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Upper Bound on the Order of τ-Critical Hypergraphs

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The right order of magnitude for the maximal number of vertices in an r-uniform τ-critical hypergraph H is achieved by obtaining an upper bound of $O(\tau(H)^{r-1})$.

An r-uniform hypergraph H is a set $V(H)$ with a collection, $E(H)$, of r-element subsets of $V(H)$. Sets $V(H)$ and $E(H)$ are called the vertices and edges of H; $|V(H)|$ is called the order of H.

Throughout this paper we restrict ourselves to hypergraphs where $V(H)$ and $E(H)$ are finite, a subset of $V(H)$ appears in $E(H)$ at most once (H has no multiple edges) and every vertex of H is contained in some edge of H (H has no isolated vertices). The hypergraph induced by a subset of $E(H)$ is called a partial hypergraph of H.

A set $T \subseteq V(H)$ is a transversal of the hypergraph H if $T \cap e \neq \varnothing$ for every $e \in E(H)$. The transversal-number $\tau(H)$ of the hypergraph H is defined as $\min\{|T|: T \text{ is a transversal of } H\}$.

Our paper is a contribution to the theory of τ-critical hypergraphs started with the paper [3] of P. Erdös and T. Gallai in 1961. A hypergraph H is τ-critical if the removal of any edge reduces the transversal-number of H, i.e., for every $e \in E(H)$, $\tau(H-e) = \tau(H) - 1$ ($H - e$ is the partial hypergraph of H induced by $E(H) - e$).

We define $v_{\max}(r, t)$ as $\max|V(H)|$, where H runs over the r-uniform τ-critical hypergraphs of transversal number t. The problem of determining $v_{\max}(r, t)$ appeared in [3] and the case $r = 2$ was solved there. For $r = 3$ a result of Szemerédi and Petruska [8] implies the right order of magnitude of $v_{\max}(3, t)$. As far as we know, no results have been published on $r \geqslant 4$ and $t \geqslant 3$.

The main result of this paper is Theorem 2:

$$v_{\max}(r, t) \leqslant \binom{t+r-2}{r-2} t + t^{r-1} = \left(1 + \frac{1}{(r-2)!}\right) t^{r-1} + O(t^{r-2}).$$

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On the other hand, an easy construction (Remark 1) yields $v_{\max}(r, t) \geqslant (t^{r-1}/(r-1)!) + O(t^{r-2})$, therefore Theorem 2 gives the right order of magnitude of $v_{\max}(r, t)$ for fixed r. Its consequence for $r=2$, Corollary 2, is a theorem of Erdös and Gallai [3, p. 196], while for $r=3$ it improves the upper bound of $8t^2+2t$ given in [8]. Theorem 1 is a generalization of a result on τ-critical graphs proved independently by Surányi [6] and Lovász [5, Ex. 22, p. 57]. It has a corollary used in the proof of Theorem 2.

Let H be an r-uniform hypergraph. A set $S \subset V(H)$ is called strongly stable if $|e \cap S| \leqslant 1$ for every $e \in E(H)$. The degree of a vertex x is defined as the number of edges containing x and it is denoted by $d(x)$. We shall denote by $\Gamma(X)$ the "$(r-1)$-neighbours" of a set $X \subset V(H)$, defined as $\Gamma(X) = \{e - \{x\} : x \in e \in E(H),\ x \in X\}$. It is worth noting that the elements of the set $\Gamma(X)$ are $(r-1)$-element subsets of $V(H)$. Clearly, $d(x) = |\Gamma(x)|$, where we write $\Gamma(x)$ instead of $\Gamma(\{x\})$.

THEOREM 1. *If S is a strongly stable set in a τ-critical hypergraph H, then* $\mathrm{d}(x) \leqslant |\Gamma(S)| - |S| + 1$ *for every* $x \in S$.

Proof. Suppose that the statement is not true and let S be a strongly stable set of minimal cardinality for which

$$d(x) > |\Gamma(S)| - |S| + 1 \tag{1}$$

holds for some $x \in S$.

Step 1. If $Y \subset S - \{x\}$ and

$$|\Gamma(Y) - \Gamma(x)| < |Y|, \tag{2}$$

then $\Gamma(Y) \cap \Gamma(x) \neq \varnothing$. (Otherwise, $|Y| > |\Gamma(Y)| \geqslant |\Gamma(Y)| - d(y) + 1$ would hold for every $y \in Y$, hence Y would satisfy (1) which contradicts the minimality of S.)

Step 2. Let $Y \subset S - \{x\}$ be a minimal set satisfying (2). Such a Y exists because (2) holds for $S - \{x\}$: from (1) we have

$$\begin{aligned} |S - \{x\}| = |S| - 1 > |\Gamma(S)| - d(x) &= |\Gamma(S) - \Gamma(x)| \\ &= |\Gamma(S - \{x\}) - \Gamma(x)|, \end{aligned}$$

on the other hand, $Y \neq \varnothing$ is trivial. Now, because of $\Gamma(Y) \cap \Gamma(x) \neq \varnothing$, there exists a $y \in Y$ and an edge $e \in E(H)$, $y \in e$, such that $e - \{y\} \in \Gamma(x)$. Let $f = (e - \{y\}) \cup \{x\}$.

