is a matching with 2 triples. Indeed, each nontrivial 3 uniform hypergraph contains at least two edges that form one of *bow*, *kite* or M, and Theorem 2 gives $r_k(M) = k + 5$.

Theorem 10. *Let $r_k = r_k(kite)$. Then*

$$r_k = \begin{cases} k+1, & \text{if } k \equiv 3 \pmod 6 \\ k+1 \;\; or \;\; k+2, & \text{if } k \equiv 4 \pmod 6 \\ k+2, & \text{if } k \equiv 0, 2 \pmod 6 \\ k+3, & \text{if } k \equiv 1, 5 \pmod 6, k \neq 5 \\ 6 & \text{if } k = 5, \\ 5 & \text{if } k = 4 \end{cases}$$

Let a, b be positive integers. Denote by $F(a, b)$ the 3-uniform hypergraph with vertex set $V = A \cup B$, $A \cap B = \emptyset$, $|A| = a, |B| = b$ and edge set consisting of all triples with one vertex in A and two vertices in B (for example, $F(2, 2)$ is the kite).

Proposition 11. *For any $a \geq 2$, we have*

$$k(a-1) < r_k(F(a,2)) \leq k(a-1)+3.$$

In general, $r_k(F(a,b))$ grows slower than double exponential in k and possibly faster than exponential in k. (Recall that Theorems 5 and 6 give double-exponential bounds.)

Theorem 12. *Given $3 \leq a \leq b$, we have, for positive constants $c = c(a,b)$ and $c' = c(a,b)$*

$$2^{c'k} < r_k(F(a,b)) < r_t(K_b) + m < 2^{ck^{a+1}\log k},$$

where $m = (a-1)k+1$, and $t = k\binom{m}{a}$.

The **windmill** W with *center edge abc* is the hypergraph with six vertices and edge set $\{abc, abd, bce, acf\}$.

Theorem 13.

$$(1-o(1))3k \leq r_k(W) \leq 3k+3.$$

It is interesting to compare Theorem 13 with Proposition 3. In fact, the upper bounds in both cases come from the corresponding Turán-type results. Indeed, $\mathrm{ex}(n, C_3^3) = \binom{n-1}{2}$ (Frankl-Füredi [12] for large n, Csákány-Kahn [6] for $n \geq 6$) while $\mathrm{ex}(n,W) \leq \binom{n}{2}$ ([12]).

The ideas giving the asymptotic of $r_k(W)$ can be also used for the **tight path** $P_3^3 = \{abc, bcd, cde\}$.

Theorem 14. $2k(1-o(1)) \leq r_k(P_3^3) \leq 2k+3.$

The rest of the paper will be organized as follows. In Section 2, we give some auxiliary results and prove Proposition 4. Theorems 6 - 14 will be proved in Sections 3-6. Section 7 is devoted to exact values of Ramsey numbers for small number of colors and Section 8 contains remarks, conjectures and problems.

In some later sections we give lower bounds on Ramsey numbers based on block designs. A $t-(v,k,\lambda)$ **design** is a subset of $\binom{[v]}{k}$, called blocks, such that each t element subset of $[v]$ is contained in exactly λ blocks.

2 General bounds and auxiliary results

In this section we prove some general bounds on $r_k(H)$ and obtain some consequences including Proposition 4. Recall that the density of an r-uniform hypergraph F with n vertices and e edges is $d(F) = \frac{e}{\binom{n}{r}}$.

Lemma 15. *Let H be a fixed r-uniform hypergraph and F be an r-uniform hypergraphs with n vertices, density $d(F) = d$, and not containing copies of H as a subhypergraph. Then*

(i) $r_k(H) \leq 1 + \max\{n : \lceil \binom{n}{r} / \text{ex}(n, H) \rceil \leq k\}$,
(ii) If $\binom{n}{r}(1-d)^k < 1$ *then* $r_k(H) \geq n$.

Proof. (i) Consider a coloring of K_n^r with k colors and no monochromatic copy of H. Then each color class has at most $\text{ex}(n, H)$ edges.

(ii) Consider k copies of hypergraph F obtained by mapping its vertices randomly to a given set V of n vertices. Here, we choose vertex permutations uniformly. Assign the edges of the ith copy of F color i, $i = 1, \ldots, k$. If an edge belongs to several copies of F, assign the smallest available label. We claim that with positive probability, each edge of $K = \binom{V}{r}$ belongs to some copy of F. Indeed, the probability that a given edge of K uncovered is $(1-d)^k$. Thus, the probability that there is an uncovered edge of K is at most $\binom{n}{r}(1-d)^k < 1$. Therefore, with positive probability, all edges are covered and the resulting coloring of K contains no monochromatic copy of H. □

Proof of Proposition 4. (i) The proposition follows from Lemma 15(i) by using the fact that $\text{ex}(n, H) < n^{r-c}$ for some positive constant $c = c(H)$, when H is r-partite, see [8]. So, $k \geq \binom{n}{r}/\text{ex}(n, F) \geq Cn^r/n^{r-c} = Cn^c$, for a constant $C = C(r)$. Thus $n \leq C^{-1/c}k^{1/c}$.

(ii) Let H be non-r-partite. Apply Lemma 15(ii) with F being a complete r-uniform r-partite balanced hypergraph on $n = 2^{c'k}$ vertices (and $r|n$). Clearly H is not contained in F as a subgraph. Moreover, $d(F) \geq \frac{(n/r)^r}{\binom{n}{r}} > \frac{(n/r)^r}{(en/r)^r} = e^{-r}$. Hence for $k = c \log n$ and $c > e^r(r+1)$,

$$\binom{n}{r}(1-d)^k = \binom{n}{r}(1-d)^{c \log n} < n^r e^{-cd \log n} = e^{(r-cd)\log n} < 1. \qquad \square$$

The *trace* of a 3-uniform hypergraph H at vertex v is the graph on vertex set $V(H) - \{v\}$ and with edge set $\{e - \{v\} : \; e \in H, \; v \in e\}$. A transversal of a hypergraph is a set of vertices non-trivially intersecting each edge.

