Rainbow matchings in bipartite multigraphs

János Barát
MTA-ELTE Geometric and Algebraic Combinatorics Research Group
barat@cs.elte.hu

András Gyárfás*
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, P.O. Box 127
Budapest, Hungary, H-1364
gyarfas.andras@renyi.mta.hu

Gábor N. Sárközy†
Alfréd Rényi Institute of Mathematics
Hungarian Academy of Sciences
Budapest, P.O. Box 127
Budapest, Hungary, H-1364
sarkozy.gabor@renyi.mta.hu
and
Computer Science Department
Worcester Polytechnic Institute
Worcester, MA, USA 01609
gsarkozy@cs.wpi.edu

January 11, 2016

Abstract

Suppose that k is a non-negative integer and a bipartite multigraph G is the union of

$$N = \left\lfloor \frac{k+2}{k+1} n \right\rfloor - (k+1)$$

matchings $M_1, \dots, M_N$, each of size n. We show that G has a rainbow matching of size $n-k$, i.e. a matching of size $n-k$ with all edges coming from different M_i's. Several choices of the parameter k relate to known results and conjectures.

Suppose that a multigraph G is given with a proper N-edge coloring, i.e. the edge set of G is the union of N matchings $M_1, \dots, M_N$. A *rainbow matching* is a matching whose edges are from different M_i's.

A well-known conjecture of Ryser [10] states that for odd n every 1-factorization of $K_{n,n}$ has a rainbow matching of size n. The companion conjecture, attributed to Brualdi [4] and Stein [12] states that for every n, every 1-factorization of $K_{n,n}$ has a rainbow matching of size at least $n-1$. These conjectures are known to be true in an asymptotic sense, i.e. every 1-factorization of $K_{n,n}$ has a rainbow matching containing $n - o(n)$ edges. For the $o(n)$ term, Woolbright [13] and independently Brouwer et al. [5] proved $\sqrt{n}$. Shor [11] improved this to $5.518(\log n)^2$, an error was corrected in [8].

*Research was supported in part by OTKA K104343.
†Research was supported in part by OTKA K104343.

There are several results for the case when $K_{n,n}$ is replaced by an arbitrary bipartite multigraph. The following conjecture of Aharoni et al. [3] strengthens the Brualdi-Stein conjecture.

Conjecture 1. *If a bipartite multigraph G is the union of n matchings of size n, then G contains a rainbow matching of size $n-1$.*

As a relaxation, Kotlar and Ziv [9] noticed that the union of n matchings of size $\frac{3}{2}n$ contains a rainbow matching of size $n-1$. Conjecture 1 would follow from another one posed by Aharoni and Berger:

Conjecture 2. *If a bipartite multigraph G is the union of n matchings of size $n+1$, then G contains a rainbow matching of size n.*

Recently, there has been gradual progress on this question. Aharoni et al. proved that matchings of size $\frac{7}{4}n$ suffice [3]. Kotlar and Ziv [9] improved it to $\frac{5}{3}n$ and Clemens and Ehrenmüller to $(\frac{3}{2}+\varepsilon)n$.

One needs a lot more matchings of size n to guarantee a rainbow matching of size n. Aharoni and Berger [2] and (in a slightly weaker form) Drisko [7] proved the following.

Theorem 1. *If a bipartite multigraph G is the union of $2n-1$ matchings of size n, then G contains a rainbow matching of size n.*

The (unique) factorization of a cycle on $2n$ vertices with edges of multiplicity $n-1$ shows that in the statement $2n-1$ cannot be replaced by $2n-2$ (see [7]). We merge Conjecture 1 and Theorem 1 into a unified context and ask the following. (We note that this question was also raised independently in [6].)

Question 1. *For integers $0 \le k < n$, what is the smallest $N = N(n,k)$ such that any bipartite multigraph G that is the union of N matchings of size n, contains a rainbow matching of size $n-k$?*

Conjecture 1 claims that $N(n,1) = n$ and Theorem 1 states that $N(n,0) = 2n-1$. In this note we give the following upper bound on $N(n,k)$.

