Star versus two stripes Ramsey numbers and a conjecture of Schelp

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October 21, 2011

Abstract

R. H. Schelp conjectured that if $G$ is a graph with $|V(G)| = R(P_n, P_n)$ is such that $\delta(G) > \frac{3|V(G)|}{4}$ then in every 2-coloring of the edges of $G$ there is a monochromatic $P_n$. In other words, the Ramsey number of a path does not change if the graph to be colored is not complete but has large minimum degree.

Here we prove Ramsey type-results that imply the conjecture in a weakened form, first replacing the path by a matching, showing that the star-matching-matching Ramsey number $R(S_n, nK_2, nK_2) = 3n - 1$ which extends $R(nK_2, nK_2) = 3n - 1$, an old result of Cockayne and Lorimer. Then we extend this further from matchings to connected matchings and outline how this

*Research supported in part by NSF Grant DMS-0968699.
implies Schelp’s conjecture in asymptotic sense through a standard application of the Regularity Lemma.

It is sad that we are unable to hear Dick Schelp’s reaction to our work generated by his conjecture.

1 Introduction

The path-path Ramsey number was determined in [7] and its diagonal case (stated for convenience for even paths) is that \( R(P_{2n}, P_{2n}) = 3n - 1 \), i.e. in every 2-coloring of the edges of \( K_{3n-1} \), the complete graph on \( 3n - 1 \) vertices, there is a monochromatic \( P_{2n} \), a path on \( 2n \) vertices. An easy example shows that \( K_{3n-2} \) can be 2-colored with no monochromatic \( P_{2n} \). It is a natural question to ask whether a similar conclusion is true if \( K_{3n-1} \) is replaced by some subgraph of it. One such result was obtained in [10] where it was proved that in every 2-coloring of the edges of the complete 3-partite graph \( K_{n,n,n} \) there is a monochromatic \( P_{(1-o(1))2n} \). The following conjecture of Schelp [15] states that \( K_{3n-1} \) can be replaced by a graph \( G \) of large minimum degree \( \delta(G) \).

**Conjecture 1.** Suppose that \( n \) is large enough and \( G \) is a graph on \( 3n - 1 \) vertices with minimum degree larger than \( \frac{3|V(G)|}{4} \). Then in any 2-coloring of the edges of \( G \) there is a monochromatic \( P_{2n} \).

Schelp’s conjecture is stated in its original form as in [15] but it is probably true for every \( n \geq 1 \). In fact, apart from Theorem 6, all results we prove here are valid for every \( n \).

Schelp also noticed that the condition on the minimum degree in Conjecture 1 is close to best possible. Indeed, suppose that \( 3n - 1 = 4m \) for some \( m \) and consider a graph whose vertex set is partitioned into four parts \( A_1, A_2, A_3, A_4 \) with \( |A_i| = m \). Assume there are no edges from \( A_1 \) to \( A_2 \) and from \( A_3 \) to \( A_4 \); edges in \( [A_1, A_3] \), \( [A_2, A_4] \) are red, edges in \( [A_1, A_4] \), \( [A_2, A_3] \) are blue and edges within \( A_i \)-s are colored arbitrarily. In this coloring the longest monochromatic path has \( 2m = \frac{3n-1}{2} \) vertices, much smaller than \( 2n \), while the minimum degree is \( 3m - 1 = \frac{3(3n-1)}{4} - 1 \). Thus, and this makes the conjecture surprising, even a miniscule increase in the minimum degree results in a dramatic increase in the length of the longest monochromatic path. Schelp notes in [15] that he proved (and he referred [16]) that there exists a \( c < 1 \) for which Conjecture 1 holds if the minimum degree is raised to \( c|V(G)| \).

We will prove Ramsey type results leading to an asymptotic version of Conjecture 1. As a first step, we have Theorem 2 and its diagonal case, Corollary 3, a weaker form of Conjecture 1, where paths are replaced by matchings. This is a “traditional” 3-color Ramsey-type result which strengthens significantly (the 2-color case of) a well-known result of Cockayne and Lorimer [3].
Let $nK_2$ denote a matching of size $n$, i.e. $n$ pairwise disjoint edges, and let $S_t$ be a star with $t$ edges. The Ramsey number for two matchings (in fact for any number of matchings) was determined in [3] as $R(n_1K_2, n_2K_2) = 2n_1 + n_2 - 1$ for $n_1 \geq n_2$. The next result extends this, as it implies that the Ramsey number for two matchings does not change if a graph of maximum degree $n_1 - 1$ is deleted from $K_{2n_1+n_2-1}$. It is worth noting that the Ramsey number for many stars and one matching was determined in [4].