Lemma 16. *Let H be a 3-uniform hypergraph with a single-vertex transversal $\{v\}$. Let G be a trace of H with respect to v. Then $r_k(H) \leq r_k(G) + 1$.*

Proof. Given a k-coloring c of $\binom{[n]}{3}$ with no monochromatic H, let c' be the k-coloring of $\binom{[n-1]}{2}$ defined by $c'(ij) = c(ijn)$. Then c' has no monochromatic G and consequently $r_k(G) \geq r_k(H) - 1$ as required. □

3 K_s^r for $s > r \geq 2$

In this section we prove Theorem 6 using a variant of the stepping-up lemma of Erdős and Hajnal.

Proof of Theorem 6. It suffices to prove the result for $s = r+1$ since $r_k(K_s^r) \geq r_k(K_{r+1}^r)$ for any $s > r$. We use induction on r to show that $r_k(K_{r+1}^r) \geq t_r(k/2^{r-2} - 2r)$ for all $k \geq r2^r$. Since $k \geq r2^r$, we have $k/2^{r-2} - 2r \geq k/2^r$ and the result follows.

The base case $r = 2$ is given by $r_k(K_3) > 2^k > 2^{k-4} = t_2(k-4)$. Assume the result holds for some $r \geq 2$ and let $n = r_k(K_{r+1}^r) - 1$. By the inductive hypothesis

$$n \geq t_r(k/2^{r-2} - 2r) - 1.$$

Let $\phi : \binom{[n]}{r} \to [k]$ be a coloring with no monochromatic K_{r+1}^r. We will construct a coloring $\psi : \binom{[2^n]}{r+1} \to [2k + 2r - 4]$ with no monochromatic K_{r+2}^{r+1}. This shows that

$$r_{2k+2r-4}(K_{r+2}^{r+1}) \geq 1 + 2^n \geq 1 + \frac{1}{2} t_{r+1}(k/2^{r-2} - 2r).$$

Now suppose we are given $k' \geq (r+1)2^{r+1}$. If $k' - 2r + 4$ is odd, then let $k'' = k' - 1$ and if $k' - 2r + 4$ is even then let $k'' = k'$. Set $k = (k'' - 2r + 4)/2$ (which is an integer) and observe that $k \geq r2^r$ and $k'' = 2k + 2r - 4$. Then

$$k/2^{r-2} - 2r \geq k''/2^{r-1} - 2(r+1) + 1$$

and $r_{k'}(K_{r+2}^{r+1})$ is at least

$$r_{k''}(K_{r+2}^{r+1}) \geq 1 + \frac{1}{2} t_{r+1}(k/2^{r-2} - 2r) \geq 1 + \frac{1}{2} t_{r+1}(k''/2^{r-1} - 2(r+1) + 1) > t_{r+1}(k'/2^{r-1} - 2(r+1)).$$

Now we shall construct a coloring ψ of $\binom{[2^n]}{r+1}$ using the coloring ϕ of $\binom{[n]}{r}$ that has no monochromatic K_{r+1}^r. Represent the elements of $[2^n]$ with 0-1-sequences on n coordinates. For a vertex u and integer i, we denote $u(i)$ the ith coordinate of u in this representation. Given two vertices $u, v \in [2^n]$, say that $u < v$ if $u(i) < v(i)$ and $u(j) = v(j)$ for $j < i$. Denote such an i by $f(uv)$. Given any $u_1 < \cdots < u_{r+1}$, let $f_i := f(u_i u_{i+1})$, for every $1 \leq i \leq r$. Observe crucially that

(1) $f_i \neq f_{i+1}$, for every $1 \leq i \leq r-1$;

(2) $f(u_1 u_{r+1}) = \min_{1 \leq i \leq r} \{f_i\}$ and the minimum is reached by a unique i.

We define coloring ψ as follows:

$$\psi(u_1 \ldots u_{r+1}) = \begin{cases} (\phi(f_1, \ldots, f_r), 1) & \text{if } (f_1, \ldots, f_r) \text{ is an increasing sequence,} \\ (\phi(f_1, \ldots, f_r), 2) & \text{if } (f_1, \ldots, f_r) \text{ is a decreasing sequence,} \\ (i, 3) & \text{if } f_1 < f_2 < \cdots < f_i > f_{i+1}, 2 \leq i \leq r-1, \text{ for } r \geq 3, \\ (i, 4) & \text{if } f_1 > f_2 > \cdots > f_i < f_{i+1}, 2 \leq i \leq r-1, \text{ for } r \geq 3. \end{cases}$$

Suppose to the contrary that there is a monochromatic copy of K_{r+2}^{r+1} under ψ on vertex set $U = \{u_1, \ldots, u_{r+2}\}$ with $u_1 < \cdots < u_{r+2}$. Without loss of generality, we distinguish two cases.

Case 1: The second coordinate of ψ on each $(r+1)$-tuple is 1. First notice that the second coordinate of ψ on $u_1, ..., u_{r+1}$ and $u_2, ..., u_{r+2}$ being 1 implies $f_1 < f_2 < \cdots < f_r < f_{r+1}$ and together with (2), we have $f(u_1u_i) = f(u_1u_2) = f_1$ for all $3 \le i \le r+2$. Similarly from $u_2, ..., u_{r+2}$, we have that for every $2 \le p < q \le r+2$, $f(u_pu_q) = f_p$. Recall that the color of the $(r+1)$-set $\{u_1, ..., u_{r+2}\} - \{u_i\}$ under ψ is determined by the color of the r-set $\{f_1, ..., f_{r+1}\} - \{f_i\}$ under ϕ. Let $F := \{f_1, ..., f_{r+1}\}$ and $U = \{u_1, \dots, u_{r+2}\}$. Let us denote the above implication by

$$U \setminus \{u_i\} \Rightarrow F \setminus \{f_i\}.$$

Thus a monochromatic K^{r+1}_{r+2} on U under ψ yields a monochromatics K^r_{r+1} on F under ϕ, a contradiction.

Case 2: Each $(r+1)$-tuple gets color $(i, 3)$ for some i with $2 \le i \le r-1$. Then $\psi(u_1, ..., u_{r+1}) = (i, 3)$ implies $f_i > f_{i+1}$. On the other hand, $\psi(u_2, ..., u_{r+2}) = (i, 3)$ implies $f_i < f_{i+1}$, a contradiction.

