New Bounds on the Grundy Number of Products of Graphs

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Abstract: The Grundy number of a graph G is the largest k such that G has a greedy k-coloring, that is, a coloring with k colors obtained by applying the greedy algorithm according to some ordering of the vertices of G. In this article, we give new bounds on the Grundy number of the product of two graphs.

Keywords: *coloring; product; Grundy*

1. INTRODUCTION

Graphs considered in this article are undirected, finite and contain neither loops nor multiple edges (unless stated otherwise). The definitions and notation used in this article are standard and may be found in any textbook on graph theory; see [4] for example. Given two graphs G and H, the *direct product* $G\times H$, the *lexicographic product* $G[H]$, the *Cartesian product* $G\Box H$, and the *strong product* $G\boxtimes H$ are the graphs with vertex set $V(G)\times V(H)$ and the following edge sets:

$$E(G\times H)=\{(a,x)(b,y)|ab\in E(G) \text{ and } xy\in E(H)\};$$

$$E(G[H])=\{(a,x)(b,y)| \text{ either } ab\in E(G) \text{ or } a=b \text{ and } xy\in E(H)\};$$

$$E(G\Box H)=\{(a,x)(b,y)| \text{ either } a=b \text{ and } xy\in E(H) \text{ or } ab\in E(G) \text{ and } x=y\};$$

$$E(G\boxtimes H)=E(G\times H)\cup E(G\Box H).$$

A *k-coloring* of a graph G is a surjective mapping $\psi: V(G)\to\{1,\dots,k\}$. It is *proper* if for every edge $uv\in E(G)$, $\psi(u)\neq\psi(v)$. A *proper k-coloring* may also be seen as a partition of the vertex set of G into k disjoint non-empty *stable sets* (i.e. sets of pairwise non-adjacent vertices) $C_i=\{v|\psi(v)=i\}$ for $1\le i\le k$. For convenience (and with a slight abuse of terminology), by proper k-coloring we mean either the mapping ψ or the partition $\{C_1,\dots,C_k\}$. The elements of $\{1,\dots,k\}$ are called *colors*. A graph is *k-colorable* if it admits a k-coloring. The *chromatic number* $\chi(G)$ is the least k such that G is k-colorable.

Many upper bounds on the chromatic number arise from algorithms that produce colorings. The most basic one is the greedy algorithm. A *greedy coloring* relative to a vertex ordering $v_1<v_2<\cdots<v_n$ of $V(G)$ is obtained by coloring the vertices in the order $v_1,\dots,v_n$, assigning to v_i the smallest positive integer not already used on its lower indexed neighbors. Trivially, a greedy coloring is proper. Denoting by C_i the stable set of vertices colored i, a greedy coloring has the following property:

$$\text{For every } i<j, \text{ every vertex in } C_j \text{ has a neighbor in } C_i \qquad (\star)$$

for otherwise the vertex in C_j would have been colored i or less. Conversely, a coloring satisfying Property ($\star$) is a greedy coloring relative to any vertex ordering in which the vertices of C_i precede those of C_j whenever $i<j$. The *Grundy number* $\Gamma(G)$ is the largest k such that G has a greedy k-coloring.

Let $\Delta(G)$ denote the maximum degree in a graph G. Let K_n denote the complete graph on n vertices and $K_{p,q}$ denote the complete bipartite graph with parts of size p and q. Let S_n denote the edgeless graph on n vertices.

In [1], Asté et al. investigated the Grundy number of several types of graph products. They showed that the Grundy number of the lexicographic product of two graphs is bounded in terms of the Grundy numbers of these graphs.

Theorem 1 (Asté et al. [1]). *For any two graphs G and H, $\Gamma(G[H]) \leq 2^{\Gamma(G)-1}(\Gamma(H)-1)+\Gamma(G)$.*

Moreover, when the graph G is a tree, they obtained an exact value.

Theorem 2 (Asté et al. [1]). *Let T be a tree and H be any graph. Then $\Gamma(T[H]) = \Gamma(T)\Gamma(H)$.*

They also showed that, in contrast with the lexicographic product, there is no upper bound of $\Gamma(G\Box H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$; for example, $\Gamma(K_{p,p})=2$ and $\Gamma(K_{p,p}\Box K_{p,p}) \geq p+1$. Nevertheless, they showed that $\Gamma(G\Box H)$ is bounded by a function of $\Delta(G)$ and $\Gamma(H)$.

Theorem 3 (Asté et al. [1]). *For any two graphs G and H, $\Gamma(G\Box H) \leq \Delta(G)\cdot 2^{\Gamma(H)-1}+\Gamma(H)$.*

However, they conjectured that this upper bound is far from being tight.

