Discrete Mathematics 30 (1980) 235–244.

INDUCED SUBTREES IN GRAPHS OF LARGE CHROMATIC NUMBER

A. GYÁRFÁS
Computer and Automation Institute of the Hungarian Academy of Sciences, Budapest, Hungary

E. SZEMEREDI
Mathematical Institute of the Hungarian Academy of Sciences, Budapest, Hungary

Zs. TUZA
Computer and Automation Institute of the Hungarian Academy of Sciences, Budapest, Hungary

Received 29 October 1979

Our paper proves special cases of the following conjecture: for any fixed tree T there exists a natural number $f=f(T)$ so that every triangle-free graph of chromatic number $f(T)$ contains T as an induced subgraph. The main result concerns the case when T has radius two.

1. Introduction

Our paper gives a reexposition and some partial results on the following conjecture of A. Gyárfás:

There exists an integer-valued function f defined on the finite trees with the property that every triangle-free graph with chromatic number $f(T)$ contains T as an induced subgraph.

The crucial point in the conjecture that it concerns *induced subtrees*—trees as partial graphs can be found easily in graphs of large chromatic number (cf. Section 2).

The conjecture was posed in [1] for K_n-free graphs but it seems to us that the special case $n=3$ contains all the difficulties. We restrict ourselves to triangle-free graphs throughout this paper.

Our main result is Theorem 5 which proves the conjecture for trees of radius two and replaces the ad hoc proofs known by us for various special trees. The only other case when we can prove the conjecture occurs if T is a "mop" (Theorem 4). A "mop" is a path with a star at the end.

Let $G=(V,E)$ and $G'=(V',E')$ be two graphs. G' is a *partial graph* of G if $V'\subseteq V$ and $E'\subseteq E$. G' is an *induced subgraph* of G if $V'\subseteq V$ and for $x, y\in V'$ $(x,y)\in E'$ if and only if $(x,y)\in E$. If G' is an induced subgraph of G then G' is determined by V'—sometimes we say that G' is induced by V'. The subgraph of G induced by $X\subseteq V(G)$ is denoted by G_X.

2. A walk around the conjecture

Partial versus induced subtrees

The conjecture becomes true with $(f(T) = |V(T)|)$ if we want T to appear only as a partial graph instead of induced subgraph. This can be seen from the corollary of Theorem 1 below, which certainly belongs to the graph theoretic folklore. We do not know how well-known is Theorem 1 itself.

Theorem 1. *Let G be a k-chromatic graph whose vertices are labeled with $1, 2, \ldots, k$ according to a good k-coloring. If T is a labeled tree on k vertices, then G contains a partial tree isomorphic to T. (Isomorphy is understood between labeled graphs.)*

Corollary. *A k-chromatic graph contains every tree on k vertices as a partial graph.*

Proof of Theorem 1. We use induction on k. The case $k = 1$ is clear. We prove that the theorem follows from $k-1$ to k. Let P be a vertex of T with degree one and with label $l(P)$. P is connected with the vertex Q of T which is labeled with $l(Q)$. Let A be the set of vertices of G in color-class $l(Q)$ so that every vertex in A is connected with at least one vertex of color-class $l(P)$. A is not empty because G is k-chromatic. If we remove from $V(G)$ the vertices of color-class $l(P)$ and the vertices of color-class $l(Q)$ which are not in A, we have a $(k-1)$-chromatic graph with a good $(k-1)$-coloring. The inductive hypothesis guarantees a partial tree T' (label-) isomorphic to $T-P$. The edge $(x, y) \in E(G)$ where $x = A \cap V(T')$ and y is from color-class $l(P)$, completes T' to a partial tree (label-) isomorphic to T. □

Graphs without complete bipartite subgraphs

While trying to prove the conjecture, Rödl and Hajnal got (independently) the following result:

Theorem. *For every tree T and $k \geq 1$ there exists a $g = g(T, k)$ with the property: if a graph G contains no $k-k$ complete bipartite subgraph as a partial graph and $\chi(G) \geq g$, then G contains T as an induced subgraph.*

As for $k = 2$ the complete k-k bipartite graph is the quadrangle, the above result shows the conjecture to be true if "triangle-free" is replaced by "quadrangle-free". Combining these two properties, it is easy to prove the following:

Theorem 2. *A k-chromatic graph without triangles and rectangles contains every tree on k vertices as an induced subgraph.*

Remark. It is interesting to compare Theorem 2 with the corollary of Theorem 1.