If the second coordinate is 2 or 4 the arguments are almost identical to those in Case 1 or 2. □

4 $K^3_4 - e$ and F_5

Notice that in contrast to the double-exponential growth for K^3_4, $r_k(K^3_4 - e)$ is single-exponential in the number of colors k. Indeed, since $K^3_4 - e$ is not 3-partite, Proposition 4 yields $r_k(K^3_4 - e) > 2^{ck}$. For the upper bound, one can use a variation of the classical Erdős-Rado pigeonhole argument to obtain $r_k(K^3_4 - e) < 2^{(k+1)\log k}$. We will, however, use a different approach to prove this fact, which also shows some connection between the multicolor Ramsey number of $K^3_4 - e$ and the multicolor Ramsey number of a triangle.

Proof of Theorem 7. For the lower bound, let $n = r_k(K_3) - 1$ and $\phi : \binom{[n]}{2} \to k$ be a k-coloring of $\binom{[n]}{2}$ with no monochromatic triangles. We will construct a coloring ψ of $\binom{[n]}{3}$ with $4k$ colors with no monochromatic $K^3_4 - e$. This then would imply that $r_{4k}(K^3_4 - e) \ge n + 1 = r_k(K_3)$ as desired. Let ψ be the following coloring of the triples $i < j < k$. If P is a path with vertices i, j, k, denote by $\phi'(P)$ the color under ϕ of the edge in $\{i, j, k\}$ that is not in P. For such a path P, let the type of P, $t(P) = 1, 2$, or 3 if i, j or k is its center, respectively. If $\{i, j, k\}$ is a rainbow triangle, let $\psi(ijk) = (0, \phi(jk))$. If $\{i, j, k\}$ induces a monochromatic path P, let $\psi(ijk) = (t(P), \phi'(P))$.

Suppose there is a monochromatic copy $K = \{abc, abd, acd\}$ of $K^3_4 - e$, we will show a contradiction when the first coordinate is 0, namely all three triples $\{abc, abd, acd\}$ span rainbow triangles under ϕ. The cases when the first coordinate is $1, 2$ or 3, can be proved using a similar argument. Notice that when the first coordinate is 0, by the definition of ψ, the color of a triple depends on the color, under ϕ, of the edge spanned by the two largest elements in that triple. Since b, c, d play a symmetric role, we can assume that $b < c < d$.

If a is the smallest, then $\psi(abc) = \psi(abd) = \psi(acd)$ implies $\phi(bc) = \phi(bd) = \phi(cd)$, i.e. bcd is monochromatic under ϕ. Thus b is the smallest. But then $\psi(abc) = \psi(abd)$ implies $\phi(ac) = \phi(ad)$, which means acd is not a rainbow triangle under ϕ, a contradiction.

For the upper bound, simply notice that $K_4^3 - e = \{abc, abd, acd\}$ has a single vertex transversal $\{a\}$, and the trace of a is a triangle on $\{b, c, d\}$. Thus the upper bound follows from Lemma 16. The case with 2 colors is treated in Section 7. □

Proof of Theorem 8. The cases $k = 2, 3$ are treated in Section 7. The general lower bound follows from Proposition 4(ii), since F_5 is not 3-partite.

The upper bound follows by induction with basis $k = 4$. Suppose that the edges of K_{24}^3 with vertex set V can be 4-colored so that there is no monochromatic F_5. There are 22 triples uvx containing a fixed pair uv. Assume that uvx_1, uvx_2 are red triangles. Then x_1x_2y cannot be red for $y \in Y = V - \{u, v, x_1, x_2\}$. Thus we have a set S, $S \subseteq Y$, $|S| \geq \lceil (|V| - 4)/3 \rceil = 7$ and x_1x_2y are blue triples for all $y \in S$. Therefore, no triple in S is colored blue, and thus $\binom{S}{3}$ uses $k - 1 = 3$ colors. Applying Theorem 25 to the 3-colored subhypergraph spanned by S, we get a contradiction.

The inductive step is simply repeating the argument above in general. Suppose we already know $r_k(F_5) \leq k!$ for some $k \geq 4$ and we have a K_n^3 with a $(k+1)$-coloring such that there is no monochromatic F_5. Selecting u, v, x_1, x_2 as above and applying the same argument, we get $n - 4 \leq k(k! - 1) < (k+1)! - k$, thus $n \leq (k+1)! - k + 4 \leq (k+1)!$. This implies $r_{k+1}(F_5) \leq (k+1)!$. □

Remark. The above results slightly suggests that $r_k(F_5) \leq r_k(K_3)$ might hold. Although the bound $r_k(F_5) \leq k!$ in Theorem 8 can be improved slightly, this improvement still does not show that $r_k(F_5) \leq r_k(K_3)$.

5 Bow, Kite, $F(a, b)$

The next lemma (without the statements on the extremal configurations) is referred in [27] as an unpublished remark of Erdős and Sós.

Lemma 17.

$$\mathrm{ex}(n, bow) = \begin{cases} n & \text{if } n \equiv 0 \pmod 4 \\ n - 1 & \text{if } n \equiv 1 \pmod 4 \\ n - 2 & \text{if } n \equiv 2, 3 \pmod 4. \end{cases}$$

When $n \equiv 0, 1 \pmod 4$, the extremal configurations are unique, all components are K_4^3-s, (apart from a possible one vertex component). When $n \equiv 2 \pmod 4$, the extremal configuration is either $\frac{n-2}{4}$ copies K_4^3-s and two isolated vertices or any number of K_4^3-s and one star component. Similarly, when $n \equiv 3 \pmod 4$, the extremal configuration is either $\frac{n-3}{4}$ copies K_4^3-s and component with a single edge or any number of K_4^3-s and one star component.

Proof. Suppose C is the vertex set of a nontrivial connected component of a 3-uniform hypergraph without a *bow*. Then either C spans only one edge or there are two edges e_1, e_2 in C, intersecting in two vertices, u, v. Suppose that $|C| > 4$. Then every edge f that is not covered by $e_1 \cup e_2$ and intersecting $e_1 \cup e_2$ must contain u, v and a vertex w not covered by $e_1 \cup e_2$. It is easy to see that these vertices w cover C and C has no other edges, thus C has $|C| - 2$ edges, all containing u, v. Such a component is called a star component.