Conjecture 4 (Asté et al. [1]). *For any two graphs G and H, $\Gamma(G\Box H) \leq (\Delta(G)+1)\Gamma(H)$.*

This conjecture generalizes the following conjecture of Balogh et al. [3].

Conjecture 5 (Balogh et al. [3]). *For any graph H, $\Gamma(K_2\Box H) \leq 2\Gamma(H)$.*

Here is another conjecture that would imply the preceding one.

Conjecture 6 (Havet and Zhu). *If G is any graph and M is a matching in G, then $\Gamma(G) \leq 2\Gamma(G\setminus M)$.*

In [7], Havet et al. proved Conjecture 4 in the case when one of G,H is a tree.

Theorem 7 (Havet et al. [7]). *For any graph G and tree T, $\Gamma(G\Box T) \leq (\Delta(G)+1)\Gamma(T)$.*

Here, we investigate further the relation between the Grundy number of the direct product, lexicographic product or Cartesian product of two graphs and the invariants Γ and Δ of the two graphs. We first show that $\Gamma(G\Box H) \leq \Gamma(H[K_{\Delta(G)+1}])$. Together, with Theorems 1 and 2, this implies Theorems 3 and 7, respectively. In particular, we obtain a shorter proof of Theorem 7.

We then show that $\Gamma(G[K_2]) = \Gamma(G[S_2]\Box K_2)$. As a corollary, we give an example of a graph that disproves Conjectures 4–6: there is a graph H such that $\Gamma(H)=3$ and $\Gamma(K_2\Box H)=7$. Together with Theorem 3 this yields $\max\{\Gamma(K_2\Box H) | \Gamma(H)=3\}=7$.

Regarding the direct and strong product, we answer a question raised as the last sentence in [1]. There cannot be any bound on $\Gamma(G\times H)$ and $\Gamma(G\boxtimes H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$ if $\Gamma(G)$ and $\Gamma(H)$ are both ≥ 3 (Theorem 15). It is also impossible to bound $\Gamma(G\times H)$ in terms of $\Delta(G)$ and $\Gamma(H)$ when G is any graph with at least one edge and $\Gamma(H)\geq 5$ (Theorems 17). Similarly, it is impossible to bound $\Gamma(G\boxtimes H)$ in terms of $\Delta(G)$ and $\Gamma(H)$ when $\Gamma(H)\geq 5$ unless G is the disjoint union of complete graphs (Theorem 18 and Proposition 19).

2. THE CARTESIAN AND LEXICOGRAPHIC PRODUCTS

A. Common Proof of Theorems 3 and 7

Theorem 8. *For any two graphs G and H*, $\Gamma(G\Box H)\leq\Gamma(G[K_{\Delta(H)+1}])$.

Proof. We shall prove that if $G\Box H$ has a greedy q-coloring for some integer q, then so does $G[K_{\Delta(H)+1}]$. Hence, consider a greedy q-coloring φ of $G\Box H$.

Let $x_1,x_2,\ldots,x_n$ be an ordering of the vertices of G such that $\varphi(x_1,y)\leq\varphi(x_2,y)\leq\cdots\leq\varphi(x_n,y)$. Let $z_0,\ldots,z_{\Delta(H)}$ be the vertices of $K_{\Delta(H)+1}$. So every vertex of $G[K_{\Delta(H)+1}]$ is a pair (x_i,z_j) for some $i\in\{1,\ldots,n\}$ and $j\in\{0,1,\ldots,\Delta(H)\}$. Let (x,y) be a vertex of $G\Box H$ with color q.

For every i in $\{1,\ldots,n\}$, we assign color $\varphi(x_i,y)$ to vertex (x_i,z_0) of $G[K_{\Delta(H)+1}]$. Then for $i=1$ to n, we do the following. Let L_i be the set of all colors ℓ, with $\ell<\varphi(x_i,y)$, that have not been assigned to any neighbor of (x_i,z_0) in $G[K_{\Delta(H)+1}]$. Since φ is a greedy coloring and color $\varphi(x_j,y)$ is assigned to (x_j,z_0) for each j, L_i is a subset of $\{\varphi(x_i,u)|u\in N(y)\}$. Therefore, $|L_i|\leq\Delta(H)$. Hence, we can assign all the colors of L_i to distinct vertices in $\{(x_i,z_j)|1\leq j\leq\Delta(H)\}$.