Theorem 2. *For $0 \le k < n$, $N(n,k) \le \lfloor \frac{k+2}{k+1}n \rfloor - (k+1)$.*

In the range $\lfloor n/2 \rfloor \le k < n$ Theorem 2 gives $N(n,k) \le n-k$ which is obviously best possible, therefore $N(n,k) = n-k$. When $k=0$ it gives $N(n,0) \le 2n-1$, the bound of Theorem 1, so this is best possible as well. The case $k=1$ gives a result towards Conjecture 1: if a bipartite multigraph is the union of $\lfloor \frac{3}{2}n \rfloor - 2$ matchings of size n, then there is a rainbow matching of size $n-1$. As far as we know this is the best result in this direction. If $N = \lfloor (1+\epsilon)n \rfloor$ for some $\epsilon > 0$, we get a partial rainbow matching of size $n-c$ where c is a constant depending on ϵ ($c = \lfloor 1/\epsilon \rfloor$), this goes beyond the best error term known for Ryser's conjecture ([8]), but the price is the increment in the number of colors. Also, when $k = \lfloor \sqrt{n} \rfloor$, Theorem 2 extends (from factorizations of $K_{n,n}$ to colorings of bipartite multigraphs) Woolbright's result [13], namely that a factorization of $K_{n,n}$ contains a rainbow matching of size at least $n - \sqrt{n}$.

Proof of Theorem 2. We use Woolbright's argument [13]. Set $N = \lfloor \frac{k+2}{k+1} n \rfloor - (k+1)$. Let the edge set of a bipartite multigraph $G = [A, B]$ be the union of matchings $M_1, \ldots, M_N$ each of size n and let R_1 be a maximum rainbow matching of G with t edges. Suppose to the contrary that $t \leq n - k - 1$.

We assume the edges of $M_1, \ldots, M_{N-t}$ are not used in R_1. For any subset $S \subset B$, define

$$f(S) = \{v \in A : (v, w) \in R_1 \text{ for some } w \in S\}.$$

Set $B_0 = B \setminus V(R_1), A_0 = A \setminus V(R_1)$. For every $j \in \{1, \ldots, N - t\}$ a matching $F_j \subset M_j$ of size $j(n - t)$ will be defined with the following property.

- Property 1: $V(F_j) \cap B_0 = \emptyset$.

Let $F_1 \subset M_1$ be a matching of size $n-t$ such that $V(F_1) \cap A \subseteq A_0$, since $|M_1| - |R_1| = n - t$, such F_1 exists. Set $B_1 = V(F_1) \cap B$. Since R_1 is a maximum rainbow matching, $V(F_1) \cap B_0 = \emptyset$, so Property 1 holds and $|F_1| = 1 \times (n - t)$. Set $A_1 = f(B_1)$.

Suppose that for some $i \geq 1$ the matchings F_i, R_i and the pairwise disjoint $(n - t)$-element sets $A_1, \ldots, A_i, B_1, \ldots, B_i$ have already been defined, where $|F_i| = i(n - t)$. Define the rainbow matching R_{i+1} by removing from R_i the edges that go from B_i to A_i.

To define $F_{i+1} \subset M_{i+1}$, take $(i + 1)(n - t)$ edges of M_{i+1} incident to $A \setminus V(R_{i+1})$. There exist sufficiently many edges in M_{i+1} since

$$|M_{i+1}| - |R_{i+1}| = n - (t - \sum_{j=1}^{i} |B_j|) = (i + 1)(n - t).$$

We show that Property 1 is maintained. Suppose to the contrary that we find $(a_0, b_0) \in F_{i+1}$, $a_0 \in A_j$ for some $1 \leq j \leq i$, $b_0 \in B_0$ (clearly $j \neq 0$). Then $b_1 = f^{-1}(a_0) \in B_j$, and there exists an a_1 such that $(a_1, b_1) \in F_j$ and this generates an alternating path

$$Q = (b_0, a_0), (a_0, f^{-1}(a_0)), (f^{-1}(a_0), a_1), (a_1, f^{-1}(a_1)), (f^{-1}(a_1), a_2), \ldots$$

ending in A_0 allowing us to replace all edges of $R_1 \cap E(Q)$ by edges in different F_js $(j \leq i + 1)$ contradicting the choice of t. Note that Q is a simple path, since with some $j > j_1 > \cdots > j_k > 0$, its edges go between the disjoint sets

$$(B_0, A_j), (A_j, B_j), (B_j, A_{j_1}), (A_{j_1}, B_{j_1}), (B_{j_1}, A_{j_2}), \ldots, (A_{j_k}, B_{j_k}), (B_{j_k}, A_0).$$

Now F_{i+1} is defined and by Property 1

$$|V(F_{i+1}) \cap (B \setminus (\cup_{k=0}^{i} B_k))| \geq n - t,$$

therefore we can define B_{i+1} as an $(n - t)$-element subset of $V(F_{i+1}) \cap (B \setminus (\cup_{k=0}^{i} B_k))$. Finally, set $A_{i+1} = f(B_{i+1})$.