On the other hand, if $|C| = 4$ then we have two, three or four edges in C. From this analysis the lemma follows. $\square$

Lower bounds of $r_k(bow)$ follow from the existence of resolvable designs. A $3-(n, 4, 1)$ design is a set of 4-element subsets (blocks) of an n-element set V such that each 3-element subset of V is in precisely one block. Hanani [15] showed that $3-(n, 4, 1)$ designs exist if and only if $n \equiv 2, 4 \pmod 6$. A $3-(n, 4, 1)$ design is called *resolvable* if its blocks can be grouped so that each group (parallel class) gives a partition of V. Resolvable $3-(n, 4, 1)$ designs exist if and only if $n \equiv 4, 8 \pmod{12}$, see [18, 19], and [21].

Proof of Theorem 9. When $n \equiv 4, 8 \pmod{12}$, $k = \frac{\binom{n}{3}}{n}$, $\text{ex}(n, bow) = n$, thus Lemma 15 (i) gives $r_k \leq n + 1$. This is sharp, since K_n^3 can be partitioned into k matchings. The statement $r_k(bow) \approx \sqrt{6k}$ follows from considering the construction for largest n, $n \equiv 4, 8 \pmod{12}$, $k \geq \frac{\binom{n}{3}}{n}$ for the lower bound and applying the Lemma 15(i) for the upper bound. The statements about the small values are proved in Section 7. $\square$

Proof of Theorem 10. Let $H = F(2, 2)$ be the kite. Then $\text{ex}(n, H)$ corresponds to the maximum number of triples on n elements such that any two triples intersect in at most one element, i.e. the maximum number of edges in a linear 3-uniform hypergraph. A well-known result of Schönheim [36] and others (the cases $n \equiv 0, 1, 2, 3 \pmod 6$ go back even to Kirkman [22]) is $\text{ex}(n, H) = \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - \epsilon$, where $\epsilon = 1$ for $n \equiv 5 \pmod 6$, otherwise $\epsilon = 0$. Lemma 15(i) gives, after some calculations, the upper bounds.

The lower bound for the cases $k \equiv 3, 4 \pmod 6$ is easy. Given $K_n^3 = (V, E)$, consider $V = Z_n$ and color triple ijk with color $i + j + k \pmod n$. Clearly this coloring yields no monochromatic H, hence $r_k(H) > k$.

For the cases $k \equiv 0, 1, 2, 5 \pmod 6$ the (difficult) constructions of J. X. Lu [29, 30] finished by Teirlinck [38] are needed: for $n > 7, n \equiv 1, 3 \pmod 6$, K_n^3 can be partitioned into $n - 2$ Steiner triple systems (called a *large set* of STS).

Indeed, for $k \equiv 0, 2 \pmod 6$ we need a kite-free k-coloring of K_{k+1}^3 i.e. $(n-1)$-coloring of K_n^3 when $n \equiv 1, 3 \pmod 6$. This can be done even with $n - 2$ colors according to the cited result of Lu. However, the case $k = 6$ is exceptional because Lu's theorem does not hold for $n = 7$. Nevertheless, there is a 6-coloring of K_7^3 without a monochromatic kite as shown in Proposition 22. Similarly, for $k \equiv 1, 5 \pmod 6$ we need a kite-free k-coloring of K_{k+2}^3 i.e. $(n-2)$-coloring of K_n^3 when $n \equiv 3, 1 \pmod 6$. This is provided by Lu's theorem, apart from the case $k \equiv 5$ $(n = 7)$ which is indeed exceptional, in Proposition 22 we prove that

$r_5(kite) = 6$ (together with the case $k = 4$). □

Proof of Proposition 11. In an $F(a,2)$-free coloring of K_n^3 any pair of vertices is in at most $a-1$ edges of the same color. Thus $n \le 2+k(a-1)$, proving the upper bound. (One can also use Lemma 16 and the multicolor Ramsey number for stars (see [1]): $r_k(K_{1,a}) \le k(a-1)+2$.)

For the lower bound, set $n = k(a-1)$ and consider $K_n^3 = (V,E)$ with $V = Z_n$. Color a each edge with the sum of its vertices mod k. Then a monochromatic copy of $F(a,2)$ would require that for some $y,z \in V$, $y+z+x_1, \ldots, y+z+x_a$ are all equal (mod k) i.e. we have a different positive x_s, all equal (mod k), which is impossible. Hence $r_k(F(a,2)) > k(a-1)$. □

Proof of Theorem 12. For the upper bound, let $N = r_t(K_b) + m$. Consider a k-coloring ϕ of the triples of K_N. Fix a set S of m vertices and define a t-coloring c on the pairs of the remaining $N-m$ vertices as follows. Let $c(xy) = (\phi(xys_i), s_1, s_2, \ldots, s_a)$, where $\phi(xys_i)$ is the majority color on triples containing x and y, and $s_1, s_2, \ldots, s_a \in S$ is the lexicographically first a-tuple in S such that $\phi(xys_i) = \phi(xys_j)$ for every $1 \le j \le a$ (by the choice of m there is such an a-tuple). Since c is a t-coloring of a complete graph on $N - m = r_t(K_b)$ vertices, there is monochromatic K_b in c, which gives a monochromatic $F(a,b)$ in ϕ.

A lower bound for $r_k(F(a,b))$ is obtained from Proposition 4 (i) since $F(a,b)$ is not 3-partite, for $b \ge 3$. □

6 Windmill and tight path

The following result (conjectured by Kalai) is a special case of a theorem of Füredi and Frankl ([12], Theorem 3.8). We give their proof also, since it is extremely short in this special case.

Theorem 18. $\mathrm{ex}(n,W) \le \binom{n}{2}$ *with equality for every* $n \equiv 1,5$ *(mod 20).*

Proof. The lower bound comes from the following construction. Let $n \equiv 1,5 \pmod{20}$ and consider a Steiner system S, a $2-(n,5,1)$ design, i.e., a set of 5-element blocks on n elements such that every pair lies in precisely one block. Its existence is proved by Hanani [16, 17]. Then the number of blocks is $\binom{n}{2}/10$. Now place 10 triples inside each block of S. The resulting triple system, H, has $\binom{n}{2}$ triples and is W-free. Indeed, a copy of W would have to be contained in one of the blocks, but each block has less vertices than the number of vertices in W.