Let us show that the obtained partial q-coloring of $G[K_{\Delta(H)+1}]$ is a greedy coloring. It is proper since colors already assigned to neighbors of (x_i,z_0) are not in L_i. In L_i we add every color $\ell<\varphi(x_i,z_0)$ such that (x_i,z_0) had no neighbor colored ℓ before Step i. Hence, after Step i, vertex (x_i,z_0) has a neighbor of each color less than $\varphi(x_i,y)$. Now every colored vertex (x_i,z) has a color ℓ less than $\varphi(x_i,y)$. But, by the definition of the lexicographic product, all neighbors of (x_i,z_0), except (x_i,z) itself, are neighbors of (x_i,z). Hence, (x_i,z) has a neighbor of each color less than ℓ. So the coloring is greedy. ∎

B. Disproof of Conjecture 4

Asté et al. [1] proved the following:

Lemma 9 (Asté et al. [1])**.** *For any graph G and any integer n*, $\Gamma(G[S_n])=\Gamma(G)$.

Now we prove:

Theorem 10. *Let G be a graph. Then* $\Gamma(G[K_2])=\Gamma(G[S_2]\Box K_2)$.

Proof. Let us show that the left-hand side is at most the right-hand side. Consider a greedy coloring φ of $G[K_2]$. Every vertex v of G corresponds to two adjacent vertices of $G[K_2]$. Let us denote by $\varphi_1(v)$ and $\varphi_2(v)$ the two distinct colors assigned by φ to these vertices. In the graph $G[S_2]\Box K_2$, every vertex v corresponds to four vertices a_v, b_v, a'_v and b'_v inducing two edges a_vb_v and $a'_vb'_v$, and so that if uv is any edge of G, then $G[S_2]\Box K_2$ has all edges between $\{a_u,a'_u\}$ and $\{a_v,a'_v\}$ and all edges between $\{b_u,b'_u\}$

and $\{b_v, b'_v\}$. Assign color $\varphi_1(v)$ to a_v and b'_v and color $\varphi_2(v)$ to b_v and a'_v. Doing this for every vertex, it is easy to check that we obtain a greedy coloring of $G[S_2]\square K_2$. Hence, $\Gamma(G[K_2]) \leq \Gamma(G[S_2]\square K_2)$.

Let us now show that the right-hand side is at most the left-hand side. By Theorem 8, we have $\Gamma(G[S_2]\square K_2) \leq \Gamma(G[S_2][K_2])$. We claim that $\Gamma(G[S_2][K_2]) \leq \Gamma(G[K_2])$. To see this, consider any greedy coloring φ of $G[S_2][K_2]$ with q colors. In $G[S_2][K_2]$, every vertex v of G corresponds to four vertices a_v, b_v, c_v, d_v with two edges $a_v b_v, c_v d_v$, and for every edge uv of G, there are all edges between $\{a_u, b_u, c_u, d_u\}$ and $\{a_v, b_v, c_v, d_v\}$. Suppose that φ assigns at least three different colors in $\{a_v, b_v, c_v, d_v\}$ for some v, say $\varphi(a_v)=i$, $\varphi(b_v)=j$, $\varphi(c_v)=k$, where, up to symmetry, $i<j$ and $k \notin \{i,j\}$. Note that b_v has no neighbor of color k, because its neighbors are either a_v or adjacent to c_v. So $j<k$. At least one color $h \in \{i,j\}$ is not the color of d_v, so c_v has no neighbor of color h, a contradiction. So φ uses exactly two colors in $\{a_v, b_v, c_v, d_v\}$ for every vertex v of G. It follows that the restriction of φ on the subgraph of $G[S_2][K_2]$ induced by $\{a_v, b_v | v \in V(G)\}$, which is isomorphic to $G[K_2]$, is a greedy coloring with q colors. So the claim that $\Gamma(G[S_2]\square K_2) \leq \Gamma(G[K_2])$ is established. This completes the proof. ■

Remark 11. Theorem 10 can be generalized in a straightforward manner to the following result: *Let G be any graph and p be any integer. Then* $\Gamma(G[K_p]) = \Gamma(G[S_p]\square K_p)$.

Theorem 10 implies that Conjectures 4–6 do not hold, as follows.

Corollary 12. *There is a graph H such that* $\Gamma(H)=3$ *and* $\Gamma(K_2\square H)=7$.

Proof. Let G_3 be the graph that consists of a cycle of length 6 plus one vertex g adjacent to a vertex a of the cycle and one vertex h adjacent to another vertex b of the cycle, where a and b are adjacent. Let $H = G_3[S_2]$. Asté et al. [1] showed that $\Gamma(G_3)=3$ and $\Gamma(G_3[K_2])=7$. Hence, Lemma 9 yields $\Gamma(H)=3$ and Theorem 10 yields $\Gamma(K_2\square H)=7$. This proves the corollary.

