JGT	jgt21694	Dispatch: August 6, 2012	CE:
Journal	MSP No.	No. of pages: 11	PE: XXXXX

Disconnected Colors in Generalized Gallai-Colorings

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Received October 16, 2010; Revised June 11, 2012

Contract grant sponsor: Grant-In-Aid for Young Scientists (B) Contract grant number: 20740068; Contract grant sponsor: the Everett Pitcher Fund (to S. F.); Contract grant sponsor: NSF; Contract grant sponsor: ARC; Contract grant number: DP1096525 (to Á. S.).

Published online in Wiley Online Library (wileyonlinelibrary.com).
DOI 10.1002/jgt.21694

Abstract: Gallai-colorings of complete graphs—edge colorings such that no triangle is colored with three distinct colors—occur in various contexts such as the theory of partially ordered sets (in Gallai's original paper), information theory and the theory of perfect graphs. A basic property of Gallai- Q1 colorings with at least three colors is that at least one of the color classes must span a disconnected graph. We are interested here in whether this or a similar property remains true if we consider colorings that do not contain a rainbow copy of a fixed graph F. We show that such graphs F are very close to bipartite graphs, namely, they can be made bipartite by the removal of at most one edge. We also extend Gallai's property for two infinite families and show that it also holds when F is a path with at most six vertices. J. Graph Theory 00: 1–11, 2012 Q2

1 INTRODUCTION

Edge colorings of complete graphs in which no triangle is colored with three distinct colors were called Gallai-partitions in [9], and Gallai-colorings in [5,7]. Here, we briefly call these colorings *G-colorings*and always assume that G-colorings are on the edges of a complete graph. More than just the term, the concept occurs in relation to deep structural properties of fundamental objects. An important result, Theorem 1, from Gallai's original paper [4]—translated to English and endowed by comments in [10]—can be reformulated in terms of G-colorings. Further occurrences are related to generalizations of the perfect graph theorem [1,2], or applications in information theory [8].

Our starting point is the following result of Gallai [4], see an explicit proof in [5]. We say that a color class of an edge-coloring of G is *connected* if it together with all vertices of G forms a connected graph. Otherwise, the color class is called *disconnected*.

Theorem 1. *In every G-coloring with at least three colors, at least one of the color classes must be disconnected.*

What is the role of forbidding a rainbow triangle? Call a subgraph *rainbow* if all colors on the edges of the subgraph are distinct. Can we extend Theorem 1 in some way to colorings where a rainbow copy of some other fixed graph F is forbidden? This question is the central topic of this work.

An edge coloring of a complete graph K is *connected* if every color class in K is connected. Let us say that a graph F has the *disconnection property*, *DP*, if there exists a natural number $m = m(F)$ (note that $m(F)$ does not depend on the order of K) such that the following holds: in every edge coloring of a complete graph with at least m colors, either there is a rainbow F or at least one color class is disconnected. Equivalently, F has the disconnection property if, in every connected coloring with at least $m(F)$ colors, there is a rainbow copy of F. Notice that $m(F) \geq |E(F)|$ because complete graphs which are large enough have connected colorings using $|E(F)| - 1$ colors with no rainbow F. If a graph F has *DP* with $m(F) = |E(F)|$, we say that it has the *Gallai property*, *GP*.

Sometimes GP and DP are just identified with the class of graphs having the property. For example, we can then say that, by Theorem 1, $K_3 \in GP$.

Observation 1. *If $F \in GP$, then $F \in DP$.*

Observation 2. *Assume that F_1 is a subgraph of F_2 and $F_2 \in DP$. Then $F_1 \in DP$.*

We now show that some short paths are in GP. Let P_n denote a path of order n. For this proof and others throughout this work, we will denote by $c(uv)$ the color of the edge uv.

Proposition 1. $P_3, P_4, P_5 \in GP$.

Proof. The result is trivial for P_3 and almost trivial for P_4 so we show only that $P_5 \in GP$. Let G be a complete graph whose edges are colored with four colors and assume all color classes are connected. We shall reach a contradiction by finding a rainbow P_5.

Since $P_4 \in GP$, we have a path $P = v_1v_2v_3v_4$ colored with three distinct colors, say $c(v_iv_{i+1}) = i$. Both v_1 and v_4 are incident to an edge of color 4 and, to avoid a rainbow P_5, the other ends of these edges must be in P. Suppose first that those edges coincide, i.e., $c(v_1v_4) = 4$. Observe that no edge of color i ($i = 1, 2, 3$) can go from an endpoint of an i-colored edge of P to $V(G) \setminus V(P)$. Therefore, since colors 1 and 3 are connected, we have $c(v_1v_3) = 1$ and $c(v_2v_4) = 3$ or the other way around. But now, by the same argument, the edge v_2v_3 is isolated in color 2, a contradiction.