To prove the upper bound, suppose that H is a 3-uniform hypergraph with no W. For $x,y \in V(H)$, the *codegree* $d(x,y)$ is the number of edges of H containing both x,y. Let a,b,c be codegrees of three pairs of vertices from a edge of H, $1 \le a \le b \le c$. If $a = 2$, $b \ge 3$ and $c \ge 4$, then H contains a copy of W. Thus either $a = 1$ or $a = b = 2$ or $a = 2, b = 3, c = 3$.

In each of these cases we have that $1/a+1/b+1/c \geq 1$. For each edge $e=uvw$ of H, let

$$w(e)=\frac{1}{d(u,v)}+\frac{1}{d(v,w)}+\frac{1}{d(u,w)}.$$

We see that $w(e) \geq 1$. Let $s=\sum_{e\in H} w(e)$. Notice that $s \leq \binom{n}{2}$, since a term $\frac{1}{d(u,v)}$ appears exactly $d(u,v)$ times for each pair uv that belongs to at least one edge of H. Now, $|H| \leq |H| \min_{e\in H} w(e) \leq s \leq \binom{n}{2}$. □

For the next proof we need the following decomposition result:

Theorem 19 (Pippenger and Spencer [33])**.** *Let r be fixed and D be sufficiently large. Let H be an r-uniform hypergraph with $d(v)=(1+o(1))D$ for every $v \in V(H)$ and codegree of $o(D)$ for every pair $\{u,v\} \subseteq V(H)$. Then $E(H)$ can be partitioned into $(1+o(1))D$ matchings.*

Proof of Theorem 13. To prove the lower bound, let S be a $3-(n,5,1)$ design, i.e. a set of 5-element blocks of an n-element set such that each 3-element set is in precisely one block. The existence of such designs are known for infinitely many n, for example for $n=4^s+1, s\geq 2$ [20], see also [32]. Construct an auxiliary 10-uniform hypergraph H where $V(H)$ is the set of $\binom{n}{2}$ pairs in $V(S)$, and ten of these pairs form an edge of H if and only if they are the ten pairs in a block of S. Since every pair in $V(S)$ is in exactly $(n-2)/3$ blocks of S, H is an $(n-2)/3$-regular hypergraph. On the other hand, the codegree of any two vertices in H is at most one. Indeed, any two vertices in H (two pairs in $V(S)$) contain at least three vertices in $V(S)$, and they can be in at most one block of S. With large enough n, and with $r=10, D=n/3$, the conditions of Theorem 19 hold so we can decompose $E(H)$ into $m=(1+o(1))n/3$ matchings $M_i, i=1,2,\dots,m$. Each M_i corresponds to a subset of blocks S_i of S and any two blocks in S_i share at most one element in $V(S)$. The set of triples covered by the blocks of any S_i form a W-free triple system (the center edge of a windmill W in a block $B\in S_i$ would force the other three edges of W to B, similarly as in Theorem 18). Thus K_n^3 is decomposed into $m=(1+o(1))n/3$ W-free triple systems, showing $r_k(W) \geq (1-o(1))3k$.

The upper bound follows from Theorem 18: in a k-coloring of K_n^3 with no monochromatic W, each color class has at most $\mathrm{ex}(n,W)=\binom{n}{2}$ edges. Thus $\binom{n}{3}/k \leq \binom{n}{2}$, implying $n \leq 3k+2$. So by Lemma 15(i), $r_k(W) \leq 3k+3$. □

We need the following result for tight path.

Proposition 20. *$ex(n,P_3^3) \leq \frac{n(n-1)}{3}$ with equality for $n \equiv 1,4$ (mod 12).*

Proof of Proposition. For a P_3^3-free hypergraph T on n vertices and a vertex v, the degree $d(v) \leq ex(n-1,P_4) \leq n-1$. Thus $3|E(T)| = \sum_v d(v) \leq n(n-1)$. The statement for equality comes from a $2-(n,4,1)$ design by replacing all blocks by K_4^3-s. □

Proof of Theorem 14. Observe that the trace of P_3^3 at its transversal vertex is P_4, the path on four vertices. The upper bound can be obtained in two ways.

Applying and Lemma 15 (i) with proposition 20, we have $r_k(P_3^3) \leq 2k+3$. On the other hand, we may apply Lemma 16 as well: $r_k(P_3^3) \leq r_k(P_4)+1 \leq 2k+3$ ([34]).

For the lower bound we start with a $3-(n,4,1)$ design F (already used in the proof of Theorem 9) and follow the construction in the proof of Theorem 13. Consider the 6-uniform hypergraph H with vertex set being the set of pairs of vertices of F and edges formed by the sets of pairs within the blocks of F. The degree of any vertex in H is $d=(n-2)/2$, the codegree of any pair of vertices is at most one, so the conditions for Pippenger-Spencer Theorem are satisfied, giving a decomposition of H into $(1+o(1))d=(1+o(1))n/2$ matchings, M_i. Each M_i corresponds to a set F_i of blocks of F, intersecting each other in at most one element. Let T_i be the set of triples covered by the blocks of F_i. The T_i-s provide the required P_3^3-free coloring of K_n^3 with $(1+o(1))n/2$ colors. □

7 Small Ramsey numbers

The only known non-trivial classical Ramsey number for triples is $r_2(K_4^3)=13$, due to McKay and Radziszkowski [31].

It is proven in ([34] that $13 \leq r_3(K_4^{(3)}-e) \leq 16$ and stated as an easy fact without a proof that $r_2(K_4^3-e)=7$. Here we prove this for completeness.

Proposition 21. $r_2(K_4^3-e)=7$.

Proof. Consider the following coloring C of K_6^3. Fix the set of five vertices, V, and let c be the 2-coloring of K_5 on vertex set V with two monochromatic C_5's. Let v be the remaining vertex of K_6^3. For any triple containing v, let $C(\{v,u,w\})=c(uw)$.
For each triple xyz, not containing v, let $C(\{x,y,z\})$ be the color different from $c(V-\{x,y,z\})$. Under coloring C, there are two triples of each color in every 4-set, hence there is no monochromatic K_4^3-e. □

The following proposition determines the small undecided cases from Theorem 10. A hypergraph is linear if every two edges share at most one vertex.

Proposition 22. $r_4(kite)=5, r_5(kite)=6, r_6(kite)=8$.

Proof. It is obvious that $r_4(kite)>4$. The fact that $r_4(kite)\leq 5$ follows by observing that any 4-coloring of the edges of K_5^3 contains three edges of the same color.

