Partition of graphs and hypergraphs into monochromatic connected parts

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Abstract

We show that two results on *covering* of edge colored graphs by monochromatic connected parts can be extended to *partitioning*. We prove that for any 2-edge-colored non-trivial r-uniform hypergraph H, the vertex set can be partitioned into at most $\alpha(H) - r + 2$ monochromatic connected parts, where $\alpha(H)$ is the maximum number of vertices that does not contain any edge. In particular, any 2-edge-colored graph G can be partitioned into $\alpha(G)$ monochromatic connected parts, where $\alpha(G)$ denotes the independence number of G. This extends König's theorem, a special case of Ryser's conjecture.

Our second result is about Gallai-colorings, i.e. edge-colorings of graphs without 3-edge-colored triangles. We show that for any Gallai-coloring of a graph G, the vertex set of G can be partitioned into monochromatic connected parts, where the number of parts depends only on $\alpha(G)$. This extends its cover-version proved earlier by Simonyi and two of the authors.

1 Introduction

In this paper we prove two results about partitioning edge-colored graphs (and hypergraphs) into monochromatic connected parts. Let k be a positive integer. A k-edge-colored (hyper)graph is a (hyper)graph whose edges are colored with k colors. It was observed in [5] that a well-known conjecture of Ryser which was stated in the thesis of his student Henderson [11] can be formulated as follows.

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Conjecture 1. *If the edges of a graph are colored with k colors then $V(G)$ can be covered by the vertices of at most $\alpha(G)(k-1)$ monochromatic connected components (trees).*

Ryser's conjecture (thus Conjecture 1) is known to be true for $k = 2$ (when it is equivalent to König's theorem). After partial results [9], [13], the case $k = 3$ was solved by Aharoni [1], relying on an interesting topological method established in [2]. Recently Király [12] showed, somewhat surprisingly, that an analogue of Conjecture 1 holds for hypergraphs: for $r \geq 3$, in every k-coloring of the edges of a complete r-uniform hypergraph, the vertex set can be covered by at most $\lfloor \frac{k}{r} \rfloor$ monochromatic connected components (and this is best possible). The authors in [4] will consider extensions of Király's result for non-complete hypergraphs.

The strengthening of Conjecture 1 from covering to partition was suggested in [3] (and proved for $k = 3, \alpha(G) = 1$). In this paper we extend the $k = 2$ case of Conjecture 1 for hypergraphs and for partitions instead of covers (Theorem 4).

Our second partition result (Theorem 6) is about *Gallai-colorings* of graphs where the number of colors is not restricted but 3-edge-colored triangles are forbidden. This extends the main result of [8] from cover to partition.

We consider hypergraphs H with edges of size at least two, i.e. we do not allow singleton edges. Let $V(H)$, $E(H)$ denote the set of vertices and the set of edges of H, respectively. A hypergraph is r-uniform if all edges have $r \geq 2$ vertices (graphs are 2-uniform hypergraphs). When there is no fear of confusion in context, we just say hypergraphs briefly. A hypergraph H without any edge is called *trivial.* The *cover graph* G_H of a hypergraph H is the graph defined by the pairs of vertices covered by some hyperedge; namely, G_H is the graph on $V(H)$ such that $e \in E(G_H)$ if and only if e is covered by some hyperedge of H.

The definition of independence number of hypergraphs is not completely standard. The *independence number* $\alpha(H)$ is the cardinality of a largest subset S of $V(H)$ that does not contain any edge of H (i.e., the maximum number of vertices in an induced trivial subhypergraph of H). Another useful variant important in this paper is the *strong independence number* $\alpha_1(H)$, the cardinality of a largest subset S of vertices such that any edge of H intersects S in at most one vertex. In fact, $\alpha_1(H) = \alpha(G_H)$. For example, if H is the Fano plane, $\alpha_1(H) = 1, \alpha(H) = 4$. For a complete r-uniform hypergraph H, $\alpha_1(H) = 1, \alpha(H) = r - 1$. For r-uniform hypergraphs these numbers are linked by the following inequality.

