Cliques in C_4-free graphs of large minimum degree

András Gyárfás*

Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, P.O. Box 127
Budapest, Hungary, H-1364

gyarfas.andras@renyi.mta.hu

Gábor N. Sárközy †
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, P.O. Box 127
Budapest, Hungary, H-1364
and
Computer Science Department
Worcester Polytechnic Institute
Worcester, MA, USA 01609
gsarkozy@cs.wpi.edu

September 22, 2015

Abstract

A graph G is called C_4-free if it does not contain the cycle C_4 as an induced subgraph. Hubenko, Solymosi and the first author proved (answering a question of Erdős) a peculiar property of C_4-free graphs: C_4 graphs with n vertices and average degree at least cn contain a complete subgraph (clique) of size at least $c'n$ (with $c' = 0.1c^2n$). We prove here better bounds ($\frac{c^2n}{2+c}$ in general and $(c-1/3)n$ when $c \leq 0.733$) from the stronger assumption that the C_4-free graphs have minimum degree at least cn. Our main result is a theorem for regular graphs, conjectured in the paper mentioned above: $2k$-regular C_4-free graphs on $4k+1$ vertices contain a clique of size $k+1$. This is best possible shown by the k-th power of the cycle C_{4k+1}.

1 Introduction

A graph is called here C_4-free, if it does not contain cycles on four vertices as an induced subgraph. The class of C_4-free graphs have been studied from many points

*Research supported in part by the OTKA Grant No. K104343.

†Research supported in part by the National Science Foundation under Grant No. DMS-0968699 and by OTKA Grant No. K104343.

of view, for example they appear in the theory of perfect graphs (as families containing chordal graphs). Sometimes the complements of C_4-free graphs are investigated, they are the graphs that do not contain $2K_2$ as an induced subgraph, sometimes called a *strong matching* of size two. Extremal properties of these graphs emerged in works of Bermond, Bond, Pauli and Peck [1], [2] on interconnection networks, popularized by Erdős and Nesetril, and generated extremal results, many on the *strong chromatic index*, for example [3, 4, 5, 6, 7].

In this paper we revisit [5] where the the following problem (raised by Erdős) was investigated: how large is $\omega(G)$, the size of the largest complete subgraph (clique) in a dense C_4-free graph G? It was proved in [5] that in a C_4-free graph with n vertices and at least cn^2 edges, $\omega(G) \geq c'n$, where c' depends on c only. The interest in this result is that as shown in [5], C_4 is the only graph with this property (apart from subgraphs of C_4). Let $f(c)$ denote the largest c' for which every C_4-free graph with n vertices and at least cn^2 edges contains a clique of size at least $c'n$. There is no conjecture on $f(c)$, apart from the question in [5] whether $f(1/4) = 1/4$ which is still open. Our main result, Theorem 1 gives a positive answer to the the special case of this question for regular graphs (asked also in [5]).

Theorem 1. *Every $2k$-regular C_4-free graph on $4k+1$ vertices contains a clique of size $k+1$.*

As shown in [5], Theorem 1 is sharp, the cycle on $4k+1$ vertices with all diagonals of length at most k is a $2k$-regular C_4-free graph where the largest clique is of size $k+1$. The proof of Theorem 1 follows from understanding the work of Paoli, Peck, Trotter and West [7] on regular $2K_2$-free graphs.

Our other results are improvements over the estimates of [5] under the stronger assumption that the minimum degree $\delta(G)$ is given instead of the average degree.

Theorem 2. *For C_4-free graphs $\omega(G) \geq \frac{\delta^2(G)}{2n+\delta(G)}$.*

Theorem 2 improves the estimate $\omega(G) \geq \frac{0.1a^2}{n}$ in [5] where a is the average degree of G. For a certain range of $\delta(G)$, one can do better.

Theorem 3. *Suppose that G is a C_4-free graph with $\delta(G) \leq \frac{11n}{15} \approx 0.733n$. Then $\omega(G) \geq \delta(G) - \frac{n}{3}$.*

Note that for $\delta(G) \geq n/2$, Theorem 2 gives $\omega(G) \geq n/12$ while Theorem 3 gives $\omega(G) \geq n/6$. It seems that the remark "the best estimate we know is $n/6$" in [5] comes from this and it seems an open problem whether $\omega(G) \geq n/6$ follows from $|E(G)| \geq n^2/4$. We also note that for $0.382n \approx \frac{2n}{3+\sqrt{5}} \leq \delta(G)$ the bound of Theorem 3 is better than that of Theorem 2.