Coloring the triple ijk, $1\leq i<j<k\leq 5$ by color $i+j+k$ (mod 5) gives $r_5(kite)>5$. To show that $r_5(kite)\leq 6$, we need the result of Cayley [2], stating that the maximum number of pairwise disjoint Fano planes in K_7^3 is 2. Suppose K_6^3 on vertex set V is 5-colored so that each color class i is a linear hypergraph P_i. Since the average number of edges in a color class is four and no linear hypergraphs on 6 vertices can have more than four edges, it follows that each P_i must be a Pasch configuration. Therefore the pairs uncovered by the triples of P_i form a matching M_i in the complete graph on V. The M_i-s must form a factorization on V otherwise some pair in V would be covered by at most three P_i-s instead of the required four. These P_i-s can be extended by a new vertex to a decomposition of K_7^3 into five Fano planes, contradicting Cayley's theorem stated above.

The upper bound $r_6(kite) \leq 8$ is already proved (see the proof of Theorem 10). For the lower bound we need a partition of K_7^3 into six linear hypergraphs, see Figure 1. Set $V = [7]$ and let F_1, F_2 be the two Fano planes generated by shifts of $124, 134 \pmod 7$. The next two sets are isomorphic to a Fano plane from which one line is deleted:

$$F_3 = \{135, 167, 236, 257, 347, 456\}, F_4 = \{123, 146, 247, 256, 345, 367\}$$

and $F_6 = \{127, 136, 145, 246, 567\}$ (Fano plane from which two lines are deleted), $F_7 = \{125, 147, 234, 357\}$ (a Pasch configuration).

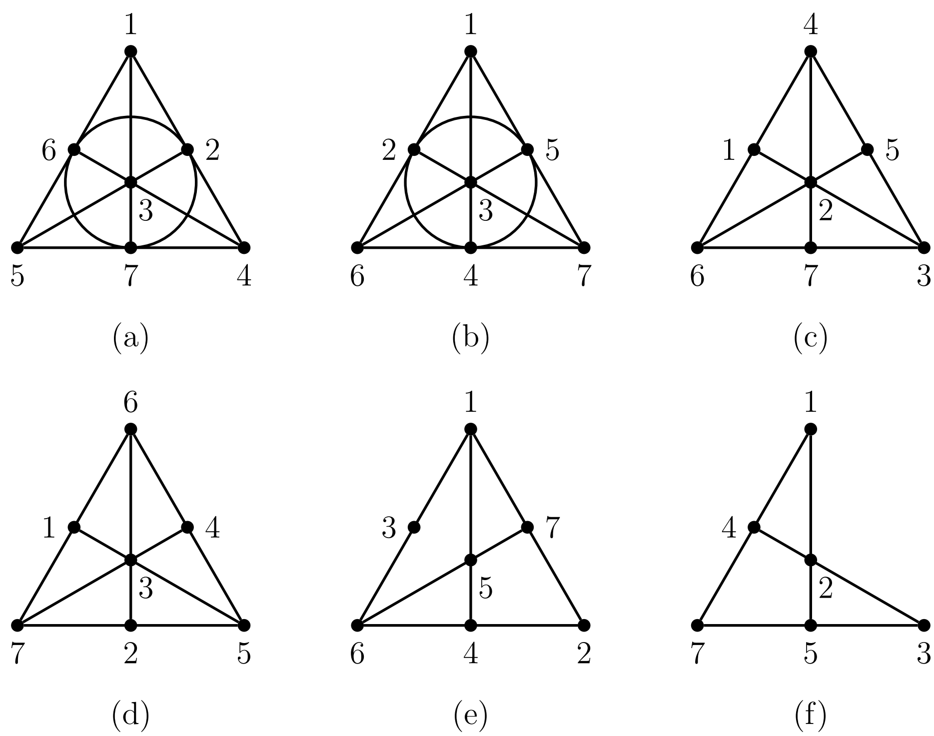

Figure 1: Partition of K_7^3 into two Fano, two Fano$-e$, Fano$-2e$, Pasch

$\square$

Proposition 23. *Set* $r_k = r_k(bow)$*, then* $r_1 = r_2 = 5, r_3 = r_4 = r_5 = 6$*,* $r_6 = 7, r_7 = r_8 = r_9 = r_{10} = 9$*,* $9 \le r_{11} \le r_{12} \le r_{13} \le r_{14} \le 10, r_{15} = 11$.

Proof. All but one upper bounds are obtained from Lemma 15(i). The exceptional case is when Lemma 15(i) gives $r_5(bow) \le 7$. Here we improve it as follows. Suppose K_6^3 is 5-colored without monochromatic bow. From Lemma 17 each color class is either a K_4^3 (type A) or four triples pairwise intersecting in the same base pair (type B). There are at most three type A colors. The base pairs for different type B colors must be vertex disjoint. Thus there are at least two type A color classes, w.l.o.g. $abcd, cdef$. But then only the base pairs ae, af, be, bf are available for type B colors. Therefore we have two type B and three type A colors, the third is the K_4^3 spanned by $abef$. Now there is no base pair available for type B color classes since every pair of vertices is covered by a type A K_4^3.

Lower bounds should be exhibited for $r_1, r_3, r_6, r_7, r_{15}$ only. Coloring all triples of K_4^3 with the same color, $r_1 > 4$ follows. Coloring the triples of $\{1,2,3,4\}$ with color 1, the triples $125, 135, 235$ with color 2, the triples $145, 245, 345$ with color 3, $r_3 > 5$ follows. Then $r_6 > 6$ comes from the following 6-coloring with color classes $\binom{\{1,2,3,4\}}{3}$, $\binom{\{3,4,5,6\}}{3}$, $\binom{\{1,4,5,6\}}{3} - \{4,5,6\}$, $\binom{\{2,4,5,6\}}{3} - \{4,5,6\}$, $\binom{\{1,2,3,5\}}{3} - \{1,2,3\}$, $\binom{\{1,2,3,6\}}{3} - \{1,2,3\}$. The 7-coloring of K_8^3 is the 7 parallel classes of the unique $3-(8,4,1)$ design. Finally, the 15-coloring of K_{10}^3 comes from the unique $3-(10,4,1)$ design whose 30 blocks can be partitioned into 15 disjoint pairs. □

Proposition 24. $r_2(F_5) = 6$.