Since $V(F_{N-t}) \cap B \subseteq B \setminus B_0$, we get

$$(N - t)(n - t) \leq t.$$

Dividing by $n - t$ (using $t \leq n - k - 1 < n$) this can be rewritten as

$$N - t \leq \frac{t}{n - t} = \frac{n - n + t}{n - t} = \frac{n}{n - t} - 1$$

or

$$N \le \frac{n}{n-t} + t - 1.$$

Using this, the definition of N and $t \le n-k-1$, we get

$$\left\lfloor \frac{k+2}{k+1} n \right\rfloor - (k+1) = N \le \frac{n}{n-t} + t - 1 \le \frac{n}{k+1} + n - k - 1 - 1$$

and this leads to

$$\left\lfloor \frac{n}{k+1} \right\rfloor \le \frac{n}{k+1} - 1,$$

a contradiction, finishing the proof. $\square$

Remark. A natural variant of Question 1 is to allow arbitrary multigraphs (instead of bipartite ones). Denote the corresponding function by $N'(n,k)$. For $k=0$ we have an example showing $N'(n,0) > 2n-1$ and recently Aharoni informed us [1] that they proved $N'(n,0) \le 3n-2$. Indeed, our example is the following. Let the vertices be denoted as $1,2,\ldots,4k$, where $2n=4k$. Let $M_1 = \cdots = M_{n-1} = \{12, 34, \ldots, (2n-1)2n\}$, $M_n = \cdots = M_{2n-2} = \{23, 45, \ldots, (2n)1\}$ and $M_{2n-1} = \{13, 24, 57, 68, \ldots, (2n-3)(2n-1), (2n-2)2n\}$. As it was remarked before, there is no full rainbow matching without using an edge of M_{2n-1}. We may assume that we use the edge 24. Now any edge of M_i that covers the vertex 3, where $1 \le i \le 2n-2$, uses either vertex 2 or 4. Therefore, there is no full rainbow matching.

References

[1] R. Aharoni, personal communication.

[2] R. Aharoni, E. Berger, Rainbow matchings in r-partite r-graphs. *Electron. J. Combin.* **16** (2009), no. 1, Research Paper 119, 9 pp.

[3] R. Aharoni, P. Charbit and D. Howard. On a generalization of the Ryser-Brualdi-Stein conjecture, *J Graph Theory* **78** Issue 2 (2015), 143–156.

[4] R.A. Brualdi and H.J. Ryser, *Combinatorial Matrix theory*, Cambridge University Press, Cambridge, UK, 1991.

[5] A.E. Brouwer, A.J. de Vries, R.M.A. Wieringa, A lower bound for the length of partial transversals in a Latin square, *Nieuw Arch. Wisk.* (3) **24** (1978), 330–332.

[6] D. Clemens, J. Ehrenmüller, An improved bound on the sizes of matchings guaranteeing a rainbow matching, arXiv:1503.00438.

[7] A. A. Drisko, Transversals in row-latin rectangles, *J. Combin. Theory Ser. A* **84** (1998), 181–195.

[8] P.Hatami and P.W. Shor, A lower bound for the length of a partial transversal in a Latin square, *J. Combin. Theory Ser. A* **115** (2008), 1103–1113.

[9] D. Kotlar, R. Ziv, Large matchings in bipartite graphs have a rainbow matching. *European J. Combin.* **38** (2014), 97–101.

[10] H.J. Ryser, Neuere probleme der kombinatorik, *Vorträge über Kombinatorik Oberwolfach*, Mathematisches Forschungsinstitut Oberwolfach, July 1967.

[11] P.W. Shor, A lower bound for the length of a partial transversal in a Latin square, *J. Combin. Theory Ser. A* **33** (1982), 1–8.

[12] S. K. Stein. Transversals of latin squares and their generalizations. *Pacific J. Math.* **59** (1975), 567–575.

[13] D.E. Woolbright, An $n \times n$ Latin square has a transversal with at least $n - \sqrt{n}$ distinct symbols, *J. Combin. Theory Ser. A* **24** (1978), 235–237.