Proof. We prove a stronger statement: if G is a triangle- and quadrangle-free graph and every vertex of G has degree at least $k-1$, then G contains every tree of k vertices as an induced subgraph. We prove by induction. The case $k=2$ is obvious. The inductive step runs as follows.

If G is a graph without triangles and rectangles and $d(x) \geq k$ for every $x \in V(G)$ and T is a tree on $k+1$ vertices then G contains an induced T' which we get from T by removing the edge AB and the vertex A where A is of degree 1. The set of vertices connnected with B in G are divided into two parts:

$$X_1 = \{x : x \in V(T'), (B, x) \in E(G)\}, \qquad X_2 = \{x : x \notin V(T'), (B, x) \in E(G)\}$$

No vertex of X_2 is connected to any vertex of X_1 as G is triangle-freee. No two vertices of X_2 are connected to the same vertex of $V(T') - X_1 - \{B\}$ since G is quadrangle-free. $|V(T') - X_1 - \{B\}| = k - |X_1| - 1 < |X_2|$ because $k-1 < |X_1| + |X_2| = d(B) \geq k$, therefore there exists a $y \in X_2$ which is not connected to $V(T') - \{B\}$. The subgraph of G induced by $V(T') \cup \{y\}$ is isomorphic to T. □

Triangle-free graphs of diameter two

It would be very desirable to prove the special case of the conjecture when G contains no triangles but the addition of any new edges destroys this property. It is easy to see that these graphs are the triangle-free graphs of diameter two. $R(k, 3)$ denotes the classical Ramsey-number, i.e. the smallest m for which every graph of m vertices contains a triangle or its complement contains K_k. $T_{k,1}$ is a tree where k paths of three vertices start from a common center. (The notation $T_{k,1}$ is introduced in Section 3, that is the reason behind the notation.)

Theorem 3. *If G is a triangle-free graph of diameter two, $\chi(G) \geq R(k, 3) + 1$ and $P \in V(G)$, then G contains $T_{k,1}$ as an induced subgraph, so that P is the center of $T_{k,1}$.*

Proof. We decompose $V(G) - \{P\}$ into two disjoint sets:

$$A = \{x : x \in V(G) - \{P\}, (x, P) \in E(G)\},$$
$$B = \{x : x \in V(G) - \{P\}, (x, P) \notin E(G)\}.$$

For every $a \in A$ we define $B_a = \{b : b \in B, (a, b) \in E(G)\}$. Let s be the smallest number for which we have an $A' \subseteq A$, $|A'| = s$ for which $\bigcup_{a \in A'} B_a = B - s$ exists and $s \leq |A|$ because $\bigcup_{a \in A} B_a = B$ (G has diameter two). If $A' = \{a_1, \ldots, a_s\}$, then the definition of s guarantees $b_1, \ldots, b_s$ where $b_i \in B_{a_i}$ and $b_i \notin B_{a_j}$ for $1 \leq i, j \leq s$, $i \neq j$. The sets $B_{a_1}, B_{a_2}, \ldots, B_{a_s} \cup \{P\}$, A induce empty subgraphs in G which means $s + 1 \geq \chi(G) \geq R(k, 3) + 1$ i.e. $s \geq R(k, 3)$. The subgraph of G induced by $\{b_1, b_2, \ldots, b_s\}$ is triangle-free so it contains k vertices, say $b_1, b_2, \ldots, b_k$ which induce an empty subgraph of G. The set $\{P, a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_k\}$ induces $T_{k,1}$ in G. □

3. Mops and trees of radius two

Mops

An (m, n)-mop is defined by identifying one extreme vertex of a path of m vertices with the center of a star of $n+1$ vertices. The other extreme vertex of the path is called the top of the mop. We assume $m \geq 2$, $n \geq 1$. A $(2, n)$-mop is a star of $n+2$ vertices and an $(m, 1)$-mop is a path of $m+1$ vertices.

