Packing trees into n-chromatic graphs

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Let T_i denote a tree with i edges. The author's tree packing conjecture [3] states that K_n has an edge disjoint decomposition into any given sequence $T_1, T_2, \dots T_{n-1}$ of trees. Gerbner, Keszegh and Palmer extended the conjecture by replacing K_n with an arbitrary n-chromatic graph (Conjecture 2 in [1]). Here we show that the extended conjecture follows from the original one. We need the following, perhaps folklore result (Theorem 1 in [4]) and for convenience we include its simple proof.

Lemma 1. *Let G be a k-chromatic graph with a proper k-coloring with k distinct colors. Suppose T is a tree on k vertices and each vertex of T is labeled with a different color from the same set of k colors. Then G contains a subtree that is label-isomorphic to T (labeled exactly the same way as T).*

Proof. The proof is by induction, the base step is trivial for $k = 1$. Let S_t denote the vertices of color t in a proper k-coloring of a k-chromatic graph G with a k-element color set C. Select a leaf vertex P in a tree T labeled with the k colors of C, suppose its label is i and assume P is adjacent in T with vertex Q labeled with j. Let A denote the set of vertices in S_j adjacent to at least one vertex of S_i. Observe that A is nonempty, otherwise G would be $(k-1)$-chromatic. The subgraph $G^* \subset G$ obtained by removing S_i and $S_j - A$ from $V(G)$ is $(k-1)$-chromatic, since the removed vertices form an independent set. Also, G^* is colored with the color set $C - i$. By induction, G^* contains a label-isomorphic copy of the tree $T-P$, its vertex with color j is in A, thus adjacent to a vertex in S_i, extending $T-P$ to a label-isomorphic copy of T. $\square$

Theorem 2. *Suppose that K_n has an edge disjoint decomposition into a given sequence $T_1, T_2, \dots T_{n-1}$ of trees and G is an n-chromatic graph. Then G contains edge disjoint copies of $T_1, T_2, \dots T_{n-1}$.*

Proof. Let $S_1, S_2, \dots, S_n$ be a partition of $V(G)$ into independent sets where G is an n-chromatic graph and color all vertices of S_i with color i. By assumption the complete graph on vertex set $V = \{1, 2, \dots, n\}$ can be decomposed into $T_1, T_2, \dots T_{n-1}$. Let G_i be the subgraph of G induced by

$$\cup_{j \in V(T_i)} S_j.$$

The graph G_i is obviously $(i+1)$-chromatic since it has $i+1$ color classes and a proper coloring of G_i with at most i colors could be obviously extended to a proper coloring of G with at most $n-1$ colors, leading to a contradiction. Applying Lemma 1 to G_i, we find a copy F_i of T_i labeled exactly the same way as T_i is labeled in K_n. Repeating this for $i = 1, 2, \dots, n-1$, we obtain edge disjoint copies of $F_1, F_2, \dots, F_{n-1}$ in G, in fact they are not only edge disjoint but the union of their edge sets takes exactly one edge from each bipartite graph $\{[S_i, S_j] : 1 \le i < j \le n\}$. $\square$

Theorem 2 allows to transfer known tree-packing results from K_n to n-chromatic graphs. In particular, since any sequence of trees containing only paths and stars are known to be packable to K_n ([3, 5]), we get the following, conjectured in [2].

Corollary 3. *Any sequence $T_1, T_2, \dots T_{n-1}$ of stars and paths is packable into any n-chromatic graph.*

References

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