Proof. The lower bound is obvious, color triples of K_5^3 containing a fixed vertex with color 1 and other triples by color 2. For the upper bound, consider a 2-colored K_6^3 on vertex set $\{1,2,3,4,5,6\}$ and its 2-colored trace $K = K_5^2$ with respect to vertex 6. There is a monochromatic, say red odd cycle C in $V(K) - \{6\}$. If $C = 1,2,3,1$ then either there is a red triple in K with two vertices on C and one vertex not in C or all such triples are blue. The former gives a red, the latter a blue F_5. If $C = 1,2,3,4,5,1$ then either there is a red triple with vertices non-consecutive on C or all the five such triples are blue. Again, the former gives a red, the latter a blue F_5. □

Theorem 25. $r_3(F_5) = 7$.

Proof. For the lower bound, color the triples of K_6^3 containing v with color 1, color uncolored triples containing vertex $w \ne v$ with color 2 and color all other edges with color 3.

To prove the upper bound, call a graph G *nice* if for every triple $T = \{v_1, v_2, v_3\}$ of vertices at least one of the following holds:
1. There are two vertex disjoint edges of G, such that one of them is in T and the other meets T. 2. There is a path of length two in G connecting two vertices of T with midpoint not in T.

Observation 26. *If H is an F_5-free 3-uniform hypergraph, such that the trace of v for a vertex v is a nice graph, then all edges of H within $V(G) \cup \{v\}$ contain v.*

Indeed, otherwise from the definition of a nice graph we find F_5 in H. Thus finding a large nice subgraph in a trace one can reduce the number of colors. More generally, a graph is i-nice if the property holds for all but at most i triples of vertices.

We need a lemma on 6-vertex graphs. Since its proof is routine but lengthy, we state it without proof.

Lemma 27. *Suppose G has six vertices. If $|E(G)| \geq 9$ then G is nice. If $|E(G)| = 8$ then G is 1-nice, if $|E(G)| = 7$ then G is 2-nice. If $|E(G)| = 6$ then G is 5-nice, except in one case, when G is $K_{2,3}$ plus an isolated vertex (in this case it is 6-nice).*

With these preparations we are ready to prove the upper bound. The majority color, say red in a 3-colored K_7^3, has at least 12 edges. Some vertex v has red degree at least 6. Let G be the trace of a red hypergraph at v. We get a contradiction from Lemma 27 (and from the fact that we have 12 edges) except when G has exactly six edges and the trace is $K_{2,3} + w$. This case implies that the red color class has 12 edges forming $K_{2,2,3}$, a complete 3-partite hypergraph with parts of sizes 2, 2, and 3. However, among the $35 - 12 = 23$ edges of other colors, one color, say blue, has at least 12 edges. Repeating the argument for the blue hypergraph, we conclude that the blue hypergraph is also a $K_{2,2,3}$. However, as one can easily check, there is no way to place two edge disjoint $K_{2,2,3}$-s on 7 vertices. □

8 Concluding remarks

We determined, for 3-uniform hypergraphs, r_k ranges from $\sqrt{k}$ to double exponential in k, and showed a jump in r_k when H changes from r-partite to non-r-partite. This leads to the following question.

Problem 28. *For which 3-uniform hypergraphs F, is $r_k(F)$ double exponential? Are there other jumps that the Ramsey function r_k exhibits?*

The ramsey-numbers $r_k(bow), r_k(kite)$ are closely connected to block designs. In case of the kite the only uncertainty is whether $r_k(kite)$ is $k+1$ or $k+2$ when $k \equiv 4 \pmod 6$. This leads to the following problem.

Problem 29. *Suppose $n \equiv 5 \pmod 6$. Is it possible to partition the triples of an n-element set into $n-1$ partial triple systems, i.e. into parts so that distinct triples in each part intersect in at most one vertex? By Theorem 10, this is not possible for $n = 5$ but perhaps for large enough n (possibly for $n \geq 11$) such partitions exist.*

In case of the bow, the problems related to sharper bounds of $r_k(bow)$ are not purely design theoretic, since color classes can be star components as well. We state just one of those problems.

Problem 30. *Suppose* $n \equiv 6, 10$ *(mod 12). Is it possible to partition the triples of an* n*-element set into* $\frac{n(n-1)}{2}$ *classes so that each class is the union of some disjoint* K_4^3*-s and at most one star component? (Any color class has* $n-2$ *triples.) For* $n = 6$ *there is no solution.*

Concerning $r_k(K_3 - e)$ the most challenging (perhaps difficult) problem is to decrease the upper bound of Theorem 7 by one.

Problem 31. $r_k(K_4^3 - e) < r_k(K_3) + 1$ *for every* $k \geq 3$*?*

A challenging open problem is to improve the estimates of $r_k(P)$ (and/or $ex(n, P)$) where P is the *Pasch configuration* with edges $\{abc, bde, cef, adf\}$. (It can be obtained from the Fano plane by deleting a vertex.) Presently only the following is known.

Proposition 32. *For positive constants* c, c', $\quad c\left(\frac{k}{\log k}\right)^2 < r_k(P) < c'k^4$.

Proof. The lower bound is based on the following P-free hypergraph, showing that $\text{ex}(n, P) = \Omega(n^{5/2})$, [26]. Take an incidence graph G of a projective plane with n points and n lines. It has $\Omega(n^{3/2})$ edges. Add n new vertices $x_1, ..., x_n$ and add all triples of the form $x_i \cup e$, where e is an edge of G. The resulting 3-uniform hypergraph, call it H, has $3n$ vertices and $\Omega(n^{5/2})$ edges.

Notice that the edge-density of H is $d(H) = cn^{-1/2}$ for some constant $c > 0$. From Lemma 15(ii) we see that there is a coloring of K_n^3 with $(c'n^{1/2} \log n)$ colors and no monochromatic P. Thus $r_k(P) > n$ with $k = c'n^{1/2} \log n$. Expressing n in terms of k gives the desired lower bound.

The upper bound follows from Lemma 15(i) and the fact that $\text{ex}(n, P) = O(n^{11/4})$ [26]. This is based on the claim that $\text{ex}(n, K(2, 2, 2)) = O(n^{11/4})$ proved by Erdős [8], where $K(2, 2, 2)$ is the complete 3-partite 3-uniform hypergraph with two vertices in each part. $\square$

9 Acknowledgments

Thanks to Zoli Füredi and Roman Glebov for conversations on the subject of this paper, and Stefan Walzer for improving the lower bound in Theorem 6. The authors would also like to thank the organizers of the 2012 Midsummer Combinatorial Workshop that took place at Charles University.