**Theorem 2.** Suppose that $n_1 \geq n_2 \geq 1$ and $t \geq 1$. Then

$$R(S_t, n_1K_2, n_2K_2) = \begin{cases} 2n_1 + n_2 - 1 & \text{if } t \leq n_1, \\ n_1 + n_2 - 1 + t & \text{if } t \geq n_1. \end{cases}$$

**Corollary 3.** $R(S_n, nK_2, nK_2) = 3n - 1$.

Next we have Theorem 4 which is still weaker than Conjecture 1, but it gives a monochromatic connected matching of the right size. This is the main result of this paper.

**Theorem 4.** Suppose that a graph $G$ has $3n - 1$ vertices and $\delta(G) > \frac{3\nu(G)}{4}$. Then, in every 2-coloring of the edges of $G$ there is a monochromatic connected matching of size $n$.

It is worth mentioning the following lemma that is used in the proof of Theorem 4. A well-known remark of Erdős and Rado says that in a 2-colored complete graph there is a monochromatic spanning tree. For a survey of results grown from this remark, see [8]. Lemma 5 extends the remark from complete graphs (where $\delta(G) = |V(G)| - 1$) to graphs of large minimum degree.

**Lemma 5.** Suppose that the edges of a graph $G$ with $\delta(G) \geq \frac{3\nu(G)}{4}$ are 2-colored. Then there is a monochromatic component with order larger than $\delta(G)$. This estimate is sharp.

In Section 4 we outline how Theorem 4 and the Regularity Lemma imply Theorem 6, the asymptotic form of Conjecture 1. This technique is established by Łuczak in [13] and used successfully in many recent results, see e.g. [2], [6], [9], [10],[11].

**Theorem 6.** For every $\eta > 0$ there is an $n_0 = n_0(\eta)$ such that the following is true. Suppose that $G$ is a graph on $n \geq n_0$ vertices with $\delta(G) > \frac{2}{3} + \eta)n$. Then in every 2-coloring of the edges of $G$ there is a monochromatic path with at least $\frac{2}{3} - \eta)n$ vertices.

We note that Benevides, Łuczak, Scott, Skokan and White recently [1] proved Conjecture 1.
2 Proof of Theorem 2

To see that the claimed Ramsey number cannot be less than claimed in Theorem 2, consider a partition of \( n_1 + n_2 + \max\{t, n_1\} - 2 \) vertices into three sets, \( A, B, C \) of size \( n_1 - 1, n_2 - 1, \max\{t, n_1\} \), respectively. Color all edges incident to some vertex of \( B \) blue. From the remaining uncolored edges color red those that are incident to \( A \). If \( t > n_1 \) then all edges within \( C \) remain uncolored (or might be viewed as the ‘star-color’). If \( t \leq n_1 \) then \( |C| = n_1 \) and in this case color all edges red within \( C \). (In fact this is the 2-coloring of \( K_{2n_1 + n_2 - 1} \) that does not have monochromatic matching of size \( n_i \) in color \( i \).

Clearly, there is no \( S_t \) in the star-color, there is no red \( n_1K_2 \) and no blue \( n_2K_2 \).

To prove the other direction, Consider a graph \( G \) with \( f(n_1, n_2, t) \) vertices, where

\[
f(n_1, n_2, t) = \begin{cases} 
2n_1 + n_2 - 1 & \text{if } t \leq n_1 \\
n_1 + n_2 - 1 + t & \text{if } t \geq n_1
\end{cases}
\]

and consider an arbitrary red-blue coloring of the edges of \( G \). We show that either there is a vertex nonadjacent to at least \( t \) vertices or a red matching of size \( n_1 \) or a blue matching of size \( n_2 \). Notice that the case \( t < n_1 \) obviously follows from the case \( t = n_1 \) so we may assume that \( |V(G)| = n_1 + n_2 - 1 + t \) and \( t \geq n_1 \geq n_2 \). We use induction on \( n_1 \), for \( n_1 = 1 \) (thus \( n_2 = 1 \)), the statement is obvious for every \( t \).

In the inductive step we reduce the triple \( (t, n_1, n_2) \) to \( (t, n_1-1, n_2) \) if \( n_1 > n_2 \) and to \( (t, n_1-1, n_1-1) \) if \( n_1 = n_2 \). In both cases we assume that every vertex of \( G \) is nonadjacent to at most \( t-1 \) vertices. Depending on which case we have, either there is a red matching of size \( n_1 - 1 \) or a blue matching of size \( n_2 \) or a blue matching of size \( n_1 - 1 \). If there is a blue matching of size \( n_2 \) there is nothing to prove. Otherwise, by switching colors if necessary, we may assume that there is a red matching of size \( n_1 - 1 \) and our goal is to find a blue matching of size \( n_2 \).