Step 3. As Y is minimal, the König–Hall theorem (see, e.g., [1, p. 134]) guarantees the existence of a bijection b: $Y - \{y\} \to \Gamma(Y - \{y\}) - \Gamma(x)$ such that $b(z) \in \Gamma(z)$ for every $z \in Y - \{y\}$. Define $b(y) = e - \{y\}$. (Of course, $\Gamma(y) \subset \Gamma(x) \cup \Gamma(Y - \{y\})$, because Y is minimal.) For every $z \in Y$, fix another vertex $z' \in b(z)$.

Step 4. Since H is τ-critical, the partial hypergraph $H - f$ has a $(t-1)$-element transversal T which intersects every edge with the exception of f. Thus $y \in T$, $x \notin T$ and T meets every element of $\Gamma(x)$ different from $f - \{x\}$.

Step 5. Set $T' = (T - Y) \cup \bigcup_{z \in T \cap Y} \{z'\}$. Since $|T'| \leqslant t - 1$, in order to obtain a contradiction, it is enough to show that T' is a transversal of H.

Obviously, T' meets every edge disjoint from $Y \cup \{x\}$. Also, $T' \cap f \neq \varnothing$, because $y \in e \cap f$, therefore T' meets each set $h \in \Gamma(x)$. For the remaining edges it is enough to mention that b is a *bijection* between $Y - \{y\}$ and $\Gamma(Y) - \Gamma(x)$, therefore $\Gamma(Y) - \Gamma(x) = \bigcup_{z \in Y - \{y\}} \{b(z)\}$. Consequently, if a set $h = b(z)$ is disjoint from T, then $z \in T$ and $z' \in T'$, that is, $h \cap T' \neq \varnothing$. ∎

Since $1 \leqslant d(x)$ for any vertex x of a hypergraph, we have

COROLLARY 1. *$|S| \leqslant |\Gamma(S)|$ for every strongly stable set S of a τ-critical hypergraph.*

Remark 1. That $v_{\max}(r, t) \geqslant \binom{t+r-2}{r-1} + t + r - 2$ is shown by the hypergraph, where $V(H) = X \cup Y$, $X \cap Y = \varnothing$, $|X| = t + r - 2$, $|Y| = \binom{t+r-2}{r-1}$ and $E(H)$ is constructed by adding distinct vertices of Y to the $(r-1)$-element subsets of X.

THEOREM 2. $v_{\max}(r, t) \leqslant \binom{t+r-2}{r-2} t + t^{r-1}$.

Proof. Let H be an r-uniform τ-critical hypergraph with $\tau(H) = t$. We proceed in five steps.

Step 1. Let T be the t-uniform hypergraph whose edges are the t-element transversals of H. If T' is a partial hypergraph of T and $e \in E(H)$, we define $m(e, T') = \min\{|X|: X \subset e, X \cap f \neq \varnothing$ whenever $f \in E(T')\}$, that is, $m(e, T')$ is the cardinality of the smallest subset of e that meets every t-element transversal from T'. Since every transversal of H meets every edge of H, $m(e, T') \leqslant r$. The τ-critical property of H implies that $m(e, T) = r$ for every $e \in E(H)$. Clearly, if $f \in E(T')$, then $m(e, T' - f) \geqslant m(e, T') - 1$ for every $e \in E(H)$, so we can remove edges from T successively until we reach a T^0 with the following properties:

(i) $r - 1 \leqslant m(e, T^0) \leqslant r$ for every $e \in E(H)$,

(ii) for every $f \in E(T^0)$ there exists an $e \in E(H)$ with $m(e, T^0 - f) = r - 2$.

Step 2. If $|X| \leqslant r - 2$ and $X \subset e$ for some $e \in E(H)$, then (i) guarantees an $f=f(X)\in E(T^0)$ such that $f\cap X=\varnothing$. Let $f_0\in E(T^0)$ be fixed. For each $e\in E(H)$ we choose an $(r-1)$-element subset $\{x_1,\dots,x_{r-1}\}\subset e$ satisfying

$$\begin{aligned}&x_1\in e\cap f_0\\&x_2\in e\cap f(\{x_1\})\\&x_3\in e\cap f(\{x_1,x_2\})\\&\qquad\vdots\\&x_{r-1}\in e\cap f(\{x_1,x_2,\dots,x_{r-2}\}).\end{aligned}$$

Let H_1 be the $(r-1)$-uniform hypergraph induced by the different subsets $\{x_1,x_2,\dots,x_{r-1}\}$ as e runs over $E(H)$.