Alternately, let G'_3 be the graph obtained from G_3 by identifying the two vertices g and h (i.e., replacing them by one vertex adjacent to a and b), and let $H' = G'_3[S_2]$. Then one can also check that $\Gamma(H')=3$ and $\Gamma(K_2\square H')=7$. ■

Clearly, the two graphs H and H' mentioned in the preceding proof are counterexamples to Conjectures 4 and 5. Note also that if v is any vertex of H and a_v, b_v are the corresponding two vertices in $K_2\square H$, then the set $M = \{a_v b_v | v \in V(H)\}$ is a matching in $K_2\square H$, and $(K_2\square H)\setminus M$ consists of two disjoint copies of H with no edge between them; so $\Gamma((K_2\square H)\setminus M)=3$. This shows that $K_2\square H$ is a counterexample to Conjecture 6. The same holds for $K_2\square H'$.

Corollary 12 shows that Conjecture 4 does not hold if $\Gamma(H)=3$. On the other hand, we now show that Conjecture 4 holds if $\Gamma(H)=2$.

Proposition 13. *Let G and H be two graphs. If* $\Gamma(H)=2$, *then* $\Gamma(G\square H) \leq 2(\Delta(G)+1)$.

Proof. If H is not connected, and has components $H_1, \ldots, H_p$ $(p \geq 2)$, then $G\square H$ is the disjoint union of $G\square H_1, \ldots, G\square H_p$, and it suffices to prove the proposition for each graph $G\square H_i$. Therefore, we may assume that H is connected. If $\Gamma(H)=2$, then H is a complete bipartite graph [10]. Let (A,B) be its bipartition. For every vertex $v \in V(G)$, define $A_v = \{(v,a)|a \in A\}$ and $B_v = \{(v,b)|b \in B\}$, so A_v and B_v are the two sides of the copy of H indexed by v in $G\Box H$. Let φ be a greedy coloring of $G\Box H$. We claim that:

$$\text{For any v in } V(G),\ |\varphi(A_v)| \le \Delta(G)+1 \text{ and } |\varphi(B_v)| \le \Delta(G)+1.$$

Assume for a contradiction, and up to symmetry, that $|\varphi(A_v)| \ge \Delta(G)+2$. Let α be the largest color of $\varphi(A_v)$ and let $x=(v,a)$ be a vertex colored α. The neighborhood of x in $G\Box H$ is $B_v \cup \{(w,a)|w \in N_G(v)\}$. But the colors of $\varphi(A_v)$ do not appear on B_v because it is complete to A_v, and $|\{(w,a)|w \in N_G(v)\}| = d_G(v) \le \Delta(G)$. Hence, at most $\Delta(G)$ colors of $\varphi(A_v)$ may appear on the neighborhood of x, and so at least one color of $\varphi(A_v)\setminus\{\alpha\}$ does not. This contradicts the fact that φ is a greedy coloring and proves the claim.

Let $y=(v,b)$ be a vertex such that $\varphi(y)$ is maximum. Without loss of generality, we may assume that $b \in B$. At most $2\Delta(G)+1$ colors appear in the neighborhood of y: at most $\Delta(G)+1$ on A_v according to the claim, and at most one more for each of its neighbors not in B_v, whose number is $d_G(y) \le \Delta(G)$. Hence, $\varphi(y) \le 2\Delta(G)+2$. ■

Remark 14. Proposition 13 can easily be generalized to complete multipartite graphs in a straightforward manner to obtain the following result: *if H is a complete multipartite graph, then* $\Gamma(G\Box H) \le (\Delta(G)+1)\Gamma(H)$.

3. THE DIRECT AND STRONG PRODUCTS

Here we show that $\Gamma(G\times H)$ and $\Gamma(G\boxtimes H)$ cannot be bounded by a function of $\Gamma(G)$ and $\Gamma(H)$ if $\Gamma(G), \Gamma(H) \ge 3$ (Theorem 15). It is also a natural question to bound $\Gamma(G\times H)$ or $\Gamma(G\boxtimes H)$ in terms of $\Delta(G)$ and $\Gamma(H)$. For $\Delta(G)=1$, a non-trivial construction of [2] shows that $3\lceil\Gamma(H)/2\rceil - 1 \le \Gamma(K_2\times H)$. Somewhat surprisingly, we show in Theorem 17 that there is no upper bound on $\Gamma(K_2\times H)$ in terms of $\Gamma(H)$ if $\Gamma(H)\ge 5$. Moreover, we show in Theorem 18 that there is no upper bound on $\Gamma(P_3\boxtimes H)$ in terms of $\Gamma(H)$ if $\Gamma(H)\ge 5$. In fact, Theorem 18 implies that there is no upper bound on $\Gamma(G\boxtimes H)$ as a function $\Delta(G)$ and $\Gamma(H)$ for $\Gamma(H)\ge 5$ unless G is the disjoint union of complete graphs. In Proposition 19, we show that there is an upper bound in such a case.