Thus, we may assume that edges v_1v_3, v_2v_4 are both colored with color 4. Now we get the same contradiction as before—the edge v_2v_3 is isolated in color 2. ■

We will also show later that $P_6 \in GP$ (see Theorem 4). Following this result, an obvious question would be the following.

Problem 1. *Are all paths in GP?*

The next natural question is perhaps whether C_4, the cycle on 4 vertices, is in GP? This question has been asked some years ago by Simonyi and the second author [6] but the fourth author has found a counterexample which will be presented here in a somewhat more general form. However, it might still be true that $C_4 \in DP$.

Problem 2. *Is $C_4 \in DP$?*

The construction in Section 2 shows that graphs in DP can be made into a bipartite graph by deleting at most one edge (Corollary 2). Moreover, even bipartite graphs in DP must be very sparse, their connected components must be acyclic (containing no cycle) or unicyclic (containing exactly one cycle) graphs or two such components joined by a bridge (Corollary 1).

2 NECESSARY CONDITIONS FOR *GP* AND *DP*

To get necessary conditions for $F \in GP$ or $F \in DP$, we define a specific m-colored complete graph, $K(m, k)$ for every $m \geq 2, k \geq 1$.

Construction 1. *Let A, B be disjoint sets, $|A| = |B| = 2(m-1)k + 1$. Define*

$$A = \cup_{i=1}^{m-1} A_i \cup \{a\}, \quad B = \cup_{i=1}^{m-1} B_i \cup \{b\},$$

where the sets are all disjoint and $|A_i| = |B_i| = 2k$. *Color* 0 *is distinguished, the edge ab and the edges within A and B are colored with* 0. *For* $i = 1, 2, \ldots, m-1$, *the edges of the complete bipartite graphs* $[a, B_i], [b, A_i], [A_i, B_i]$ *are colored with color i. Split each* A_i, B_i *into two disjoint equal parts,* $A_i = X_i \cup Y_i$, $B_i = U_i \cup W_i$ *(k vertices in each). For any* $i \neq j \in \{1, 2, \ldots, m-1\}$, *color* $[X_i, U_j], [Y_i, W_j]$ *with color i and color* $[X_i, W_j], [Y_i, U_j]$ *with color j. This colors all edges of the complete graph induced by* $A \cup B$, *we shall refer to it as* $K(m, k)$.

A graph is called *unicyclic*if it has exactly one cycle, i.e., it has exactly one component that is not a tree and that component can be obtained by adding a single edge to a tree. A graph is called *acyclic* if it is a forest, namely each component is a tree.

Lemma 1. *Suppose that H is a rainbow connected bipartite subgraph of Construction 1. Then H is either acyclic or unicyclic or can be obtained from two vertex disjoint copies of such graphs by connecting them with a bridge of color* 0. *An edge of H with color* 0 *can not be on a cycle of H.*

Proof. Every edge of a rainbow subgraph H of color $i \neq 0$ can be oriented from its endpoint in A_i or B_i (in the case of an edge with endpoints in both A_i and B_i, we orient it arbitrarily). Note that all edges oriented out of a vertex have a single color. Since H is rainbow, each vertex of H has out-degree at most one in this orientation. Therefore, if no edge of H has color 0, each component of H is acyclic or unicyclic. If H has an edge e of color 0 and $e = ab$ then all edges of H incident to e must be oriented toward e and the component of H containing e must be acyclic. If e is different from ab then, since H is bipartite, e cannot be in a cycle of H. Hence, e is a bridge connecting two components of $H \setminus e$, and these components are acyclic or unicyclic by the already proven first part of our argument. ■

Corollary 1. *Suppose that* $F \in DP$ *is connected and bipartite. Then F is either acyclic or unicyclic or two such components joined by an edge.*

Proof. Suppose that $F \in DP$ is bipartite. Then, the connected coloring of Construction 1 with $m \geq m(F)$ colors must contain a rainbow F. By Lemma 1, the proof is finished. ■

Corollary 2. *Suppose that* $F \in DP$. *Then for some* $e \in E(F)$, *the graph* $F - e$ *is bipartite.*

Proof. Suppose that $F \in DP$. Consider the connected coloring of Construction 1 with $m \geq m(F)$. This coloring must contain a rainbow F. At most, one edge of F can be colored with color 0 and all other edges connect a vertex in A with a vertex in B. Thus, by the removal of at most one edge, F becomes bipartite. ■

Combining Corollaries 1 and 2, we have the following.