Step 3. Now $|E(H_1)|\leqslant t^{r-1}$, since there are at most t choices for x_1 and for fixed $x_1,\dots,x_i$, $1\leqslant i\leqslant r-2$, there are at most t choices for x_{i+1}.

Step 4. We shall now prove that $|V(H_1)|\leqslant\binom{t+r-2}{r-2}t$. For every $f\in E(T^0)$ property (ii) guarantees an $(r-2)$-element subset $X=X(f)$ of some $e\in E(H)$ such that $f\cap X(f')=\varnothing$ if and only if $f=f'$. It has been proved by Bollobás (cf. [2; 5, Ex. 32, p. 81]) that the number of these pairs f, $X(f)$ is at most $\binom{|f|+|X(f)|}{|X(f)|}$ implying $|E(T^0)|\leqslant\binom{t+r-2}{r-2}$. Since $V(H_1)\subset V(T^0)$, we have

$$|V(H_1)|\leqslant|V(T^0)|\leqslant|E(T^0)|\,t\leqslant\binom{t+r-2}{r-2}t.$$

Step 5. Clearly, $S=V(H)-V(H_1)$ is a strongly stable set of H (every $e\in E(H)$ contains an edge of H_1) and $\Gamma(S)\subset E(H_1)$. Using Corollary 1 we see $|V(H)|=|V(H_1)|+|S|\leqslant|V(H_1)|+|\Gamma(S)|\leqslant|V(H_1)|+|E(H_1)|\leqslant\binom{t+r-2}{r-2}t+t^{r-1}$ and our proof is complete. ∎

A hypergraph H is called vertex-critical if any of its vertices is contained in some $\tau(H)$-element transversal of H. The following proposition is obvious:

PROPOSITION. *A hypergraph H is vertex-critical if and only if every τ-critical partial hypergraph H' of H with $\tau(H')=\tau(H)$ satisfies $|V(H')|=|V(H)|$.*

The proposition implies that $v_{\max}(r,t)$ gives the maximal number of vertices for the more general class of vertex-critical r-uniform hypergraphs with $\tau(H)=t$. Therefore, Theorem 2 holds for vertex-critical hypergraphs. For $r=2$ and $r=3$ we obtain

COROLLARY 2 (Erdös–Gallai [3, p. 196]). *Every vertex-critical graph G satisfies $|V(G)|\leqslant 2\tau(G)$.*

COROLLARY 3. *Every vertex-critical 3-uniform hypergraph H satisfies* $|V(H)| \leqslant 2\tau(H)^2 + \tau(H)$.

Remark 2. We finally show how the determination of $v_{\max}(r, t)$ fits into a general class of problems introduced by Erdös.

An important case of the so-called arrow symbol problems posed in [4] can be formulated as follows:

Find the maximal $m = m(r, t, k, u)$ for which there exists an r-uniform hypergraph H with m vertices satisfying:

(i) $t - u \leqslant \tau(H) \leqslant t$,

(ii) every k-element vertex set is contained in some t-element transversal of H.

So far only the case $r = 2$ has been extensively investigated (cf. [7] for results and further references); for larger r no results have been published except [8] concerning the case $r = 3$.

It is clear, however, that for every $r \geqslant 2$ and $1 \leqslant k \leqslant t$, $m(r, t, k, 0)$ is equal to the maximal order of r-uniform hypergraphs with transversal number t and critical in the stronger sense expressed by (ii). For $k = 1$, Theorem 2 gives

$$m(r, t, 1, 0) = v_{\max}(r, t) \leqslant \binom{t + r - 2}{r - 2} t + t^{r-1}.$$

REFERENCES

1. C. BERGE, "Graphs and Hypergraphs," North-Holland, Amsterdam, 1973.
2. B. BOLLOBÁS, On generalized graphs, *Acta Math. Acad. Sci. Hungar.* **16** (1965), 447–452.
3. P. ERDÖS AND T. GALLAI, On the maximal number of vertices representing the edges of a graph, *Közl.-MTA Mat. Kutató Int. Budapest* **6** (1961), 181–203.
4. P. ERDÖS, On a lemma of Hajnal–Folkman, in "Combinatorial Theory and Its Applications," Coll. Math. Soc., No. 4, pp. 311–316, Bolyai, 1970.
5. L. LOVÁSZ, "Combinatorial Problems and Exercices," joint ed., Akad. Kiadó, Budapest and North-Holland, Amsterdam, 1979.
6. L. SURÁNYI, On line critical graphs I., *in* "Infinite and Finite Sets," Coll. Math. Soc. No. 10, pp. 1411–1444, Bolyai, 1973.
7. L. SURÁNYI, Large α-critical graphs with small deficiency (On line critical graphs II.), *Studia Sci. Math. Hungar.* **10** (1975), 397–412.
8. E. SZEMERÉDI AND GY. PETRUSKA, On a combinatorial problem I., *Studia Sci. Math. Hungar.* **7** (1972), 363–374.

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