Using the Gallai-Edmonds structure theorem (in fact the Tutte-Berge formula suffices) for the subgraph \( G_R \subset G \) with the red edges, we can find \( X \subset V = V(G) = V(G_R) \) such that \( V \setminus X \) has \( d + |X| \) odd connected components in \( G_R \), where \( d \) is the deficiency of \( G_R \). Using that \( d = |V(G_R)| - 2 \nu(G_R) = n_1 + n_2 - 1 + t - 2(n_1 - 1) = n_2 - n_1 + t + 1 \), the number of odd components of \( V \setminus X \) in \( G_R \) is \( t - n_1 + n_2 + 1 + |X| \). We consider the union of all even connected components of \( V \setminus X \) as one special component and label the components as \( C_0, C_1, \ldots, C_m \) so that \( |C_0| \) is the largest component and either \( m = t - n_1 + n_2 + |X| \) (if all components are odd) , or \( m = t - n_1 + n_2 + 1 + |X| \) (if there are nonempty even components). Note that \( m \geq 1 \).

Let \( H \) be the graph with vertex set \( V(G) \setminus X \) and with edge set as those edges of \( G \) that connect different \( C_i \)'s. Obviously all edges of \( H \) are blue. We are going to prove that \( H \) has a (blue) matching of size \( n_2 \). Notice that \( X \) together with one vertex from each odd component must be in \( V(G) \), thus \( |X| + t - n_1 + n_2 + 1 + |X| \leq n_1 + n_2 - 1 + t \).
implying that \(|X| \leq n_1 - 1\). Therefore \(|V(H)| = |V(G)| - |X| \geq n_1 + n_2 - 1 + t - |X| \geq n_1 + n_2 - 1 + t - (n_1 - 1) \geq 2n_2\). If \(H\) has minimum degree at least \(n_2\) then (using that \(|V(H)| \geq 2n_2\)) a well-known lemma in [5] implies that \(H\) has a matching of size \(n_2\) and the proof is finished. Thus we may assume that there is a component \(C_i\) and \(y \in C_i\) such that \(d_H(y) < n_2\). Then,

\[
n_2 > d_H(y) \geq (n_1 + n_2 - 1 + t) - |X| - |C_i| - (t - 1) = n_1 + n_2 - |X| - |C_i|
\]

and we get that \(|C_i| > n_1 - |X|\) and since \(|X| \leq n_1 - 1\), we can write \(|C_i| = n_1 - |X| + k\) with some integer \(k \geq 1\). In fact, \(C_i = C_0\) because we cannot have any other component \(C_j\) as large as \(C_i\) otherwise

\[
|V| \geq |X| + |C_i| + |C_j| + t - n_1 + n_2 + |X| - 1 >
\]

\[
> |X| + n_1 - |X| + n_1 - |X| + t - n_1 + n_2 + |X| - 1 =
\]

\[
= n_1 + n_2 + t - 1 = |V|,
\]

a contradiction.

Set \(D = V(H) \setminus C_0\) and notice that \(D\) is nonempty because \(m \geq 1\). One can easily estimate the degree \(d_H(y)\) for \(y \in C_0\) in the bipartite subgraph \([C_0, D] \subset H\) as follows.

\[
d_H(y) \geq (n_1 + n_2 - 1 + t) - |X| - |C_0| - (t - 1) = n_1 + n_2 - |X| - (n_1 - |X| + k) = n_2 - k. \tag{1}
\]

On the other hand, for any \(y \in C_i\) with \(i > 0\),

\[
d_H(y) \geq |C_0| + t - n_1 + n_2 + |X| - 1 - (t - 1) = |C_0| - n_1 + n_2 + |X| =
\]

\[
= n_1 - |X| + k - n_1 + n_2 + |X| = n_2 + k \tag{2}
\]

because, apart from at most \(t - 1\) non-adjacency, \(y\) is adjacent to vertices of \(C_0\) and to at least one vertex of at least \(m - 1 \leq t - n_1 + n_2 + |X| - 1\) components.

We show, with the folkloristic argument of the lemma in [5] cited above (in fact it is credited there to Dirac) that conditions (1), (2) ensure a matching of size \(n_2\) in \(H\).

Let \(M\) be a maximum matching in the bipartite subgraph \([C_0, D] \subset H\), assume \(M\) has \(s \leq n_2 - 1\) edges. Let \(M^*\) be a matching of \(H\) such that it covers all vertices of \(C_0 \cap M\) and among those it is largest possible. Set \(Y = V(M) \cup D\).