Corollary 3. *Every connected* $F \in DP$ *can be obtained from an acyclic graph by adding at most two edges.*

Proposition 2. $C_{2i} \notin GP$.

Proof. Consider Construction 1 with $m = 2i$. If $C_{2i} \in GP$ then the coloring of $K(m, k)$ must contain a rainbow $F = C_{2i}$. However, this is possible only if color 0 is used on F but this contradicts Lemma 1. ■

Although even cycles are not in GP, it could still be possible that odd cycles, potentially a more natural extension of the triangle, are in GP.

Problem 3. *Are odd cycles in GP? Or at least in DP?*

3 SUFFICIENT CONDITIONS FOR *GP* AND *DP*

Theorem 2. *Assume that F is a unicyclic graph such that its cycle is a triangle. Then $F \in DP$.*

Let $|F|$ denote the order (number of vertices) of the graph F. Note that, since F is unicyclic, $|F| = |E(F)|$.

Proof. By Observation 2, we may assume that F is connected. If F is a triangle, $m(F) = 3$ by Theorem 1. If F is a triangle with a pendant edge then $m(F) = 4$ since in any connected 4-coloring there is a rainbow triangle and any vertex of such a triangle is incident to an edge of the fourth color. If $|F| \geq 5$ then $m(F) \leq 2|F| - 1$ follows easily by induction. Indeed, let v be a pendant vertex of F adjacent to $w \in F^* = F \setminus v$. We can find a rainbow F^* in any connected $2|F| - 3$-coloring of K_n. Since we have $k = |F| - 1$ colors used in F^* and a further $l \leq |F| - 3$ colors are present on edges wu with $u \in F^*$, there is a color used on K_n that is different from these $k + l < 2|F| - 3$ colors. Since the coloring of F is connected, this color is used on some edge incident to w and this edge extends F^* to a rainbow copy of F. ■

Corollary 4. *Acyclic graphs are in DP.*

Although it seems like the conditions for GP are much stronger than the conditions for DP, the answer to the following question is still not clear.

Problem 4. *Is $DP \setminus GP \neq \emptyset$?*

Probably many of the graphs in Theorem 2 are, in fact, in GP. Fujita and Magnant [3] showed that this is the case for a graph H obtained from a star by adding an edge between two endpoints. Indeed, in any connected $|E(H)|$-coloring of a complete graph K_n, there is a rainbow triangle T and any vertex of T is incident to edges of the other $|E(H)| - 3$ colors not used in T. A wider class of candidates would be graphs H_t^s defined as follows. Consider a triangle T and t copies of P_3. Let H_t be the graph obtained by identifying one end of each path and a fixed vertex of T into a single vertex v. More generally, let H_t^s be the graph H_t with s extra pendant edges from the central vertex v. Note that $H_t = H_t^0$ as we will use this abbreviated version in the proofs. It seems that the answer to the following should be affirmative.

Problem 5. *Is $H_t^s \in GP$?*

We show the following which answers this question in the case where $t = 1$.

Theorem 3. *$H_1^s \in GP$ for all $s \geq 0$.*

Proof. The proof is broken into two cases based on the value of s. ■

Case 1. $s = 0$.

Consider a connected 5-coloring G of K_n. We intend to show that there exists a rainbow H_1. It is known (and easy to show) that $H_0^2 \in GP$ so there exists a rainbow copy of H_0^2 in G. Label the vertices of the triangle with $\{v_1, v_2, v_3\}$ (where v_3 is the central vertex of the structure) and the leaves with v_4 and v_5. Assume the colors are $c(v_1v_3) = 1$, $c(v_1v_2) = 2$, $c(v_2v_3) = 3$, $c(v_3v_4) = 4$, and $c(v_3v_5) = 5$.

To simplify the argument, we define the following notation. For a copy of H_1 with triangle vertices u_1, u_2, u_3 and path $u_3u_4u_5$, we use the notation $H(u_1, u_2, u_3, u_4, u_5)$.

Since color 4 is connected, the vertex v_5 must have an edge of color 4. If this edge goes to a vertex $v_6 \notin \{v_1, v_2\}$ (note that possibly $v_6 = v_4$), then we have a rainbow H_1 on $H(v_1, v_2, v_3, v_5, v_6)$ so the edge of color 4 must go to either v_1 or v_2. Similarly, v_4 must have an edge of color 5 to either v_1 or v_2. Call these two edges *back edges*.