Theorem 4. *$f(T) \leq m+n$ if T is an (m, n)-mop.*

Proof. We prove a stronger statement: if G is triangle-free, $\chi(G) \geq m+n$, $P \in V(G)$, $d(x) \geq m+n-1$ for $x \in V(G)-\{P\}$, $d(P) \geq 1$ then G contains an induced (m, n)-mop with its top in P. The proof goes by induction on m. The case $m=2$ is obvious because $d(P) \geq 1$, $d(Q) \geq n+1$ where Q is a vertex connected with P.

The inductive step is made from $m-1$ to m. $X \subset V(G)-\{P\}$ denotes the set of vertices which are not connected with P. $\{x_1, x_2, \ldots, x_t\} \subseteq X$ is defined so that x_i is connected with at most $m+n-3$ vertices of $X-\{x_1, x_2, \ldots, x_i\}$ for $1 \leq i \leq t$ and t is the largest number satisfying this property. $B = X - A$. $\chi(G_X) \geq m+n-1$ since any good p-coloring of $V(G_X)$ can be extended to a good $(p+1)$-coloring of $V(G)$. We can easily define an $m+n-1$-coloring of $V(G_X)$ so that A is colored with at most $m+n-2$ colors. There exists a $Q \in V(G)$ for which $(P, Q) \in E(G)$ and Q is connected with some vertex of B, otherwise the $(m+n-1)$-coloring defined above can be extended to a good $(m+n-1)$-coloring of G which is impossible. The graph G' induced by $B \cup \{Q\}$ in G is triangle-free, $\chi(G') \geq m+n-1$, $d(x) \geq m+n-2$ for $x \in V(G')-\{Q\}$ and $d(Q) \geq 1$. The inductive hypothesis assures an induced $(m-1, n)$-mop T' with its top in Q. P completes T' to an (m, n)-mop T which is an induced subgraph of G and its top is in P. □

Trees of radius two

A graph is called of radius two if there is a vertex—the center of the graph—from which every other vertex can be reached by a path of length at most two. (The length of a path is the number of its vertices minus one.) $T_{k,l}$ is a special tree of radius two, the center of which is connected with k vertices and all these k vertices are connected with l additional vertices. $T_{k,l}$ has $kl+k+1$ vertices. If $k=1$, then $T_{k,k}$ is the k-nary tree with two levels. We write T_k instead of $T_{k,k}$. The vertices of distance one and two from the center of a tree of radius two are called "level-one" and "level-two" vertices respectively. The level-two vertices form "level-two groups"—one level-two group consists of the set of vertices connected with the same level-one vertex of the tree.

Theorem 5. *$f(T)$ exists for trees of radius two.*

Proof. *Step 1.* It is enough to prove the theorem for $T = T_k$ because every tree of radius two is a partial graph of some T_k.

The proof is presented as steps numbered with $1, 2, \ldots, 15$ for better understanding. We give here the very brief outline of the proof. $f_1(k), f_2(k), \ldots$ denote functions of k.

A Ramsey-type lemma (Lemma 1) allows us to loose the condition that T_k is wanted as an induced subgraph (steps 2 and 3). The heart of the proof is a decomposition (step 4) of G into disjoint parts $A_1, A_2, \ldots, A_t$ where A_i is a large $(f_1(k) - f_1(k))$ complete bipartite graph plus vertices which are connected with a large number (at least $f_2(k)$) of vertices of that bipartite graph. The part X of G which escapes from the decomposition is $f_3(k)$-chromatic or contains T_k (step 5). The components $A_1, \ldots, A_t$ are $f_4(k)$-chromatic (step 6). Since t certainly depends on the number of vertices of G, the structure of the edges between different A_i's must be analyzed. The set of these edges is denoted by $\mathscr{E}$. There are two possible cases:

Case A. $\mathscr{E}$ can be colored with red and blue so that the chromatic number of the graphs containing red and blue edges respectively is bounded by $f_5(k)$ and $f_6(k)$, (steps 7, 9, 10, 11, 12, 13) which implies a bound for the chromatic number of G (steps 14, 15).