Proposition 1. *For any non-trivial r-uniform hypergraph H, we have $\alpha_1(H) \leq \alpha(H) - r + 2$.*

Proof. Suppose that S is strongly independent in H. Take any $e \in E(H)$ (it satisfies $|S \cap e| \leq 1$ by the definition of S) and any $v \in e \setminus S$. Then the set $T = (S \cup e) \setminus \{v\}$ is independent and $|T| \geq |S| + r - 2$. □

We need the simplest extension of connectivity from graphs to hypergraphs (no topology involved). A hyperwalk in H is a sequence $v_1, e_1, v_2, e_2, \ldots, v_{t-1}, e_{t-1}, v_t$, where for all $1 \leq i < t$

we have $v_i \in e_i$ and $v_{i+1} \in e_i$. We say that $v \sim w$, if there is a hyperwalk from v to w. The relation $\sim$ is an equivalence relation, and the subhypergraphs induced by its classes are called the *connected components* of the hypergraph H. A vertex v that is not covered by any edge forms a trivial component with one vertex v and no edge. The vertex sets of the connected components of a hypergraph H coincide with the vertex sets of the connected components of G_H.

Let H be an edge-colored hypergraph. For a subset S of $V(H)$, the subhypergraph induced by S in H, that is the hypergraph on the vertex set S with edge set $\{e \in E(H) \mid e \subseteq S\}$, is denoted by $H[S]$. A vertex partition $\mathcal{P} = \{V_1, \ldots, V_l\}$ of $V(H)$ is called a *connected partition* if every $H[V_i]$ $(1 \leq i \leq l)$ is connected in some color. Similarly, changing partition to cover, we can define *connected cover* for every edge-colored hypergraph. (Note that, the subsets of the monochromatic connected components of a hypergraph not necessary can be used as parts of a connected partition.) Since partition into vertices is always a connected partition, we can define $cp(H), cc(H)$ for any edge-colored hypergraph H as the minimum number of classes in a connected partition or connected cover, respectively. Observe that for trivial hypergraphs $cc(H) = cp(H) = \alpha(H) = |V(H)|$.

First we will prove the following statement on coverings.

Theorem 2. *For any 2-edge-colored hypergraph H, we have $cc(H) \leq \alpha_1(H)$.*

In fact, the benefit of introducing the concept of $\alpha_1(H)$ is to provide an upper bound on $cc(H)$ in terms of $\alpha(H)$. From Proposition 1 one also gets the following important corollary:

Corollary 3. *For any 2-edge-colored non-trivial r-uniform hypergraph H, we have $cc(H) \leq \alpha(H) - r + 2$.*

One of our main results is the strengthening of Corollary 3 for partitions.

Theorem 4. *For any 2-edge-colored non-trivial r-uniform hypergraph H, we have $cp(H) \leq \alpha(H) - r + 2$.*

The previous results are sharp. To see this, consider the union of one complete r-uniform hypergraph and several isolated vertices. Observe that, the partition version of Theorem 2 does not hold. For example, for the hypergraph H having two edges of size r intersecting in one vertex, one red and one blue, we have $cc(H) = 2$ and $cp(H) = r (= \alpha(H) - r + 2)$.

It is worth noting that for $r = 2$ Theorem 4 extends the $k = 2$ case of Conjecture 1. Now we have the following general property for 2-edge-colored graphs.