Subcase 1.1. *The back edges share a vertex v_1 or v_2.*

Without loss of generality, suppose the edges both contain v_1. Consider the edge $e = v_4v_5$. As noted above, e cannot have color 4 or 5. If $c(e) = 1$ or 3, then $H(v_5, v_4, v_3, v_2, v_1)$ or, respectively, $H(v_5, v_4, v_3, v_1, v_2)$ is rainbow. Finally, if $c(e) = 2$, then $H(v_5, v_4, v_1, v_3, v_2)$ is the desired rainbow copy of H_1. Hence, regardless of the color of e, there is a rainbow H_1 in this structure.

Subcase 1.2 *The back edges do not share a vertex v_1 or v_2.*

Without loss of generality, suppose v_1v_5 has color 4 and v_2v_4 has color 5. Since color 2 is connected, either v_1 or v_2 must have an edge of color 2 to a vertex $v_6 \notin \{v_1, v_2\}$. If $v_6 \notin \{v_4, v_5\}$, then either $H(v_1, v_5, v_3, v_2, v_6)$ or $H(v_2, v_4, v_3, v_1, v_6)$ forms a rainbow H_1, a contradiction. Hence, either v_1v_4 or v_2v_5 has color 2. Without loss of generality, suppose v_2v_5 has color 2.

The vertex v_4 must have an edge of color 1 so suppose v_4v_6 has color 1 for some vertex v_6. If $v_6 \notin \{v_1, v_5\}$, then $H(v_2, v_5, v_3, v_4, v_6)$ is a rainbow copy of H_1. If $v_6 = v_1$, then $H(v_2, v_3, v_5, v_1, v_4)$ is a rainbow copy of H_1. Finally, if $v_6 = v_5$, then $H(v_4, v_5, v_3, v_2, v_1)$ is a rainbow copy of H_1, completing the proof in this case.

Case 2. $s \geq 1$.

Let G be an $(s+5)$-coloring of K_n in which each color class is connected. By induction, using the previous case as the base, we may assume there is a rainbow copy of H_1^{s-1}. Suppose this copy of H_1^{s-1} uses colors $1, 2, \ldots, s+4$. Let v be the central vertex and let u be the vertex at the end of the path of length 2 from v.

We would first like to show that v is also the center of a rainbow bowtie (here *bowtie* is used to describe the graph consisting of two triangles sharing a single vertex). Since every color class is connected, there is an edge of color $s+5$ incident to v. If this edge goes to a vertex not already in H_1^{s-1}, then this is a rainbow H_1^s so the edge of color $s+5$ must go from v to u, so v is the center of a rainbow bowtie.

We now formalize notation for the rest of the proof. Since each color class is connected, there must be an edge of every color incident to v. Let v_i be a vertex incident to v with an edge of color i for $1 \leq i \leq s+5$. Suppose $c(v_1v_2) = 3$ and $c(v_4v_5) = 6$ to form the rainbow bowtie mentioned above.

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Now consider the set $A = \{v_1, \dots, v_{s+5}\} \setminus \{v_3, v_6\}$. For $k \in \{3, 6\}$, let $A_k \subseteq A$ be the subset of vertices $a \in A$ that are connected by an edge of color k to some vertex $a' \in V(G) \setminus A$. Since the coloring is connected, $A_k \neq \emptyset$ for both k. If there is an edge $v_i v_j$ of color 6 within A and a vertex $a \in A_3 \setminus \{v_i, v_j\}$, then we have a rainbow H_1^s using $v_i v_j v$ as the triangle and vaa' as the path. Similarly, if there is an edge $v_i v_j$ in A of color 3 and some $b \in A_6 \setminus \{v_i, v_j\}$ then G contains a rainbow H_1^s. Hence, we may assume that all edges of color 3 within A are incident to all vertices in A_6 and all edges of color 6 within A are incident to all vertices in A_3. In particular, since $c(v_1v_2) = 3$ and $c(v_4v_5) = 6$, we have $A_3 \subseteq \{v_4, v_5\}$ and $A_6 \subseteq \{v_1, v_2\}$.