Case B. The structure of $\mathscr{E}$ allows us (by repetitive application of Lemma 2 in Step 8) to find T_k in $\bigcup_{i=1}^{t} A_i$.

Step 2. T_k is called quasi-induced subgraph of a triangle-free graph G if T_k is a partial graph of G and every edge of G which connects level-one and level-two vertices of T_k is an edge of T_k. In other words we can say that the only edges in G which make T_k a "non-induced" subgraph, connect level-two vertices of T_k.

The following "Ramsey-type" lemma shows that a quasi-induced $T_{(k-1)^2k+k}$ in a triangle-free graph G contains T_k which is an induced subgraph of G.

Lemma 1. *H_i denotes the complete i-partite graph, where every vertex-class contains i vertices. In every two-coloring of the edges of $H_{(k-1)^2k+k}$ there is either a triangle in the first color or H_k in the second color.*

Proof. Let us consider a two-coloring of the edges of $H = H_{(k-1)^2k+k}$ and suppose that there is no monochromatic triangle in the first color. We construct sets $A_1, A_2, \ldots, A_j$ so that $|A_i| = k$ for $1 \leq i \leq j$, A_i's are subsets of different vertex-classes of H and A_i's are spanning a complete j-partite graph which is monochromatic in the second color. Let j be maximal with respect to the above property. If $j \geq k$, we have nothing to prove. Let us suppose that $j < k$. $g(P)$ denotes for $P \in \bigcup_{i=1}^{j} A_i$ the number of vertex-classes in H which contain at least k vertices connected with P in color one. If $g(P) \geq k$, then these vertices span a monochromatic H_k in the second color and the lemma is proved. (H contains no monochromatic triangles in the first color.) If $g(P) \leq k-1$ for every $P \in \bigcup_{i=1}^{j} A_i$, then we have at most $jk(k-1) \leq (k-1)^2 k$ vertex-classes of H in which at least k vertices are connected with some vertex of $\bigcup_{i=1}^{j} A_i$ in color one—as H is $(k-1)^2 k + k$-partite and $j < k$, we can choose a vertex-class C_q different from these classes and different from the vertex-classes containing $A_1, A_2, \ldots, A_j$. C_q contains at most $jk(k-1) \leq (k-1)^2 k$ vertices connected with $\bigcup_{i=1}^{j} A_i$ in color one, but C_q contains $(k-1)^2 k + k$ vertices, therefore we can choose k vertices from C_q which are connected to all vertices of $\bigcup_{i=1}^{j} A_i$ in color two. The k vertices chosen from C_q can be added as A_{j+1} to $A_1, A_2, \ldots, A_j$ which is a contradiction. □

Step 3. In the light of steps 1 and 2 it is enough to prove the following statement: the chromatic number of G is bounded by a function of k if G is a triangle-free graph which does not contain a quasi-induced T_k. We assume G to be such a graph throughout the following steps of the proof.

Step 4. The heart of the proof is a decomposition of $V(G)$:

$$V(G) = \bigcup_{i=0}^{t} A_i \cup X$$

where the sets A_i and X is defined as follows. $A_0 = \emptyset$. If $A_0, A_1, \ldots, A_s$ are already defined, we consider two cases. If there is no k^8-k^8 complete bipartite subgraph in the graph induced by $V(G) - \bigcup_{i=0}^{s} A_i$, then $t = s$ and $X = V(G) - \bigcup_{i=0}^{s} A_i$. Otherwise B_{s+1} is defined as a vertex-set of a k^8-k^8 complete bipartite subgraph in the graph induced by $V(G) - \bigcup_{i=0}^{s} A_i$ in G. C_{s+1} denotes the set of vertices in $V(G) - \bigcup_{i=0}^{s} A_i$ which are connected to B_{s+1} with at least k^5 edges. $A_{s+1} = B_{s+1} \cup C_{s+1}$.