Our last estimate of $\omega(G)$ is for the case when G has a large independent set.

Theorem 4. *For every $\varepsilon > 0$ the following holds. Let G be a C_4-free graph on n vertices with minimum degree at least δ. Furthermore, let us assume that G contains an independent set of size $t \geq \frac{n^2-\delta^2}{\varepsilon d^2} + 1$. Then G contains a clique of size at least $(1-\varepsilon)\delta^2/n$.*

Thus we get the following corollary for Dirac graphs (graphs with minimum degree at least $n/2$).

Corollary 5. *For every $\varepsilon > 0$ the following holds. Let G be a C_4-free graph on n vertices with minimum degree at least $n/2$. Furthermore, let us assume that G contains an independent set of size $t \geq \frac{3}{\varepsilon} + 1$. Then G contains a clique of size at least $(1-\varepsilon)n/4$.*

Corollary 5 probably holds in a stronger form: C_4-free graphs with n vertices and with minimum degree at least $n/2$ contain cliques of size at least $n/4$.

2 Properties of C_4-free graphs

The following easy lemma can be essentially found in [3, 4, 7] but we prove it to be self contained. Let W_5 denote the 5-wheel, the graph obtained from a five-cycle by adding a new vertex adjacent to all vertices. A *clique substitution* into a graph G is the replacement of cliques into vertices of G so that between substituted vertices all or none of the edges are placed, depending whether they were adjacent or not in G. Substituting an empty clique is accepted as a deletion of the vertex. Clique substitutions into C_4-free graphs result in C_4-free graphs.

Lemma 6. *Suppose that G is a C_4-free graph with $\alpha(G) \leq 2$. Then one of the following possibilities holds.*

- *the complement of G is bipartite*
- *G can be obtained from W_5 by clique substitution*

Proof. If $\overline{G}$, the complement of G is not bipartite then we can find an odd cycle C in $\overline{G}$. Since C cannot be a triangle, $|C| \geq 5$. However, $|C| \geq 7$ is impossible since G is C_4-free. Thus $|C| = 5$. Since G is C_4-free and $\alpha(G) = 2$, any vertex not on C must be adjacent to exactly three consecutive vertices of C or to all vertices of C. This procedure naturally allows to place all vertices not on C into one of six groups and one can easily check that the groups must be cliques forming the claimed structure. $\square$

Corollary 7. *Suppose that G is a C_4-free graph with $\alpha(G) \leq 2$. Then $\omega(G) \geq \frac{2n}{5}$.*

In the proof of Theorem 1 we shall use the following result which is a special case of a more general result on regular C_4-free graphs (in [7] Theorem 4 and Lemma 7). A set $S \subset V(G)$ is *dominating* if every vertex of $V(G) \setminus S$ is adjacent to some vertex of S.

Theorem 8. *(Paoli, Peck, Trotter, West [7], (1992)) Suppose that G is a $2k$-regular C_4-free graph on $4k+1$ vertices with $\alpha(G) \geq 3$. Then G contains a pair (u, w) of non-adjacent vertices forming a dominating set.*

3 Proofs

Proof of Theorem 1. The proof comes from Theorem 8 and the analysis of Theorem 3 in [7]. We may suppose that $\alpha(G) \geq 3$, otherwise Corollary 7 gives a clique of size $\frac{8k+2}{5} \geq k+1$. Theorem 8 ensures a dominating non-adjacent pair (u, w) in G. Let X be the set of common neighbors of u, v. Then

$$4k - |X| = d(u) + d(w) - |X| = |V(G)| - 2 = 4k - 1,$$

implying that $|X| = 1$. Set $X = \{x\}$, $U = N(u) - \{x\}$, $W = N(w) - \{x\}$, $U_1 = N(x) \cap U$, $W_1 = N(x) \cap W$, $U_2 = U - U_1$, $W_2 = W - W_1$.
Claim. U_1, W_1 span cliques in G.
Proof of Claim. By symmetry, it is enough to prove the claim for U_1. Note that for $w_2 \in W_2, u_1 \in U_1$ we have $(w_2, u_1) \notin E(G)$ otherwise (w_2, u_1, x, w, w_2) would be an induced C_4.