Let us first recall some definitions. The *binomial tree* is the graph T_k defined recursively as follows. For $k=1$, T_1 is the one-vertex graph. For $k\ge 2$, T_k is obtained from T_{k-1} by adding, for each vertex v of T_{k-1}, one vertex v' with an edge vv'. It is easy to see that, for $k\ge 2$, T_k has two adjacent vertices r,s of degree $k-1$ and the other vertices have degree at most $k-2$, and the two components of $T_k\setminus rs$ are both isomorphic to T_{k-1}. We view T_k as rooted at vertex r. We have $\Gamma(T_k)=k$. More precisely, T_k has a greedy coloring ψ where each vertex $v\notin\{r,s\}$ has color equal to its degree, and s,r have color $k-1$ and k, respectively. Note that for each vertex v and color $i<\psi(v)$, v has a unique neighbor of color i.

The *radius* of a graph G is the smallest integer t for which there exists a vertex a of G such that every vertex of G is at distance at most t from a. Note that the radius of T_k is $k-1$. It is easy to see that every tree with radius at most 2 has Grundy number at most 3. This is also a corollary of the following result from [5, 6]: *the Grundy number of a tree is equal to the Grundy number of its largest binomial subtree*, and of the fact that the radius of a subtree of a tree T is not larger than the radius of T.

Theorem 15. *For every $k \geq 3$, there is a graph G such that $\Gamma(G) = 3$ and $\Gamma(G \times G) \geq k$ and $\Gamma(G \boxtimes G) \geq k$.*

Proof. Let G be the graph obtained from T_k by subdividing every edge once. Partition the vertex set of G into two stable sets A and B such that A contains the original vertices of T_k and B contains the subdivision vertices. Consider any greedy coloring of G. Every vertex in B has degree 2 and consequently receives a color from the set $\{1,2,3\}$. Moreover, a vertex in B receives color 3 if and only if its two neighbors have received colors 1 and 2, respectively. It follows that no vertex of A can receive color 4 or more. This implies that $\Gamma(G) \leq 3$. Since G contains a four-vertex path, $\Gamma(G) \geq 3$. Thus $\Gamma(G) = 3$. To complete the proof of the theorem, let us show that $G \times G$ and $G \boxtimes G$ have a common induced subgraph H_k isomorphic to T_k. This implies $\Gamma(G \times G) \geq k$ and $\Gamma(G \boxtimes G) \geq k$.

Let the root r of T_k become the root of G. Since G is viewed as a rooted tree, every vertex in B has one parent and one child. Consider the greedy coloring ψ of T_k with k colors as defined above, such that the root r has color k and the second vertex s of degree $k-1$ has color $k-1$. For $i \in \{1,\ldots,k\}$, let A_i be the set of vertices in A that receive color $(k+1)-i$. So $A_1 = \{r\}$ and $A_2 = \{s\}$. For each $i \in \{2,\ldots,k\}$, let B_i be the set of vertices in B whose child is in A_i. We say that a vertex v in $A_i \cup B_i$ has *label* i and denote by ℓ_v the label of v. Let q be the parent of s (i.e., q is the common neighbor of r and s). Let $d(x,y)$ denote the distance between any two vertices x and y in G. We prove by induction on $i \in \{2,\ldots,k\}$ that $G \times G$ and $G \boxtimes G$ have an induced subgraph H_i such that:

(1) H_i is isomorphic to T_i and contains vertex (r,q).
(2) Every vertex of H_i is of the form (a,b) or (b,a), with $a \in A$ and $b \in B$; moreover, $\ell_a < \ell_b \leq i$, vertices a, b lie in distinct components of $G \setminus rq$, and $d(a,r) = d(b,q)$.

For $i = 2$, the induced subgraph H_2 with vertices (r,q) and (q,r) and an edge between them is the desired copy of T_2. Now let $i \geq 3$. By the induction hypothesis, there exists a common induced subgraph H_{i-1} of $G \times G$ and $G \boxtimes G$ that satisfies (1) and (2). Let z be any vertex of H_{i-1}, and let $a \in A$ and $b \in B$ be such that z is equal to (a,b) or (b,a). Let u be the unique child of b in G. By the definition of the labels, we have $\ell_u = \ell_b$. By property (2), we have $\ell_a \leq i-1$, so (in T_k, and since ψ is a greedy coloring) a has a neighbor of color $(k+1)-i$, and (in G) a has a neighbor $v \in B$ with label i. Clearly, u and v lie in distinct components of $G \setminus rq$ since a and b do. Now, either (v,u) or (u,v) is a neighbor of (a,b) in $G \times G$ and we call this neighbor the *leaf* of z, and z is called the *support* of its leaf. Note that any leaf-support edge is also an edge in $G \boxtimes G$ as $E(G \times G) \subseteq E(G \boxtimes G)$. Since v has label i, the leaf of z is not a vertex in H_{i-1}. Since $\ell_u = \ell_b \leq i-1$ and $\ell_v = i$, we have $\ell_u < \ell_v \leq i$. Since u is a child of b and v is a child of a, we have $d(u,r) = d(v,q)$. (More precisely: if a lies in the component G_r of $G \setminus rq$ that contains r and b lies in the other component G_q, then $d(u,r) = d(b,q)+2$ and $d(v,q) = d(a,r)+2$; if on the contrary a lies in G_q and b lies in G_r, then $d(u,r) = d(b,q)$ and $d(v,q) = d(a,r)$.)