Theorem 5. *Any 2-edge-colored graph G can be partitioned into $\alpha(G)$ monochromatic connected parts.*

An edge-coloring of a graph is called a *Gallai-coloring* if there is no rainbow triangle in it, i.e. every triangle is colored by at most two colors. Gallai-colorings are natural extensions of

2-colorings and have been recently investigated in many papers (for references see [6]). It is known that, any Gallai-colored complete graph has a monochromatic spanning tree (see e.g. [7]). So we have $cp(G) = cc(G) = 1$ if G is a Gallai-colored complete graph. Now we focus on Gallai-colored general graphs. Our result is the following:

Theorem 6. *Let G be a Gallai-colored graph with $\alpha(G) = \alpha$. Then, with a suitable function $g(\alpha)$, we have $cp(G) \leq g(\alpha)$.*

Theorem 6 extends the result proved by Gyárfás, Simonyi and Tóth [8] that in any Gallai coloring of a graph G, $cc(G)$ is bounded in terms of $\alpha(G)$. We shall also improve on a result in [8] about dominating sets of multipartite digraphs.

2 Partitions of 2-edge-colored hypergraphs, proof of Theorem 4

We first prove the cover version.

Proof of Theorem 2. Let H be a hypergraph 2-edge-colored with red and blue. For every vertex $v \in V(H)$ let $R(v)$, $B(v)$ denote the monochromatic connected components containing v in the hypergraphs of the red and blue edges, respectively. (One or both can be a single component containing v.)

From H we construct a bipartite graph $\mathcal{G}$ with bipartition $V(\mathcal{G}) = (\mathcal{R}, \mathcal{B})$, where $\mathcal{R} = \{R(v)|v \in V(H)\}$, $\mathcal{B} = \{B(v)|v \in V(H)\}$ and with edge set $E(\mathcal{G}) = \{R(v)B(v)|v \in V(H)\}$. By the construction, note that $|E(\mathcal{G})| = |V(H)|$ and $\mathcal{G}$ may contain multiple edges. Also we can regard an edge in $E(\mathcal{G})$ as a vertex in H.

Notice that for any two independent edges $e = R(v)B(v)$, $e' = R(u)B(u) \in E(\mathcal{G})$, there is no monochromatic connected component containing v and u, and hence there is no edge in H containing both v and u. Therefore the maximum number of independent edges in $\mathcal{G}$, $\nu(\mathcal{G})$, satisfies $\nu(\mathcal{G}) \leq \alpha_1(H)$.

By König's theorem, the edges of $\mathcal{G}$ have a transversal of $\nu(\mathcal{G})$ vertices, i.e., there is a subset $T \subseteq V(\mathcal{G})$ such that $|T| = \nu(\mathcal{G})$ and T intersects all edges of $\mathcal{G}$ in at least one vertex. Then the monochromatic components of H corresponding to the vertices of T form a desired covering of $V(H)$. □

Remark. Conjecture 1 for $k = 2$ (its proof is implicitely in [5, 7]) implies Theorem 2 directly as follows. The cover graph G_H of H can be covered by $\alpha(G_H) = \alpha_1(H)$ monochromatic connected components and so $cc(H) \leq \alpha_1(H)$ also holds.

Next, we turn to the proof of the partition version.

Proof of Theorem 4. Let H be a non-trivial r-uniform hypergraph with independence number $\alpha(H)$. The proof goes by induction on $\alpha(H)$. In the base case, when $\alpha(H) = r - 1$, i.e.

H is a 2-edge-colored complete r-uniform hypergraph, it follows from Corollary 3 that one monochromatic component covers the vertices.

Suppose $\alpha(H) > r - 1$. By Corollary 3, $V(H)$ can be covered by the vertices of p red components, $R_1, \ldots, R_p$, and q blue components, $B_1, \ldots, B_q$, so that

$$p + q \leq \alpha(H) - r + 2. \tag{1}$$

We may assume that p, q are both positive, since if one of them is zero, we already have the desired partition in the other color. Set $R = (\bigcup_{1\leq i\leq p} R_i) \setminus (\bigcup_{1\leq i\leq q} B_i)$ and $B = (\bigcup_{1\leq i\leq q} B_i) \setminus (\bigcup_{1\leq i\leq p} R_i)$. If R or B is empty, we have again the required partition. Thus we may assume that both R and B are non-empty, so $\alpha(H[R]) \geq 1$, and $\alpha(H[B]) \geq 1$. Observe that