If $|A_3| = 2$ or $|A_6| = 2$, say $|A_3| = 2$, then v_4, v_5 is the only edge in A of color 6. Also, since $A_6 \cap \{v_4, v_5\} = \emptyset$, $\{v_4, v_5\}$ is a component of the subgraph of G induced by color 6, a contradiction. If $|A_3| = |A_6| = 1$, then let $A_3 = \{a\} \subseteq \{v_4, v_5\}$ and $A_6 = \{b\} \subseteq \{v_1, v_2\}$. The edge ab cannot have both colors 3 and 6, say $c(ab) \neq 6$. Since the edges of color 6 in G restricted to A is a star S with center a and b is not a vertex of S, S is a component of the subgraph of G induced on color 6, a contradiction.

Next, we use Theorem 3 to show that $P_6 \in GP$.

Theorem 4. $P_6 \in GP$.

Proof. Consider a 5-coloring G of K_n in which each color is connected. We would like to show that there is a rainbow P_6 in G, thereby showing that $P_6 \in GP$. Suppose there is no rainbow P_6. ■

Claim 1. *There exists no rainbow C_5 in G.*

Proof of Claim 1. Suppose there exists a rainbow C_5 and without loss of generality, suppose the edge colors are 1, 2, 3, 4, 5 in that order around the cycle with vertices labeled so that $v_i v_{i+1}$ has color i for $1 \leq i \leq 4$ and $v_5 v_1$ has color 5. Since the edges of color 1 form a connected subgraph of G, there must be another edge of color 1 at an end-vertex of $v_1 v_2$. If this edge goes to a vertex not on the cycle, then there exists a P_6 using that edge and most of the cycle. This means there must be a chord of the rainbow C_5 in color 1. The same is true for each of the other colors so, in fact, we know that the complement of the rainbow C_5 is another rainbow C_5. Let $C = V(C_5) \subseteq V(G)$.

The remainder of this proof is broken into two cases; namely $c(v_1v_3) = 1$ (symmetrically $c(v_2v_5) = 1$) or $c(v_1v_4) = 1$ (symmetrically $c(v_2v_4) = 1$). We call these edges *acute* and *obtuse* respectively since, when the vertices of the C_5 are drawn equally distributed on a circle, the angle between edges of the same color in the first case is smaller than the angle in the second case. Suppose first that $c(v_1v_3) = 1$, see Figure 1. The edges of color 1 span a connected graph but since there are no edges of color 1 from v_1 or v_2 to $V(G) \setminus C$ and no more edges of color 1 within C, v_3 must be connected by an edge of color 1 to some $u \in V(G) \setminus C$. We know that v_1 is incident to an edge of color 2 but it cannot be inside C so it must go to a vertex $u' \in V(G) \setminus C$. Then $u' = u$ since otherwise we have a rainbow P_6 using $u'v_1v_5v_4v_3u$. Furthermore, this means that v_1 is incident to no more edges of color 2.

Since color 2 is connected, u must be incident to another edge of color 2, connecting u to a vertex w. If $w \notin C$, then $wuv_3v_4v_5v_1$ is a rainbow P_6 so $w \in \{v_4, v_5\}$. First, suppose that uv_4 has color 2. Then v_1 has no edge of color 3 connecting to a vertex $w \in V(G) \setminus C$, otherwise $wv_1v_5v_4uv_3$ would be a rainbow P_6. Since color 3 is connected, v_1v_4 must have color 3. Now there are no edges of color 3 from $\{v_1, v_3, v_4\}$ to $V(G) \setminus \{v_1, v_3, v_4\}$ so

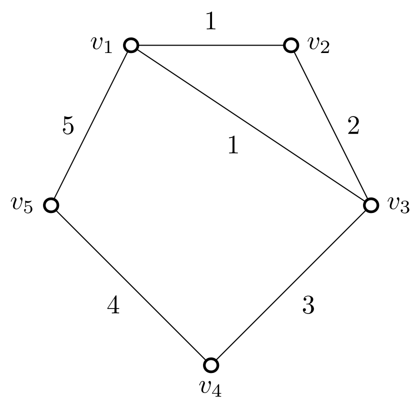

FIGURE 1. Acute case.

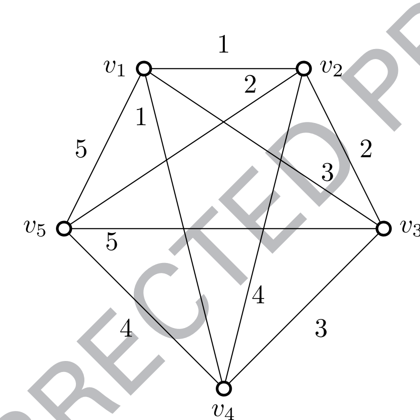

FIGURE 2. Obtuse case.

color 3 is disconnected, a contradiction. Next, we suppose that uv_5 has color 2. Then if v_1 is connected by an edge of color 4 to a vertex $w \in V(G) \setminus C$, we have a rainbow P_6 using $wv_1v_5uv_3v_4$. Hence, v_1v_4 must have color 4. Then as before, there are no edges of color 4 from $\{v_1, v_4, v_5\}$ to $V(G) \setminus \{v_1, v_4, v_5\}$ so color 4 is disconnected, a contradiction.