The graph induced by $\bigcup_{i=0}^{t} A_i$ in G is denoted by G_1. In the following step we shall prove that $\chi(G_X)$ is bounded by a function of k. Since $V(G) = V(G_X) \cup V(G_1)$, it remains to show the same for G_1. We can assume $t \geq 1$ and we can omit A_0 which was introduced only to ease the definition of the sets A_i.

Step 5. $\chi(G_X) \leq g(k)$ for some function of k. The truth of this statement follows immediately from the result of Rödl and Hajnal mentioned in Section 2 since G_X contains neither a k^8-k^8 complete bipartite graph as a partial graph nor $T = T_k$ as an induced subgraph. In order to avoid reference to an unpublished result, we give a proof of the above statement. It is enough to prove the following proposition:

Proposition. *G_X contains no $T_{a,b}$ as a partial graph if*

$$a = k^{2^{c-1}-1}c^{2^c}, \qquad b = k^c c \quad \text{and} \quad c = k^8.$$

Proof. If $T = T_{a,b}$ is a partial graph of G_X, then we apply Lemma 2. (Lemma 2 appears in Step 8. No other forward-references occur during the proof.) Lemma 2 assures a level-one vertex P_1 of T which is connected to $a_1 = \sqrt{a/k}$ level-two groups of T with $b_1 = b/k$ edges. We have no T_{a_1,b_1} as a partial tree of $T_{a,b}$ so that all level-two vertices of T_{a_1,b_1} are connected with P_1. If we iterate this argument with $a_{i+1} = \sqrt{a/k}$, $b_{i+1} = b/k$ c times in all, we find that $a_c = 1$, $b_c = c$ which shows a c-c complete bipartite subgraph of T—we have a contradiction, since $c = k^8$ and G_X contains no k^8-k^8 complete bipartite subgraph. □

Remark. Using the corollary of Theorem 1, we have the bound $|V(T_{a,b})| = ab + a + 1$ for $\chi(G_X)$. It is a poor bound, we can improve it, but we are not able to give a polynomial bound. We note that such an improvement would imply a polynomial bound for $\chi(G)$.

Step 6. The chromatic number of the graph induced by A_i in G is at most $2k^8$ since the neighborhoods of the vertices of B_i define a covering of A_i with at most $2k^8$ empty subgraphs. This observation allows us to decompose A_i into disjoint sets $A_{i,1}, A_{i,2}, \ldots, A_{i,p_i}$ so that $A_{i,j}$ induces an empty subgraph of G for $1 \leq j \leq p_i$, $p_i \leq 2k^8$ for $1 \leq i \leq t$ and for every $A_{i,j}$ we can find a $P_{i,j} \in A_i$ such that $P_{i,j}$ is connected with every vertex of $A_{i,j}$.

Step 7. **Proposition.** *For every $1 \leq i \leq t$ and $P \in A_i$*

$$|\{j : (P, Q) \in E(G) \quad \text{for some} \quad Q \in A_j\}| < k^7.$$

The proof is based on the following lemma.

Step 8. **Lemma 2.** *n, m are natural numbers and H is a graph with a partial graph $T = T_{kn^2,km}$. The subgraph of H induced by T contains either a quasi-induced T_k or a level-one vertex of T which is connected to at least m vertices of at least n level-two groups of T.*

Proof. We denote the level-one vertices of T by $x_1, x_2, \ldots, x_{kn^2}$ and the level-two groups by $S_1, S_2, \ldots, S_{kn^2}$. $g(x_i)$ denotes the number of level-two groups connected to x_i with at least m edges. We have to prove the following: if $g(x_i) < n$ for every $1 \leq i \leq kn^2$ then T contains T_k as a quasi-induced subgraph.