References

[1] S. A. Burr and J. A. Roberts, On Ramsey numbers for stars. *Utilitas Math.*, 4 (1973) 217–220.

[2] A. Cayley. On the triadic arrangements of seven and fifteen things. *London, Edinburgh and Dublin Philos. Mag. and Sci.* 37, (1850), 50–53.

[3] D. Conlon, J. Fox and B. Sudakov. An improved bound for the stepping-up lemma. *Discrete Applied Mathematics*, to appear.

[4] D. Conlon, J. Fox and B. Sudakov. Hypergraph Ramsey numbers. *J. Amer. Math. Soc.* 23(1), (2010), 247–266.

[5] F. R. K. Chung. On triangular and cyclic Ramsey numbers with k colors. *Graphs and combinatorics* (Proc. Capital Conf., George Washington Univ., Washington, D.C.), 1973, 236–242.

[6] R. Csákány an J. Kahn. A homological approach to two problems on finite sets. *J. of Algebraic Combinatorics* 9, (1999), 141–149.

[7] D. Duffus, H. Lefmann and V. Rödl. Shift graphs and lower bounds on Ramsey numbers $r_k(l;r)$. *Discrete Mathematics* 137, (1995), 177–187.

[8] P. Erdős. On extremal problems of graphs and generalized graphs. *Israel J. Math.* 2 (1964), 183-190.

[9] P. Erdős, A. Hajnal and R. Rado. Partition relations for cardinal numbers. *Acta Math. Acad. Sci. Hung.*, 16 (1965), 93–196.

[10] P. Erdős and R. Rado. Combinatorial theorems on classifications of subsets of a given set. *Proc. London Math. Soc.*, (3) 2 (1952), 417–439.

[11] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Math.*, 2 (1935), 464–470.

[12] Z. Füredi and P. Frankl. Exact solution of some Turán-type problems. *J. of Combinatorial theory A.*, 45 (1987), 226–262.

[13] G. L. Graham, B. L. Rothschild and J. H. Spencer. **Ramsey theory**, 2nd edition. John Wiley & Sons (1990).

[14] A. Gyárfás and G. Raeisi. The Ramsey number of loose triangles and quadrangles in hypergraphs. *Electronic J. of Combinatorics* 19(2) (2012) P30.

[15] H. Hanani. On quadruple systems. *Canad. J. of Mathematics* 12 (1960), 145–157.

[16] H. Hanani. The existence and construction of balanced incomplete block designs. *Annals of Math. statistics* 32 (1961), 361–386.

[17] H. Hanani. A balanced incomplete block design. *Annals of Math. statistics* 36 (1965), 711.

[18] A. Hartman. Parallelism of Steiner Quadruple systems. *Ars Combinatoria* 6 (1978), 27–37.

[19] A. Hartman. Resolvable Steiner Quadruple systems. *Ars Combinatoria* 9 (1980), 263–273.

[20] D. R. Hughes. On t-designs and groups. *American J. of Mathematics* 87 (1965), 761–778.

[21] L. Ji and L. Zhu. Resolvable Steiner quadruple systems for the last 23 orders. *SIAM J. of Discrete Mathematics*, 19 (2005), 420–430.

[22] T. P. Kirkman. On a problem of combinatorics. *Cambridge and Dublin Math. J.*, 2 (1847), 191–204.

[23] T. Kővári and V. T. Sós and P. Turán. On a problem of K. Zarankiewicz. *Colloquium Math.* 3 (1954), 50–57.

[24] G. B. Khosrovshahi and B. Tayfeh-Rezaie. Large sets of t-designs through partitionable sets: a survey. *Discrete Mathematics*, 306 (2006), no. 23, 2993–3004.

[25] F. Lazebnik and D. Mubayi. New lower bounds for Ramsey numbers of graphs and hypergraphs. *Adv. in Appl. Math.* 28(3–4), (2002), 544–559.

[26] H. Leffmann, K. T. Phelps and V. Rödl. Extremal problems for triple systems. *Journal of Combinatorial Designs* 1 (1993), 379–394.

[27] D. G. Larman and C. A. Rogers. The realization of distances within sets in Euclidean space. *Mathematika* 19 (1972), 1–24.

[28] L. Lovász. Kneser's Conjecture, Chromatic Numbers and Homotopy. *J. of Combinatorial Theory A.* 25 319-324, 1978.

[29] J. X. Lu. On large sets of disjoint Steiner triple systems, I,II,III. *J. of Combinatorial Theory A.* 34 (1983), 140–182.

[30] J. X. Lu. On large sets of disjoint Steiner triple systems, IV,V,VI. *J. of Combinatorial Theory A.* 37 (1984), 136–192.

[31] B. D. McKay and S. P. Radziszkowski. The first classical Ramsey number for hypergraphs is computed. *Proceedings of the Second Annual ACM-SIAM Symposium on Discrete Algorithms, SODA'91* 304–308.

[32] H. Mohacsy and D. K. Ray-Chaudhuri. A construction of infinite families of Steiner 3-designs. *J. of Combinatorial Theory A.* 94 (2001), 127–141.

[33] N. Pippenger and J. H. Spencer. Asymptotic behavior of the chromatic index for hypergraphs. *J. Comb. Theory, Ser. A.* 51(1), (1989), 24–42.

[34] S. P. Radziszowski. Small Ramsey Numbers. *Dynamic Surveys, Electronic J. of Combinatorics.*

[35] D. K. Ray-Chaudhuri and R. M. Wilson. The existence of resolvable block designs. *Survey of combinatorial theory* (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971), (North-Holland, Amsterdam, 1973), 361–375.

[36] J. Schönheim. On maximal systems of k-tuples. *Studia Sci. Math. Hung.* 1, (1966), 363–368.

[37] A. F. Sidorenko, Turán l-graphs and Ramsey numbers. (Russian) *Dokl. Akad. Nauk SSSR*, 251 (1980), no. 4, 805–808.

[38] L. Teirlinck. Completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems. *J. of Combinatorial Theory A.* 57 (1991), 302–305.