Finally, we suppose $c(v_1v_4) = 1$. In fact, we may assume all edges in the interior of the C_5 are obtuse, see Figure 2. One may easily verify that this forces $c(v_iv_{i+3}) = i$ for all i modulo 5. There must be an edge of color 2 from v_1 to a vertex $u \in V(G) \setminus C$ but this creates a rainbow P_6 using $uv_1v_2v_4v_3v_5$, completing the proof of the claim. $\blacksquare_{Claim\ 1}$

Since $H_1 \in GP$, we know there must exist a rainbow copy of H_1. Suppose this uses vertices $v_1, \ldots, v_5$ with $c(v_iv_{i+1}) = i + 1$ for $1 \leq i \leq 4$ and $c(v_1v_3) = 1$ (see Fig. 3). By Claim 1, we know $c(v_1v_5) \neq 1$ and $c(v_2v_5) \neq 3$. In order to avoid a rainbow P_6, there is no edge of color 1 from v_1 or v_5 to any vertex $v_6 \notin V(H_1)$ and similarly there is no edge of color 3 from v_2 or v_5 to v_6. Hence, v_5 must be connected in colors 1 and 3 to the set $\{v_1, v_2, v_3\}$. By the symmetry of colors 1 and 3, by Claim 1, we may assume that $c(v_2v_5) = 1$. Conversely, since color 1 is connected, there must be another edge of color 1 incident to either v_1 or v_3. Hence, we break the remainder of the proof into three cases based on where that edge can be.

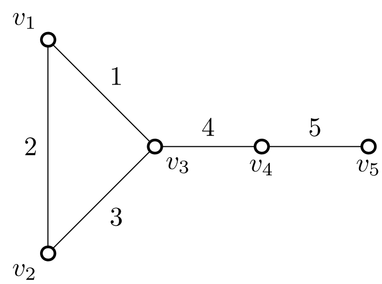

FIGURE 3. Rainbow H_1.

Case 1. *The edge v_1v_4 has color* 1.

There is no edge of color 4 connecting v_5 to a new vertex $v_6 \notin V(H_1)$ as this would produce a rainbow P_6 on $v_6v_5v_4v_1v_2v_3$. This means that there must be edges of colors 1, 3, 4 connecting v_5 to the vertices v_1, v_2, v_3. Recall that $c(v_2v_5) = 1$.

If $c(v_3v_5) = 4$, then $c(v_1v_5) = 3$. This gives us a rainbow C_5 on $v_1v_2v_3v_5v_4v_1$. If $c(v_1v_5) = 4$, then $c(v_3v_5) = 3$. We know that v_5 is incident to at least one edge of color 2 so suppose $c(v_5v_6) = 2$, where $v_6 \notin V(H_1)$. We also know that v_1 is incident to an edge of color 5 so suppose, for some vertex $v_7 \notin V(H_1)$, that v_1v_7 has color 5 (note that possibly $v_6 = v_7$). Now the path $v_6v_5v_3v_4v_1v_7$ forms a rainbow P_6 (if $v_6 = v_7$ then this forms a rainbow C_5 contradicting Claim 1), a contradiction.

Case 2. *The edge v_3v_5 has color* 1.

In this case, the only possibility for an edge of color 3 incident to v_5 is v_1v_5 (see the paragraph immediately before Case 1). Also, v_5 must be incident to an edge of color 4, and the other endpoint of this edge is a vertex $v_6 \notin V(H_1)$.

There is also an edge of color 5 connecting v_1 to some vertex v_7. If $v_7 \notin V(H_1)$, then $v_6v_5v_3v_2v_1v_7$ is a rainbow P_6 (or C_5 if $v_6 = v_7$) so we must have $v_7 = v_4$ and $c(v_1v_4) = 5$. Now notice that there must exist $v_8 \notin V(H_1)$ with $c(v_3v_8) = 2$. Then $v_8v_3v_4v_1v_5v_2$ is a rainbow P_6.