First we define a subsequence $\{x'_i\}$ of $\{x_i\}$ and a subsequence $\{S'_i\}$ of $\{S_i\}$ with kn elements as follows: $x'_1 = x_1$, $S'_1 = S_1$. If $x'_1, \ldots, x'_r$ and $S'_1, \ldots, S'_r$ are already defined and $r < kn$ then we define $S'_{r+1} = S_i$, where S_i is a level-two group which is connected to every vertex of $\{x'_1, \ldots, x'_r\}$ with at most $m-1$ edges and i is the smallest possible index with this property. We can choose such an S_i because

$g(x_1), g(x_2), \ldots, g(x_r) < n$ and $r < kn$. x'_{r+1} is the level-one vertex belonging to S'_{r+1}. The sequences $\{x'_i\}$, $\{S'_i\}$ $(i = 1, 2, \ldots, kn)$ have the property that x'_i is connected to at most $m-1$ vertices of S'_j if $i < j$.

Now we define a subsequence of $\{x'_i\}$ and $\{S'_i\}$ both with k elements. These are denoted by $\{x''_i\}$ and $\{S''_i\}$ $(i = 1, 2, \ldots, k)$ and they are defined as follows: $x''_1 = x'_{kn}$, $S''_1 = S'_{kn}$. If $x''_1, x''_2, \ldots, x''_r$ and $S''_1, S''_2, \ldots, S''_r$ are already defined and $r < k$, then $S''_{r+1} = S'_i$ where S'_i is a level-two group which is connected with at most $m-1$ edges to every element of $\{x''_1, x''_2, \ldots, x''_r\}$ and i is the largest possible index with this property. We can choose such an S'_i, because $g(x''_1), g(x''_2), \ldots, g(x''_r) < n$ and $r < k$. x''_{r+1} is the level-one vertex belonging to S''_{r+1}. The sequences $\{x''_i\}$, $\{S''_i\}$ $(i = 1, 2, \ldots, k)$ have the property that x''_i is connected to at most $m-1$ vertices of S''_j if $i \neq j$. If we omit from S''_j the vertices which are connected with x''_i for $i \neq j$, then at least $km-(k-1)(m-1) \geq k$ vertices remain in them which together with $x''_1, \ldots, x''_k$ and the center of T define a quasi-induced T_k. □

Step 9. Proof of the proposition given in step 7. We suppose that some $P_1 \in A_i$ is connected with at least k^7 different A_j. We renumber these sets A_j with indices $1, 2, \ldots, k^7$. Now we can define a partial tree T_{k^7,k^5} in G by taking the centre of T_{k^7,k^5} at P_1, choosing the level-one vertices $Q_1, Q_2, \ldots, Q_{k^7}$ so that $Q_j \in A_j$ and choosing the level-two group belonging to Q_j from the complete bipartite graph B_j for $1 \leq j \leq k^7$. (The definition of A_j makes this possible.)

Applying Lemma 2 for T_{k^7,k^5}, we have a $P_2 \in \{Q_1, Q_2, \ldots, Q_{k^7}\}$ which is connected with at least $\sqrt{k^7/k} = k^3$ level-two groups of T_{k^7,k^5}. (P_2 is connected to at least $k^5/k = k^4$ vertices of these groups, but we do not use that now.) The definition of P_2 allows us to define a partial tree T_{k^3,k^8} of G with centre in P_2 so that every level-one vertex is in the same B_j as its level-two group. Applying Lemma 2 again, we find P_3 which is connected to at least $k^8/k = k^7$ vertices of $\sqrt{k^3/k} = k$ level-two groups of T_{k^3,k^8}. The definition of P_3 makes possible to define the following partial graph G' of G (we make again a renumbering of the sets A_i in order to have simpler indices): B'_i and B''_i form a k^7-k^7 complete bipartite graph for $i = 1, 2, \ldots, k$ and P_3 is connected to every vertex of $\bigcup_{i=1}^{k} B'_i$. We construct a quasi-induced T_k in G' as follows:

$R_k \in B'_k$ and $S_k \subset B''_k$ so that $|S_k| = k$. We suppose that R_k, S_k, R_{k-1}, $S_{k-1}, \ldots, R_r, S_r$ are constructed for $r > 1$. We define R_{r-1} and S_{r-1}: R_{r-1} is a vertex of B'_{r-1} which is not connected to $\bigcup_{j=r}^{k} S_j$ and $S_{r-1} \subset B''_{r-1}$, $|S_{r-1}| = k$ so that S_{r-1} is not connected to $\bigcup_{j=r}^{k} R_j$. The definition of R_{r-1} makes sense since every vertex of B_j can be connected with less than k^5 vertices of B_i if $i < j$ according to the definition of B_i so the number of vertices in B''_{r-1} which are connected to $\bigcup_{j=r}^{k} S_j$ is less than $k(k-r+1)k^5 < |B''_{r-1}| = k^7$. The same reasoning shows that the number of vertices in B''_{r-1} connected with $\bigcup_{j=r}^{k} R_j$ is less than $(k-r+1)k^5 \leq |B''_{r-1}| - k = k^7 - k$ so S_{r-1} can be defined sensibly.

The center P_3, the level-one vertices $R_1, R_2, \ldots, R_k$ and the level-two groups $S_1, S_2, \ldots, S_k$ determine a quasi-induced T_k in G. This is a contradiction. □

Step 10. We define $m = m(i, j)$ for every $A_{i,j}$ ($1 \leq i \leq t, 1 \leq j \leq p_i$—the definition of $A_{i,j}$ is given in step 6) as follows: m is the maximal integer for which there exist $x_1, x_2, \ldots, x_m \in A_{i,j}$ $D_{i_1} \subset A_{i_1}, D_{i_2} \subset A_{i_2}, \ldots, D_{i_m} \subset A_{i_m}$ so that $i < i_1 < i_2 < i_m$, $|D_{i_1}| = |D_{i_2}| = \cdots = |D_{i_{m'}}| = k$ and x_n is connected with every vertex of D_{i_n} for $n = 1, 2, \ldots, m$. If no m exists with the required property, we set $m(i, j) = 0$ (for example $m(t, j) = 0$ for $j = 1, 2, \ldots, p_t$).

Proposition. *$m = m(i, j) < k^{15}$ for every i, j, $1 \leq i \leq t$, $1 \leq j \leq p_i$.*

Proof. If $m = m(i, j) \geq k^{15}$ for some i, j, then

$$\{P_{i,j}\} \cup \{x_1, x_2, \ldots, x_m\} \cup \bigcup_{n=1}^{m'} D_{i_n}$$

induces a subgraph of G which contains $T_{k^{15},k}$ as a partial graph. ($P_{i,j}$ is the vertex of A_i which is connected to every vertex of $A_{i,j}$—cf. step 6.) Lemma 2 gives us a vertex x_n (for some $1 \leq n \leq m$) which is connected to at least $k/k = 1$ vertex of $\sqrt{k^{15}/k} = k^7$ different sets A_i—we have a contradiction with the proposition of step 7. □

Step 11. We color the edges of G connecting different A_i's with two colors. Let (x, y) be an edge of G so that $x \in A_i$, $y \in A_{i'}$ and $i' \neq i$. We may assume $i < i'$ and $x \in A_{i,j}$ for some $1 \leq j \leq p_i$. We consider the set $\{x_1, x_2, \ldots, x_m\} \subset A_{i,j}$ defined in step 10. If $m(i, j) = 0$, then the empty set is chosen.

The edge (x, y) is colored with

red if $x \notin \{x_1, x_2, \ldots, x_m\}$ and $y \notin A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_{m'}}$,

blue otherwise.