$$\alpha(H[R]) + \alpha(H[B]) \leq \alpha(H) \tag{2}$$

since no edge of H can meet both R and B. Therefore $\alpha(H[B]) \leq \alpha(H) - 1$ and $\alpha(H[R]) \leq \alpha(H) - 1$. If $H[R]$ is non-trivial, then $cp(H[R]) \leq \alpha(H[R]) - r + 2$ by the inductive hypothesis, but if $H[R]$ is trivial then $cp(H[R]) = |R| = \alpha(H[R])$. Similarly, if $H[B]$ is non-trivial, then $cp(H[B]) \leq \alpha(H[B]) - r + 2$, if $H[B]$ is trivial then $cp(H[B]) = \alpha(H[B])$.

Case 1. $H[R]$ is non-trivial (and $H[B]$ is either non-trivial or trivial).

Thus R (the vertex set of $H[R]$) has a connected partition $\mathcal{P}_R$ into at most $\alpha(H[R]) - r + 2$ parts. The set B (the vertex set of $H[B]$) has a connected partition $\mathcal{P}_B$ into at most $\alpha(H[B])$ parts. Hence $\mathcal{P}_R \cup \{B_1, \ldots, B_q\}$ and $\mathcal{P}_B \cup \{R_1, \ldots, R_p\}$ are two connected partitions on $V(H)$. Using (1),(2) we have

$$(|\mathcal{P}_R| + q) + (|\mathcal{P}_B| + p) \leq (\alpha(H[R]) - r + 2) + \alpha(H[B]) + p + q \leq 2(\alpha(H) - r + 2),$$

therefore one of the previous connected partitions has at most $\alpha(H) - r + 2$ parts, as desired.

The case when $H[B]$ is non-trivial goes similarly.

Case 2. $H[R]$ and $H[B]$ are both trivial.

Assume $p \geq q$, and select a vertex v from R, without loss of generality $v \in R_p$. Observe that no blue edge contains v, because $H[R]$ is trivial. Hence every edge containing v is in R_p, implying that $\alpha(H \setminus R_p) \leq \alpha(H) - 1$. If $p > 1$ then $H \setminus R_p$ is non-trivial, thus by induction $H \setminus R_p$ has a connected partition with at most $(\alpha(H) - 1) - r + 2$ parts, adding R_p we obtain the required partition for H. We conclude $p = q = 1$.

Let S be a maximal (non-extendable) independent set of H in the form $R \cup B \cup M$. By definition of S (and as H is non-trivial) there exists a hyperedge intersecting $M \cup R$ or $M \cup B$ in exactly $r - 1$ vertices (since no edge can intersect both R and B), assume the former. Therefore $r \leq |M| + |R| + 1$, this yields

$$\alpha(H) - r + 2 \geq |S| - r + 2 = |R| + |B| + |M| - r + 2 \geq |R| + |B| + |M| - (|M| + |R| + 1) + 2 = |B| + 1,$$

thus the red component, R_1 and vertices of B gives a partition of H into at most $\alpha(H) - r + 2$ connected parts. □

3 Partitions of Gallai-colored graphs, proof of Theorem 6

We need some notions introduced in [8]. If D is a digraph and $U \subseteq V(D)$ is a subset of its vertex set then $N_+(U) = \{v \in V(D) | \exists u \in U\ (u,v) \in E(D)\}$ is the *outneighborhood* of U. A multipartite digraph is a digraph D whose vertices are partitioned into classes $A_1, \dots, A_t$ of independent vertices. Let $S \subseteq [t]$. A set $U = \cup_{i \in S} A_i$ is called a *dominating set* of size $|S|$ if for any vertex $v \in \cup_{i \notin S} A_i$ there is a $w \in U$ such that $(w,v) \in E(D)$. The smallest $|S|$ for which a multipartite digraph D has a dominating set $U = \cup_{i \in S} A_i$ is denoted by $k(D)$. Let $\beta(D)$ be the cardinality of the largest independent set of D whose vertices are from different partite classes of D. (We sometimes refer to them as *transversal independent sets*.) An important special case is when $|A_i| = 1$ for each $i \in [t]$. Then it follows that $\beta(D) = \alpha(D)$ and $k(D) = \gamma(D)$, the usual domination number of D, the smallest number of vertices in D whose closed outneighborhoods cover $V(D)$. In [8], the followings are shown:

Theorem 7 ([8]). *Suppose that D is a multipartite digraph such that D has no cyclic triangle. If $\beta(D) = 1$ then $k(D) = 1$ and if $\beta(D) = 2$ then $k(D) \le 4$.*

Theorem 8 ([8]). *For every integer β there exists an integer $h = h(\beta)$ such that the following holds. If D is a multipartite digraph without cyclic triangles and $\beta(D) = \beta$, then $k(D) \le h$.*

To keep the paper self-contained we give a proof for this statement with a slightly better bound than the one presented in [8].

Proof of Theorem 8. Set $h(1) = 1$, $h(2) = 4$ and $h(\beta) = \beta + (\beta+1)h(\beta-1)$ for $\beta \ge 3$. The proof goes by induction on β. By Theorem 7, we may assume that $\beta \ge 3$ and the theorem is proved for $\beta - 1$. Let D be a multipartite digraph with no cyclic triangle and $\beta(D) = \beta$. For each $x \in V(D)$, let $Z^{(x)}$ be the partite class containing x. Let $k_1, \dots, k_\beta$ be β vertices of D, each from a different partite class, such that $|N_+(\{k_1, \dots, k_\beta\}) \cup (\bigcup_{1 \le i \le \beta} Z^{(k_i)})|$ is maximal. Let $\mathcal{K}_1 = \{Z^{(k_i)} \mid 1 \le i \le \beta\}$. For each partite class $Z \notin \mathcal{K}_1$, let $Z_0 = Z \cap N_+(\bigcup_{1 \le i \le \beta} Z^{(k_i)})$. For every i with $1 \le i \le \beta$, let Z_i be the set of vertices in $Z \setminus Z_0$ that are not sending an edge to k_i, but sending an edge to k_j for all $j < i$. Finally, let $Z_{\beta+1}$ denote the remaining part of Z, the set of those vertices of Z that does not belong to $N_+(\bigcup_{1 \le i \le \beta} Z^{(k_i)})$ and send an edge to all vertices $k_1, \dots, k_\beta$. (We will refer to the set Z_i as the i-th part of Z.) The subgraph D_i of D induced by the i-th parts of the partite classes of $D \setminus (\bigcup_{1 \le i \le \beta} Z^{(k_i)})$ is also a multipartite digraph with no cyclic triangle. For every i with $1 \le i \le \beta$, since adding k_i to any transversal independent set of D_i we get a larger transversal independent set, it satisfies $\beta(D_i) \le \beta - 1$.

Suppose that $\beta(D_{\beta+1}) \ge \beta$. Let $\{l_1, \dots, l_\beta\}$ be a transversal independent set of $D_{\beta+1}$.

Claim. *For every $x \in \left(N_+(\{k_1, \dots, k_\beta\}) \cup (\bigcup_{1 \le i \le \beta} Z^{(k_i)})\right) \setminus (\bigcup_{1 \le i \le \beta} Z^{(l_i)})$, we have $x \in N_+(\{l_1, \dots, l_\beta\})$.*

Proof. Suppose that $x \in N_+(\{k_1,\dots,k_\beta\})\backslash\bigcup_{1\le i\le\beta} Z^{(l_i)}$. Then there exists an integer $1 \le i_0 \le \beta$ such that $(k_{i_0}, x) \in E(D)$. Recall that $(l_i, k_{i_0}) \in E(D)$ for every $1 \le i \le \beta$. Since $\{x, l_1,\dots,l_\beta\}$ is not independent and D has no cyclic triangle, $x \in N_+(\{l_1,\dots,l_\beta\})$, as desired. Thus we may assume that $x \in \bigcup_{1\le i\le\beta} Z^{(k_i)}$. Recall that $(x, l_i) \notin E(D)$ for every $1 \le i \le \beta$. Since $\{x, l_1,\dots,l_\beta\}$ is not independent, $x \in N_+(\{l_1,\dots,l_\beta\})$. $\square$