Case 3. *The vertex v_3 is connected by an edge of color* 1 *to a new vertex $v_6 \notin V(H_1)$.*

If we have $c(v_1v_5) = 3$, then $v_6v_3v_4v_5v_1v_2$ forms a rainbow P_6. Since there is an edge of color 3 incident to v_5 and this edge cannot have an endpoint $v_7 \notin H_1$, we must have $c(v_5v_3) = 3$. The vertex v_2 is incident to an edge of color 4. If $c(v_2v_7) = 4$ for some $v_7 \notin V(H_1)$, then $v_7v_2v_1v_3v_5v_4$ forms a rainbow P_6. This means that v_2v_4 must have color 4. Then $v_2v_4v_5v_3v_1v_2$ is a rainbow C_5, contradicting Claim 1. This completes the proof of Theorem 4.

Another result related to Theorem 3 is the following. Define $H_0^{s,1}$ to be the graph obtained from a triangle abc by adding s pendant edges incident to a and one pendant edge incident to b.

Theorem 5. *For all $s \geq 0$, $H_0^{s,1} \in GP$.*

Proof. Consider a connected $(s+4)$-coloring of a complete graph. Since $s \geq 0$, by Theorem 1, there exists a rainbow triangle $T = abc$. Since all $s+4$ colors are connected, there is an edge of each color incident to a forming a rainbow H_0^{s+1}. Let $1, 2, \ldots, s, s+1$ be the colors on the pendant edges of H_0^{s+1} and let $s+2, s+3, s+4$ be the colors on the edges ab, bc, ac of the triangle, respectively. Denote by X the set of pendant vertices and $x_i \in X$ where the edge ax_i has color i for $1 \leq i \leq s+1$.

If there is an edge bw or cw of color $i \in \{1, \ldots, s+1\}$ such that $w \notin X$ then we are done by deleting the vertex x_i from the neighbors of a. This means that for any $y \in \{a, b, c\}$, we have $\{x : \text{the color of } xy \text{ is in } \{1, 2, \ldots, s+1\}\} = X$. Moreover, if the colors of two edges from a vertex $x_i \in X$ to $\{a, b, c\}$ are the same, say x_ib has color i, then using the edge x_ib and the edges from a to X with colors $\{1, 2, \ldots, s+1\} \setminus \{i\}$ along with the triangle abc, we have the required rainbow $H_0^{s,1}$.

Since the color classes are connected, either b or c (suppose b) has another edge of color $s+3$ to a vertex $v \notin T \cup X$. Without loss of generality, suppose the color of $x_{s+1}b$ is 1. Then $x_{s+1}ab$ is a rainbow triangle using colors $1, s+1, s+2$, the vertex b has a pendant edge of color $s+3$ to $v \notin T \cup X$ and a has pendant edges of colors $2, 3, \ldots, s-1, s, s+4$ to the set $(X \cup \{c\}) \setminus x_1$ to form the desired rainbow copy of $H_0^{s,1}$. ■

Our final result provides another graph in GP, namely a triangle with a single pendant edge on each vertex, also known as a net.

Theorem 6. *Let F be the unicyclic graph consisting of a triangle $v_1v_2v_3$ with a single pendant edge on each vertex such as v_1w_1, v_2w_2, v_3w_3. Then $F \in GP$.*

Proof. Consider a connected 6-coloring K of a complete graph K_n. For any rainbow triangle $T \subset V(K)$ define a set $X(T)$ that consists of nine edges as follows: at each vertex of T, select three edges in the three distinct colors not appearing on T. Observe that T exists and also $X(T)$ is well defined and nonempty since each color appears at every vertex. Call T *tight* if $X(T)$ covers just three vertices in $V(K) \setminus V(T)$. ■

Claim 2. *If there exists a rainbow triangle T that is not tight then there is a rainbow copy of F in K.*

Proof. To prove the claim, suppose $T = abc$ is a rainbow triangle colored with colors 4, 5, 6 and T is not tight. Assume edges au_i, bv_i, cw_i are colored with color i for $i = 1, 2, 3$, respectively. If the sets $V = \{v_1, v_2, v_3\}$ and $U = \{u_1, u_2, u_3\}$ intersect in at most one vertex, then there exists a color i so that $u_i \notin V$ and $v_i \notin U$. Then the following three edges and T form a rainbow copy of F: cw_{i+1}; one of au_{i+2}, bv_{i+2} that is not incident to w_{i+1}; and one of au_i, bv_i that is not incident to w_{i+1} (the indices are taken mod 3). This argument implies that the three sets U, V, W of neighbors of a, b, c in the three colors 1, 2, 3 are pairwise intersecting in at least two vertices.