Let V_{i-1} be the vertex set of H_{i-1} and let W_{i-1} be the set of leaves of vertices in V_{i-1}. Let H_i be the subgraph of $G \boxtimes G$ induced by the vertices in $V_{i-1} \cup W_{i-1}$. As observed above, H_i satisfies property (2). In order to show that H_i is isomorphic to T_i, we need only to prove that (i) each vertex in W_{i-1} has a unique neighbor in V_{i-1} and (ii) W_{i-1} induces a stable set. Note that this also implies that H_i is an induced subgraph in $G \times G$ as $E(G \times G) \subseteq E(G \boxtimes G)$.

To show that Claim (i) is true, suppose on the contrary that the leaf $(v,u)\in W_{i-1}$ of some vertex $(a,b)\in V_{i-1}$ is adjacent to a vertex $(x,y)\in V_{i-1}$ different from (a,b). Up to symmetry we may assume that $a,u\in A$ and $b,v\in B$ and that a lies in G_r and b in G_q (the argument in the other cases is similar). We must have $x=a$, for otherwise x is either v or the child of v and $\ell_x=i$, which contradicts property (2) in H_{i-1}. Since $x\in A$, then $y\in B$ by property (2). Now, $y\neq b$, and y is a child of u. Now $d(y,q)=d(b,q)+2$, whereas $d(x,r)=d(a,r)$, so $d(x,r)\neq d(y,q)$, a contradiction.

To show that Claim (ii) is true, suppose on the contrary that (a,b) and (b',a') are two adjacent vertices in W_{i-1}. We can consider $a,a'\in A$ and $b,b'\in B$ as they could not be adjacent otherwise. Let (s_a,s_b) and $(s_{b'},s_{a'})$ be the supports of (a,b) and (b',a'), respectively. Note that $s_a,s_{a'}\in B$ and $s_b,s_{b'}\in A$, which implies that $\ell_{s_b}<\ell_{s_a}$ and $\ell_{s_{b'}}<\ell_{s_{a'}}$. By the definition of the labels, we have $\ell_{s_a}=\ell_a$ and $\ell_{s_{a'}}=\ell_{a'}$. Moreover, each of b and b' has label i and consequently has a child of label i and $\ell_a<\ell_b=i$. Thus, for (a,b) to be adjacent to (b',a'), a must be the neighbor of b' with label smaller than i, which is $s_{b'}$. In particular, $\ell_a=\ell_{s_{b'}}$, and, by a symmetric argument, $\ell_{a'}=\ell_{s_b}$. Putting this all together, we obtain that if (a,b) is adjacent to (b',a'), then $\ell_a=\ell_{s_{b'}}<\ell_{s_{a'}}=\ell_{a'}=\ell_{s_b}<\ell_{s_a}=\ell_a$ which is a contradiction. ■

To prove Theorems 17 and 18, we study the graph H_k defined as follows. We start from the binomial tree T_k whose vertex set is partitioned into three sets X_1,X_2,X_3. The root of T_k is in X_1. For every $v\in X_1\cup X_3$, the children of v are in X_2. For every $v\in X_2$, the children of v are placed according to the position of the parent w of v: if $w\in X_1$, then the children of v are in X_3; if $w\in X_3$, then the children of v are in X_1. Now H_k is obtained by adding to T_k all edges between X_1 and X_3.

Theorem 16. *For $k\geq 1$, $\Gamma(H_k)\leq 5$. Furthermore, for $k\geq 9$, $\Gamma(H_k)=5$.*

Proof. We first observe that $\Gamma(H_k)\leq 6$ for every k. Indeed, in H_k every stable set is contained either in $A_1=X_1\cup X_2$ or in $A_2=X_2\cup X_3$. If H_k admits a greedy coloring with at least seven colors, then at least four color classes are included in one of the two sets A_1 and A_2, say in A_j. This means that the subgraph H^* induced by A_j in H_k has Grundy number at least four. However, each component of H^* is a tree of radius at most 2, which implies that H^* has Grundy number at most 3.