Suppose that $y, z \in U_1$ and $(y, z) \notin E(G)$. Let N be the number of non-adjacent pairs (p, q) such that $p \in \{y, z\}, q \notin U_1$.

- every $w_1 \in W_1$ contributes at least one to N, otherwise (w_1, y, u, z, w_1) is a C_4
- every $u_2 \in U_2$ contributes at least one to N, otherwise (u_2, y, x, z, u_2) is a C_4
- every $w_2 \in W_2$ contributes two to N since $(w_2, u_1) \notin E(G)$ for every $u_1 \in U_1$
- w contributes two to N

Therefore we have

$$N \geq |W_1| + |U_2| + 2|W_2| + 2 = (|W_1| + |W_2|) + (|U_2| + |W_2|) + 2 = (2k-1) + 2k + 2 = 4k+1.$$

However, since $(y, z) \notin E(G)$, $N \leq 2(d_{\overline{G}}(y) - 1) = 2(2k-1) = 4k-2$, a contradiction, proving that U_1 spans a clique in G and the claim is proved. $\square$

Now the two cliques $U_1 \cup \{u, x\}$ and $W_1 \cup \{w, x\}$ cover $A = V(G) \setminus (U_2 \cup W_2)$. Since $|A| = 4k + 1 - 2k = 2k + 1$ and the two cliques intersect in $\{x\}$, one of the cliques has size at least $k + 1$, finishing the proof. $\square$

Proof of Theorem 2. Here we follow the proof of the corresponding theorem in [5] with replacing average degree by minimum degree. Fix an independent set $S = \{x_1, x_2, \ldots, x_t\}$. Let A_i be the set of neighbors of x_i in G and set $m = \max_{i \neq j} |A_i \cap A_j|$. Since G is C_4-free, all the subgraphs $G(A_i \cap A_j)$ are complete graphs, and thus $m \leq \omega(G)$. Using that $|A_i| \geq \delta$, we get

$$t\delta \leq \sum_{i=1}^{t} |A_i| < n + \sum_{1 \leq i < j \leq t} |A_i \cap A_j|,$$

implying that

$$\omega(G) \geq m \geq \frac{t\delta - n}{\binom{t}{2}}.$$

If $\alpha(G) \geq \frac{2n}{\delta}$ then set $t = \lceil \frac{2n}{\delta} \rceil$ and we get

$$\omega(G) \geq \frac{\lceil \frac{2n}{\delta} \rceil \delta - n}{\binom{\lceil \frac{2n}{\delta} \rceil}{2}} \geq \frac{n}{\binom{\lfloor \frac{2n}{\delta} \rfloor + 1}{2}}.$$

If $\alpha(G) \leq \frac{2n}{\delta}$ then of course $\alpha(G) \leq \lfloor \frac{2n}{\delta} \rfloor$ as well. Now we shall use the following claim: $\omega(G) \geq \frac{n}{\binom{\alpha(G)+1}{2}}$. This follows by selecting an independent set S with $|S| = \alpha(G) = \alpha$. Using the notation introduced above, the $\binom{\alpha}{2}$ sets $A_i \cap A_j$ and the α sets $\{x_i\} \cup B_i$ cover the vertex set of G where B_i denotes the set of vertices whose only neighbor in S is x_i. All of these sets span complete subgraphs because G is C_4-free and S is maximal. Now we have

$$\omega(G) \geq \frac{n}{\binom{\alpha(G)+1}{2}} \geq \frac{n}{\binom{\lfloor \frac{2n}{\delta} \rfloor + 1}{2}}.$$

Therefore in both cases we have

$$\omega(G) \geq \frac{n}{\binom{\lfloor \frac{2n}{\delta} \rfloor + 1}{2}} \geq \frac{n}{\binom{\frac{2n}{\delta} + 1}{2}} = \frac{\delta^2}{2n + \delta}.$$