Thus we have $N_+(\{k_1,\dots,k_\beta\}) \cup (\bigcup_{1\le i\le\beta} Z^{(k_i)}) \subseteq N_+(\{l_1,\dots,l_\beta\}) \cup (\bigcup_{1\le i\le\beta} Z^{(l_i)})$. Since $l_1 \in \big(N_+(\{l_1,\dots,l_\beta\}) \cup (\bigcup_{1\le i\le\beta} Z^{(l_i)})\big) \setminus \big(N_+(\{k_1,\dots,k_\beta\}) \cup (\bigcup_{1\le i\le\beta} Z^{(k_i)})\big)$, it follows

$$\left|N_+(\{k_1,\dots,k_\beta\}) \cup \left(\bigcup_{1\le i\le\beta} Z^{(k_i)}\right)\right| < \left|N_+(\{l_1,\dots,l_\beta\}) \cup \left(\bigcup_{1\le i\le\beta} Z^{(l_i)}\right)\right|,$$

which contradicts the choice of $k_1,\dots,k_\beta$. Thus $\beta(D_{\beta+1}) \le \beta - 1$.

By induction on β, D_i $(1 \le i \le \beta+1)$ can be dominated by at most $h(\beta-1)$ partite classes. Let $\mathcal{K}_2$ be the appropriate $(\beta+1)h(\beta-1)$ partite classes such that $\bigcup_{Z\in\mathcal{K}_2} Z$ dominates $\bigcup_{1\le i\le\beta+1} V(D_i)$. Hence we constructed a dominating set $\bigcup_{Z\in\mathcal{K}_1\cup\mathcal{K}_2} Z$ of D containing at most $\beta + (\beta+1)h(\beta-1)$ partite classes.

This completes the proof of Theorem 8. $\square$

To prepare the proof of Theorem 6 we need the following lemma about trees.

Lemma 9. *Let $t \ge 1$ be an integer. Let T be a tree of order at least t. Then there exist two set $R \subseteq C \subseteq V(T)$ such that $|R| = t$, $|C| \le 2t$, $T[C]$ is connected, and either $T \setminus R$ is connected or $V(T) = R$.*

Proof. If $|V(T)| = t$, then the lemma holds by choosing $R = C = V(T)$. Thus we may assume that $|V(T)| \ge t+1$. For each edge $xy \in E(T)$, let T^x_{xy} denote the component of $T\setminus xy$ containing x. Note that $|\{x\} \cup (\bigcup_{y\in N(x)} V(T^y_{xy}))| = |V(T)| \ge t+1$ for every $x \in V(T)$. We choose a vertex $x_0 \in V(T)$ and a subset $A_0 \subseteq N(x_0)$ such that

(i) $|\{x_0\} \cup (\bigcup_{y\in A_0} V(T^y_{x_0y}))| \ge t+1$, and

(ii) subject to (i), $|\{x_0\} \cup (\bigcup_{y\in A_0} V(T^y_{x_0y}))|$ is minimized.

By the definition of x_0 and A_0, we have $A_0 \ne \emptyset$. Set $a = |\{x_0\} \cup (\bigcup_{y\in A_0} V(T^y_{x_0y}))|$.