There are two different structures for the three triples of neighbors, provided that no two triples are the same. One is when these triples pairwise intersect in the same two element set $Y = \{y_1, y_2\}$ and let u, v, w be the three further vertices adjacent to a, b, c, respectively. If there are two distinct colors among the colors of au, bv, cw, say au, bv are colored with 1, 2 then au, bv, and the color 3 edge from c to $\{w\} \cup Y$ gives a rainbow F. If all three edges au, bv, cw have the same color, say 1, then two vertices from $\{a, b, c\}$ connect to a vertex $y \in Y$ in the same color, say both ay_1, by_1 have color 2. Now ay_1, by_2, cw are colored with 2, 3, 1, respectively, giving the desired rainbow F.

The other configuration is when the vertices a, b, and c are adjacent to $\{y_1, y_2, y_3\}$, $\{y_2, y_3, y_4\}$, and $\{y_1, y_2, y_4\}$, respectively. Without loss of generality, ay_i is colored with i and by_4 is colored with 1. If cy_4 is of color 1 then the edge of color 2 from c to $\{y_1, y_2\}$

and the edges ay_3, by_4 give the required F. If cy_4 has color 2 or 3, say 2, then the edges ay_1, cy_4 and the color 3 edge from b to $\{y_2, y_3\}$ define the required F.

We conclude that two triples of neighbors coincide, say the neighbors of a, b are in a three-element set Y. By assumption, c has a neighbor $y_4 \notin Y$ and we can suppose that cy_4 has color 1. Then, in order to avoid a rainbow F, we must have $y_1, y_2 \in Y$ so that ay_1 and by_2 are of color 2 and ay_2 and by_1 have color 3. Now we can find a rainbow F by taking the edge cy_1 or cy_2 (one of them must be present). By symmetry, we can suppose that this edge is cy_1, and of color 2. Then, for $y_3 \in Y \setminus \{y_1, y_2\}$, the edges cy_1, ay_2, by_3 have colors 2, 3, 1, respectively.

Thus, all the three triples must coincide and this proves the claim. $\blacksquare_{\textit{Claim 2}}$

We conclude that every rainbow triangle in K is tight. Take a rainbow triangle T; we may assume that T is colored with 4, 5, and 6. It is easy to see that if the color-1 edges in $X(T)$ have a common endpoint or they are pairwise disjoint then $X(T)$ contains a rainbow matching. Using the notation used in Claim 2, we can assume that the edges of color 1 in $X(T)$ are au_1, bu_1, cu_2. But, in this case, the triangle $Z = acu_1$ is rainbow and not tight because b can be chosen as a neighbor of a in $X(Z)$ but b cannot be chosen as such a neighbor of u_1. This contradiction completes the proof. $\blacksquare$

ACKNOWLEDGEMENT

The authors would like to thank the referees for helpful suggestions and corrections.

REFERENCES

[1] K. Cameron and J. Edmonds, Lambda composition, J Graph Theory 26 (1997), 9–16.

[2] K. Cameron, J. Edmonds, and L. Lovász, A note on perfect graphs, Periodica Math Hungar 17 (1986), 173–175.

[3] S. Fujita and C. Magnant, Extensions of Gallai-Ramsey results, J Graph Theory, In press. Q4

[4] T. Gallai, Transitiv orientierbare Graphen, Acta Math Sci Hungar 18 (1967), 25–66. English translation in Maffray and Preissmann (2001).

[5] A. Gyárfás and G. Simonyi, Edge colorings of complete graphs without tricolored triangles, J Graph Theory 46 (2004), 211–216.

[6] A. Gyárfás and G. Simonyi, Personal communication.

[7] A. Gyárfás, G. N. Sárközy, A. Sebő, and S. Selkow, Ramsey-type results for Gallai colorings, J Graph Theory 64 (2009), 233–243.

[8] J. Körner and G. Simonyi, Graph pairs and their entropies: Modularity problems, Combinatorica 20 (2000), 227–240.

[9] J. Körner, G. Simonyi, and Zs. Tuza, Perfect couples of graphs, Combinatorica 12 (1992), 179–192.

[10] F. Maffray and M. Preissmann, A translation of Gallai's paper: 'Transitiv Orientierbare Graphen', In: Perfect Graphs (J. L. Ramirez-Alfonsin and B. A. Reed, Eds.), John Wiley and Sons, 2001, pp. 25–66. Q5

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