$\square$

Proof of Theorem 3. If $\alpha(G) \leq 2$ then by Lemma 6 and by the upper bound on $\delta(G)$,

$$\omega(G) \geq \frac{2n}{5} \geq \delta(G) - \frac{n}{3}.$$

If $\alpha(G) \geq 3$, then select an independent set $\{v_1, v_2, v_3\}$ and let A_i denote the set of neighbors of x_i. Then

$$3\delta(G) \leq \sum_{i=1}^{3} |A_i| < n + \sum_{1\leq i<j\leq 3} |A_i \cap A_j|,$$

implying that for some $1 \leq i < j \leq 3$, the clique induced by $A_i \cap A_j$ is larger than $\delta(G) - \frac{n}{3}$. □

Proof of Theorem 4. Let $S = \{x_1, x_2, \ldots, x_t\}$ be an independent set in G of size $t \geq \frac{n^2-d^2}{\varepsilon d^2} + 1$. Let A_i be the set of neighbors of x_i in G. Note that being induced C_4-free implies that for every $i, j, i \neq j$ the set $A_i \cap A_j$ induces a clique in G. Thus if we show that there are $i, j, i \neq j$ such that $|A_i \cap A_j| \geq (1-\varepsilon)d^2/n$, then we are done. Assume indirectly, that for every $i, j, i \neq j$ we have $|A_i \cap A_j| < (1-\varepsilon)d^2/n$ and from this we will get a contradiction.

Consider an auxiliary bipartite graph G_b between the sets S and $V = V(G)$, where we connect each x_i with its neighbors in G. We will give both a lower and an upper bound for the quantity $\sum_{v\in V} deg_{G_b}(v)^2$. To get a lower bound we apply the Cauchy-Schwarz inequality and the minimum degree condition:

$$\sum_{v\in V} deg_{G_b}(v)^2 \geq n\left(\frac{\sum_{v\in V} deg_{G_b}(v)}{n}\right)^2 = n\left(\frac{\sum_{i=1}^{t}|A_i|}{n}\right)^2 \geq n\left(\frac{td}{n}\right)^2 = \frac{t^2d^2}{n}.$$

To get the upper bound we use the indirect assumption:

$$\sum_{v\in V} deg_{G_b}(v)^2 = \sum_{i=1}^{t}\sum_{j=1}^{t} |A_i \cap A_j| = \sum_{i=1}^{t} |A_i| + \sum_{i\neq j} |A_i \cap A_j| <$$

$$< nt + (1-\varepsilon)\frac{d^2t(t-1)}{n} = \frac{t^2d^2}{n} + nt - \frac{d^2t}{n} - \varepsilon\frac{d^2t(t-1)}{n} \leq \frac{t^2d^2}{n}$$

(using $t \geq \frac{n^2-d^2}{\varepsilon d^2} + 1$), a contradiction. □

Acknowledgment. The authors are grateful to József Solymosi for conversations and to Xing Peng for his interest in the subject.

References

[1] J. C. Bermond, J. Bond, M. Paoli, C. Peyrat, Graphs and interconnection networks: diameter and vulnerability, *Surveys in Combinatorics, London Math. Soc. Lecture Notes* **82** (1983) 1-29

[2] J. C. Bermond, J. Bond, C. Peyrat, Bus interconnection networks with each station on two buses, *Proc. Coll. Int. Alg. et Arch. Paralleles (Marseilles)* (North Holland, 1986) 155-167

[3] F. R. K. Chung, A. Gyárfás, W. T. Trotter, Zs. Tuza, The maximum number of edges in $2K_2$-free graphs of bounded degree, *Discrete Mathematics* **81** (1990) 129-135

[4] R. J. Faudree, A. Gyárfás, R. H. Schelp, Zs. Tuza, The strong chromatic index of graphs, *Ars Combinatoria* **29 B** (1990) 205-211

[5] A. Gyárfás, A. Hubenko, J. Solymosi, Large cliques in C_4-free graphs, *Combinatorica* **22** (2002) 269-274

[6] M. Molloy, B. Reed, A Bound on the Strong Chromatic Index of a Graphs, *Journal of Combinatorial Theory, Series B* **69** (1997), 103-109

[7] M. Paoli, G. W. Peck, W. T. Trotter, D. B. West, Large regular graphs with no induced $2K_2$, *Graphs and Combinatorics* **8** (1992) 165-192