Claim. *$a \le 2t$.*

Proof. Suppose that $a \ge 2t+1$. If $|A_0| = 1$, say $A_0 = \{y_0\}$, then $|\{y_0\}\cup(\bigcup_{y\in N(y_0)\setminus\{x_0\}} V(T^y_{y_0y}))| = a-1 (\ge t+1)$, which contradicts the definition of x_0 and A_0. Thus $|A_0| \ge 2$. Then there exists a vertex $y_1 \in A_0$ such that $|V(T^{y_1}_{x_0y_1})| \le (a-1)/2$. Hence

$$|\{x_0\} \cup \Big(\bigcup_{y\in A_0\setminus\{y_1\}} V(T^y_{x_0y})\Big)| = a - |V(T^{y_1}_{x_0y_1})| \ge a - \frac{a-1}{2} = \frac{a+1}{2} \ge \frac{2t+2}{2} = t+1,$$

which contradicts the definition of A_0. $\square$

Write $\bigcup_{y \in A_0} V(T^y_{x_0 y}) = \{x_1, \ldots, x_{a-1}\}$, we may assume that the elements of this set are ordered in a non-increasing order by the distance from x_0. Let $C = \{x_0\} \cup (\bigcup_{y \in A_0} V(T^y_{x_0 y}))$ and $R = \{x_i \mid 1 \leq i \leq t\}$. Then $|R| = t$, $|C| \leq 2t$ and both $T[C]$ and $T \setminus R$ are connected. $\square$

Now we are ready to prove Theorem 6. Let $g(1) = 1$ and $g(\alpha) = \max\{h(\alpha)(\alpha^2 + \alpha - 1), 2h(\alpha)g(\alpha - 1) + h(\alpha) + 1\}$ for $\alpha \geq 2$.

Proof of Theorem 6. We show that $cp(G) \leq g(\alpha(G))$ with the function g defined above. We may assume that $|V(G)| \geq g(\alpha)$. We proceed by induction on α. If $\alpha = 1$, then G is complete, and hence there is a connected monochromatic spanning subgraph of G, as desired. Thus we may assume that $\alpha \geq 2$. Let T_0 be a maximum connected spanning monochromatic subtree of G in the coloring c. We may assume that every edge of T_0 has color 1. It was proved in [7] that the largest monochromatic subtree in every Gallai-coloring of a graph G has at least $|V(G)|(\alpha^2 + \alpha - 1)^{-1}$ vertices. Using this, since $|V(G)| \geq g(\alpha) \geq h(\alpha)(\alpha^2 + \alpha - 1)$, $|V(T_0)| \geq h(\alpha)$ follows. By Lemma 9, there exist two sets R and C with $R \subseteq C \subseteq V(T_0)$ such that $|R| = h(\alpha)$, $|C| \leq 2h(\alpha)$, $T_0[C]$ is connected, and either $T_0 \setminus R$ is connected or $V(T_0) = R$. Write $C = \{u_1, \ldots, u_m\}$. Note that $h(\alpha) \leq m \leq 2h(\alpha)$. We may assume that $R = \{u_1, \ldots, u_{h(\alpha)}\}$. For every i with $1 \leq i \leq m$, let U_i be the set of vertices in $V(G) \setminus V(T_0)$ that are not adjacent to u_i, but adjacent to u_j for all $j < i$. For every i with $1 \leq i \leq m$, we have $\alpha(G[U_i]) \leq \alpha - 1$ because adding u_i to any independent set of $G[U_i]$ we get a larger independent set. By the inductive assumption, for every i with $1 \leq i \leq m$, there exists a partition $\mathcal{P}_i$ of U_i such that $|\mathcal{P}_i| \leq g(\alpha - 1)$ and, for every $U \in \mathcal{P}_i$, $G[U]$ has a connected spanning monochromatic subgraph concerning c.

Let $U_0 = V(G) \setminus \left(V(T_0) \cup \left(\bigcup_{1 \leq i \leq m} U_i\right)\right)$. Recall that $T_0[C]$ is a connected monochromatic tree and c is a Gallai-coloring of G. For every $v \in U_0$, since v is adjacent to every vertex of C, all of $E(v, C)$ are colored with the same color, say c_v. Note that $c_v \neq 1$ for every $v \in U_0$ by the definition of T_0. Let l be the number of colors used on edges of $E(U_0, C)$. We may assume that $2, \ldots, l + 1$ are the colors used on these edges. For each i with $2 \leq i \leq l + 1$, $A_i = \{v \in U_0 \mid c_v = i\}$. Note that $\{A_2, \ldots, A_{l+1}\}$ is a partition of U_0. Since c is a Gallai coloring of G, each edge between A_i and A_j is colored with either color i or j for i, j with $2 \leq i, j \leq l + 1$ and $i \neq j$.