Proposition. (a) $|\{y : y \in A_{i'},\ i < i',\ (x, y)$ *is red*$\}| < (k-1)k^7$ *for every fixed i and* $x \in A_i$,

(b) $|\{i' : x \in A_i,\ y \in A_{i'},\ i < i',\ (x, y)$ *is blue*$\}| < 2k^{30}$ *for every fixed i.*

Proof. (a) Let us suppose that $x \in A_{i,j}$. If (x, y) is red, then $x \notin \{x_1, x_2, \ldots, x_m\}$, $y \notin A_{i_1} \cup A_{i_2} \cup \cdots \cup A_{i_m}$ and the choice of $m = m(i, j)$ implies that at most $k-1$ edges go from x to $A_{i'}$. On the other hand, x is connected with less than k^7 different $A_{i'}$'s by the proposition of step 7.

(b) Let x be a vertex of $A_{i,j}$. If $x \in A_{i,j} - \{x_1, \ldots, x_m\}$ then blue edges from x to $A_{i'}$ for $i < i'$ are possible only if $i' \in \{i_1, \ldots, i_m\}$. This means that the blue edges from $A_{i,j}$ reach less than $mk^7 \leq k^{22}$ $A_{i'}$'s for $i < i'$ if i, j are fixed. (Propositions in step 7 and step 10 were used.) Since $A_i = \bigcup_{j=1}^{p_i} A_{i,j}$, $p_i \leq 2k^8$ (cf. Step 6) therefore the blue edges from A_i reach less than $k^{22} \cdot 2k^8 = 2k^{30}$ A_i's for $i < i'$ if i is fixed. □

Step 12. **Proposition.** *The graph G_2 with vertices $\bigcup_{i=1}^{t} A_i$ and with the red edges (defined in step 11) is at most $(k-1)k^7$-chromatic.*

Proof. Let $x_1, x_2, \ldots$ be the following ordering of the vertices of G_2: first we take the vertices of A_1 in any order, then the vertices of A_2 in any order, ..., finally the vertices of A_t in any order. A_i induces an empty subgraph of G_2 for all $1 \leq i \leq t$, therefore part (a) of the proposition in step 11 shows that the "forward degree" of the vertices of G_2 in the ordering given above is less than $(k-1)k^7$ which implies $\chi(G_2) \leq (k-1)k^7$ easily. □

Step 13. **Proposition.** *The graph G_3 with vertices $\bigcup_{i=1}^{t} A_i$ and with the blue edges (defined in step 11) is at most $2k^{30}$-chromatic.*

Proof. We define the graph G_3' as follows: $V(G_3') = \{w_1, w_2, \ldots, w_t\}$ and $(w_i, w_{i'})$ is an edge of G_3' if and only if there is an edge between A_i and $A_{i'}$ in G_3. Part (b) of the proposition in step 11 shows that the "forward degree" of the vertices of G_3' in the ordering $w_1, w_2, \ldots, w_t$ are less than $2k^{30}$ so $\chi(G_3') \leq 2k^{30}$ which implies $\chi(G_3) \leq 2k^{30}$ since A_i induces an empty subgraph of G_3. □

Step 14. **Proposition.** *The graph G_4 with vertices $\bigcup_{i=1}^{t} A_i$ and with the edges of G_1 which are neither blue nor red, is at most $2k^8$-chromatic.*

Proof. G_4 consists of t connected components all of which are at most $2k^8$-chromatic (step 6). □

Step 15. The proof of Theorem 5 is now complete since

$$\chi(G_1) = \chi(G_2 \cup G_3 \cup G_4) \leq \chi(G_2)\chi(G_3)\chi(G_4)$$
$$\leq (k-1)k^7 \cdot 2k^{30} \cdot 2k^8 \leq 4k^{46}$$

therefore the chromatic number of G_1 is bounded by a polynom of k as claimed in step 4.

Reference

[1] A. Gyárfás, On Ramsey covering-numbers, Coll. Math. Soc. János Bolyai 10. Infinite and finite sets, 801–816.