In order to complete the first part of the theorem, let us give a more detailed analysis to show that $\Gamma(H_k)\leq 5$. The following two properties of T_k are useful:

(1) Any vertex $v\in X_2$ has either exactly one neighbor in X_1 or exactly one neighbor in X_3 (because if the parent of v is in one of X_1,X_3, then all its children are in the other of these two sets).

(2) For $i=1,3$, no path on five vertices in $X_i\cup X_2$ has its two endvertices in X_i (because every component of $X_i\cup X_2$ consists of either the root of T_k and its children, or some vertex of X_2, its children and its grandchildren).

Suppose that there exists a greedy 6-coloring φ on H_k.

Case 1. $\varphi(v)\in\{5,6\}$ *for* $v\in X_2$. Vertex v has neighbors of colors 1, 2, 3, 4. By property (1), v is adjacent to at most one vertex of X_1 or X_3. So there is $i\in\{1,3\}$ such that v has neighbors $w_1,w_2,w_3\in X_i$ with $\varphi(w_1)<\varphi(w_2)<\varphi(w_3)\leq 4$. Then w_3 has a neighbor w_4 with $\varphi(w_4)=\varphi(w_2)$, and w_4 has a neighbor w_5 with $\varphi(w_5)=\varphi(w_1)$. Since $\{w_2,w_4\}$ and $\{w_1,w_5\}$ are stable sets, we have $w_4\in X_2$ and $w_5\in X_i$. But then the path w_1-v-w_3-w_4-w_5 contradicts property (2).

Case 2. $\varphi(v)=6$ *for some* $v\in X_1\cup X_3$. Let i be the index in $\{1,3\}$ such that $v\in X_i$. Vertex v has a neighbor w with $\varphi(w)=5$. Then $w\in X_{4-i}$, otherwise Case 1 applies. Vertices v and w have neighbors u_v and u_w of color 4, possibly $u_v=u_w$, but we cannot have one in X_1 and the other in X_3. Hence, one vertex $u\in\{u_v,u_w\}$ is in X_2. Let t be its neighbor in $\{v,w\}$ and j the index such that $t\in X_j$. Vertex u has three neighbors a, b, c such that $\{\varphi(a),\varphi(b),\varphi(c)\}=\{1,2,3\}$. By property (1), either two elements of $\{a,b,c\}$, say a, b, are in X_j, or $\{a,b,c\}\subset X_{4-j}$. If $a,b\in X_j$, we may assume $\varphi(a)<\varphi(b)$, and we pick a neighbor d of t with $\varphi(d)=\varphi(b)$ and a neighbor e of d with $\varphi(e)=\varphi(a)$. Since $\{a,e\}$ and $\{b,d\}$ are stable sets in H_k, we have $d\in X_2, e\in X_j$. But then the path e–d–t–u–a contradicts property (2). If $\{a,b,c\}\subset X_{4-j}$, we may assume that $\varphi(a)=1$, $\varphi(b)=2$ and $\varphi(c)=3$. There is a neighbor d of c with $\varphi(d)=2$ and a neighbor e of d with $\varphi(e)=1$. Since $\{a,e\}$ and $\{b,d\}$ are stable sets in H_k, we have $d\in X_2, e\in X_{4-j}$. But then the path e–d–c–u–a contradicts property (2). Thus we have shown that $\Gamma(H_k)\leq 5$, which completes the first part of the theorem.

Now, we show that $\Gamma(H_k)=5$ when $k\geq 9$. We know that $\Gamma(T_k)=k$, so T_k contains a path $a_1-a_2-\cdots-a_9$ whose vertices are colored $k,k-1,\ldots,k-8$, respectively, where a_1 is the root of T_k, and a path $a_2-b_3-b_4-b_5$ whose vertices are colored $k-1,k-3,k-4,k-5$, and a path $a_6-b_7-b_8$ whose vertices are colored $k-5,k-7,k-8$. Note that vertices a_1,a_5,a_9,b_5 are in X_1, vertices a_2,a_4,a_6,a_8,b_4,b_8 are in X_2 and vertices a_3,a_7,b_3,b_7 are in X_3. Now we can make a greedy coloring of H_k with five colors, where vertices a_2,a_5,b_5,b_8,a_9 receive color 1, vertices a_3,b_4,b_7,a_8 receive color 2, vertices b_3,a_6 receive color 3, and vertices a_1 and a_7 receive colors 4 and 5. ■

Theorem 17. *If G is a graph with at least one edge and $k\geq 1$, then $\Gamma(G\times H_k)\geq k$.*

Proof. It is enough to prove the theorem when $G=K_2$, $V(G)=\{v_1,v_2\}$. We claim that $\Gamma(G\times H_k)\geq k$. To see this, let $Y_i=\{v_1\}\times X_i$ for $i=1,3$ and $Y_2=\{v_2\}\times X_2$. Then it is easy to check that $Y_1\cup Y_2\cup Y_3$ induces a copy of T_k in $K_2\times H_k$, where Y_i plays the role of X_i in the partition of H_k. ■