We construct the multipartite digraph D on U_0 as follows:

(i) $A_2, \ldots, A_{l+1}$ are the partition classes of D.

(ii) For i, j with $2 \leq i, j \leq l + 1$ and $i \neq j$, $v \in A_i$ and $v' \in A_j$, let $(v, v') \in E(D)$ if and only if $vv' \in E(G)$ and $c(vv') = i$.

Note that $\beta(D) \leq \alpha$ and D has no cyclic triangle. By Theorem 8, there exist at most $h(\alpha)$ partite classes dominating $V(D)$, say $B_1, \ldots, B_p$. Let $B_{p+1} = \cdots = B_{h(\alpha)} = \emptyset$. For every i with $1 \leq i \leq h(\alpha)$, let B'_i be the set of vertices in $U_0 \setminus \left(\bigcup_{1\leq i\leq h(\alpha)} B_i\right)$ that are dominated by B_i, but not dominated by B_j for all $j < i$, and let $B''_i = \{u_i\} \cup B_i \cup B'_i$. For each i with $1 \leq i \leq h(\alpha)$, note that $G[B''_i]$ has a connected monochromatic spanning subgraph. Therefore $\mathcal{P} = \{V(T_0) \setminus R, B''_1, \ldots, B''_{h(\alpha)}\} \cup \left(\bigcup_{1\leq i\leq m} \mathcal{P}_i\right)$ is a partition of $V(G)$ satisfying that $G[U]$ has a connected spanning monochromatic subgraph concerning c for every $U \in \mathcal{P}$. Furthermore,

$$\begin{aligned}
|\mathcal{P}| &\leq (h(\alpha)+1) + \sum_{1\leq i\leq m} |\mathcal{P}_i| \leq (h(\alpha)+1) + \sum_{1\leq i\leq m} g(\alpha-1) = \\
&= (h(\alpha)+1) + mg(\alpha-1) \leq (h(\alpha)+1) + 2h(\alpha)g(\alpha-1).
\end{aligned}$$

This completes the proof of Theorem 6. □

4 Conclusion, open problems

The quantities $cc(G), cp(G)$ can be far apart, even for 2-edge-colored graphs. For example, let G be a star with $2t$ edges and color t edges in both colors. Then $cc(G) = 2, cp(G) = t+1$. Nevertheless, the extension of Conjecture 1 to partitions of complete graphs have been formulated in [3]. Probably this remains true for Ryser's conjecture in general.

Conjecture 2. *If the edges of G are colored with k colors then $cp(G) \leq \alpha(G)(k-1)$.*

As mentioned before, Conjecture 2 is proved for $\alpha(G) = 1, k = 3$ in [3]. Note that $cc(G) \leq \alpha(G)k$ is obvious for any k-edge-colored graph G. For k-edge-colored complete graphs K, Haxell and Kohayakawa [10] proved $cp(K) \leq k$, this is just one off from Conjecture 2. It would be interesting to attack the case $k = 3$ in Conjecture 2 since its cover version, Conjecture 1 is available ([1]).

As mentioned in the introduction, Király [12] solved completely the cover problem for complete r-uniform complete hypergraphs ($r \geq 3$). (The number of colors k can be arbitrary.) It seems that the analogue for partition is not easy. A first test case might be the following.

Problem 3. *Suppose that a complete 3-uniform hypergraph H is 6-edge-colored. Is it true that $cp(H) \leq 2$? ($cc(H) \leq 2$.)*

In general, the cover problem of hypergraphs for general α or α_1 seems difficult, even to find the right conjecture is a challenge. We shall address this question in [4].

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