Theorem 18. *If G is a connected non-complete graph and $k\geq 1$, then $\Gamma(G\boxtimes H_k)\geq k$.*

Proof. It is enough to prove the theorem when $G=P_3=v_1-v_2-v_3$ as G contains an induced subgraph isomorphic to P_3. We claim that $\Gamma(G\boxtimes H_k)\geq k$. To see this, let $Y_i=\{v_i\}\times X_i$ for $i\in\{1,2,3\}$. It is easy to check that $Y_1\cup Y_2\cup Y_3$ induces a copy of T_k in $P_3\boxtimes H_k$, where Y_i plays the role of X_i in the partition of H_k. ■

If G is a the disjoint union of complete graphs, then there is an upper bound on $\Gamma(G\boxtimes H)$ as a function of $\Gamma(G)$ and $\Gamma(H)$. It is enough to consider the case $G=K_{m+1}$. Observe that $K_{m+1}\boxtimes H=H[K_{m+1}]$. Hence, by Theorem 1 we get the following.

Proposition 19. *If $\Gamma(H)=k\geq 2$ and $m\geq 1$ then $\Gamma(K_{m+1}\boxtimes H)\leq m2^{k-1}+k$.*

4. COMMENTS AND OPEN QUESTIONS

Section 3 shows that any upper bound on the Grundy number of $G\times H$ as a function of $\Delta(G),\Gamma(H)$ is possible only if $\Gamma(H)\leq 4$. Perhaps a good test case is to decide whether

$\Gamma(K_2 \times H)$ is bounded for $\Gamma(H) \leq 4$. (On the other hand, if the maximum degree of both graphs may intervene, then we know the easy inequality $\Gamma(G \times H) \leq \Delta(G \times H) + 1 \leq \Delta(G)\Delta(H) + 1$, but this is probably not a very interesting bound.)

A referee asked whether, in replacement for the now failed Conjecture 4, the following inequality could be conjectured to hold for any two graphs G and H

$$\Gamma(G \Box H) \leq \max\{(\Delta(G)+1)\,\Gamma(H), (\Delta(H)+1)\,\Gamma(G)\} \qquad (**)$$

We can prove this inequality, as follows. First, suppose that $\Gamma(G) = 1$. Then G has no edge, so the left-hand side of $(**)$ is $\Gamma(H)$ and the right-hand side is $\max\{\Gamma(H), \Delta(H)+1\} = \Delta(H)+1$, so $(**)$ holds for every H. Now suppose that $\Gamma(G) \geq 2$ and similarly $\Gamma(H) \geq 2$. On the right-hand side of $(**)$, we have $\max\{(\Delta(G)+1)\Gamma(H), (\Delta(H)+1)\Gamma(G)\} \geq \frac{1}{2}\{(\Delta(G)+1)\,\Gamma(H) + (\Delta(H)+1)\,\Gamma(G)\} \geq \Delta(G) + 1 + \Delta(H) + 1$. On the left-hand side, we have $\Gamma(G \Box H) \leq \Delta(G \Box H) + 1 = \Delta(G) + \Delta(H) + 1$, so it is strictly smaller than the right-hand side. Actually this proof shows that $(**)$ tends to give a weak upper bound on $\Gamma(G \Box H)$ in general; indeed in all cases it is weaker than $\Delta(G \Box H) + 1$.

Concerning the lexicographic product, it was proved in [1] that if $\Gamma(H) = k$, then for any graph G, we have $\Gamma(G[H]) = \Gamma(G[K_k])$. Moreover, as mentioned in Remark 11, we have $\Gamma(G[K_k]) = \Gamma(G[S_k] \Box K_k)$. So $\Gamma(G[H]) = \Gamma(G[S_k] \Box K_k)$. Thus the Grundy number of the lexicographic product of any two graphs G and H can be seen as a particular case of the Grundy number of the Cartesian product of two graphs. Therefore, we feel that the most interesting questions in this domain are about the Cartesian product. In particular, although Conjecture 4 is now known to be false because of Corollary 12, one may still wonder whether there exists a constant λ such that any two graphs G and H satisfy $\Gamma(G \Box H) \leq \lambda(\Delta(G)+1)\,\Gamma(H)$. Note that the graph H given in the proof of Corollary 12 satisfies $\Gamma(K_2 \Box H) = \frac{7}{6}(\Delta(K_2)+1)\,\Gamma(H)$, and so does the second graph H'. We could not find a graph with a ratio larger than $\frac{7}{6}$. Is it true that $\Gamma(K_2 \Box H) \leq 2c\,\Gamma(H)$ for some constant $c \geq 7/6$?

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