Suppose first that \(M\) covers all vertices of \(C_0\). If \(M^*\) has less than \(n_2\) edges then (since \(Y = V(H)\) in this case and \(|V(H)| \geq 2n_2\)) at least two vertices, \(v, w\) of \(H\) are uncovered by \(M^*\). Now the choice of \(M^*\) implies that all edges of \(H\) from \(v, w\) must go to vertices of \(M^*\) but condition (2) implies that there exists \(e \in M^*\) such that \(u, v\) are adjacent to two ends of \(e\). Replacing \(e\) by these two edges, we get a matching of size one larger than the size of \(M^*\), a contradiction.
If $M$ does not cover $C_0$, select $z \in C_0 \setminus M$. By condition (1) $z$ is adjacent (in $H$) to a set $B$ of $n_2 - k$ vertices in $D \cap M$. Let $A$ be the set of vertices mapped by $M$ from $B$ to $C_0$. From the choice of $M$, no edges of $H$ goes from $D \setminus M$ to $A$ or to $C_0 \setminus M$.

Suppose that $D \setminus M = \emptyset$. Then $|D| = |M| = s \leq n_2 - 1$ implies that $V \setminus X$ has at most $n_2$ odd components (vertices of $D$ and $C_0$) in $G_R$. However, as we have seen above, $V \setminus X$ has $t - n_1 + n_2 + 1 + |X| > n_2$ odd components in $G_R$, contradiction.

Using (2) for every $v \in D \setminus M$, the degree of $v$ in $D$ is at least $n_2 + k - (s - (n_2 - k)) = 2n_2 - s$. This implies that $|Y| > s + 2n_2 - s = 2n_2$ which allows us to use the same argument as in the previous paragraph, to show that $M^*$ has size at least $n_2$. We conclude that $G$ has a blue matching of size $n_2$. \hfill \Box

3 Large connected matchings, proof of Theorem 4

**Proof of Lemma 5.** To see that the estimate of the lemma is sharp, consider $K_n$ from which the edges of a balanced complete bipartite graph $[A, B]$ are removed, where $|A| = |B| = m$ ($0 \leq m \leq \frac{n}{2}$). Set $C = V(K_n) \setminus (A \cup B)$, color all edges incident to $A$ red, all edges incident to $B$ blue and all edges within $C$ arbitrarily. Now $\delta(G) = n - m - 1$ and the largest monochromatic component in both colors have $n - m$ vertices. The theorem is also sharp in the sense that $\delta(G)$ cannot be lowered. Indeed, suppose that $n$ is divisible by four, consider four disjoint sets $S_i$ with $|S_i| = n/4$. Let the pairs within $S_i$ and in $[S_1, S_2], [S_3, S_4]$ be red edges and the pairs in $[S_1, S_3], [S_2, S_3]$ be blue edges. This defines a 2-colored graph $G$ with $n$ vertices, $\delta(G) = \frac{3n}{4} - 1$ and all monochromatic components have only $n/2$ vertices.

To prove that there is a monochromatic component of the claimed size, assume that $|V(G)| = n$, $\delta(G) \geq \frac{3n}{4}$ and let $v \in V(G)$. Let $R, B$ denote the vertex sets of the red and blue monochromatic components containing $v$. Observe that there are no edges in the bipartite graphs $[B \setminus R, R \setminus B], [R \cap B, V(G) \setminus (R \cup B)]$.

Clearly, from the minimum degree condition, $|V(G) \setminus (R \cup B)| < \frac{n}{4}$. If $B \setminus R$ or $R \setminus B$ is empty then $R$ or $B$ is larger than $\frac{3n}{4}$. Otherwise, both $B \setminus R$ and $R \setminus B$ are smaller than $\frac{n}{4}$. We conclude that for the largest monochromatic, say red, component $C$ of $G$,

$$|C| > \frac{n}{2}$$

holds.

We show that in fact, $|C| > \delta(G)$. Set $D = V(G) \setminus C$. Since $C$ is a red component, all edges of $[C, D]$ are blue. Moreover, because of (3) and the minimum degree condition, the set of blue neighbors of any two vertices $v, w \in D$ must intersect in $C$. This implies that $F = D \cup A$ is connected in blue, where $A = \{x \in C : \exists v \in$
By the choice of $C$, $|A \cup D| \leq |C|$, therefore

$$|D| \leq |C \setminus A| < n - \delta(G)$$

because any vertex of $D$ is nonadjacent to all vertices of $C \setminus A$. Thus $|D| < n - \delta(G)$ implying $|C| > \delta(G)$ as desired. $\square$

Now we are ready to prove Theorem 4, the extension of Corollary 3.

**Proof of Theorem 4.** Set $V = V(G)$ and let $C_1$ be a largest monochromatic, say red, component. From Lemma 5, $|C_1| > \frac{3|V(G)|}{4}$. If $U = V \setminus V(C_1) \neq \emptyset$ then $U$ is covered by a blue component $C_2$ because from the minimum degree condition the set the blue neighbors of any two vertices in $U$ intersect in $C_1$. If $U = \emptyset$, then define $C_2$ as a largest blue component in $G$. Set $p = |V(C_1) \setminus V(C_2)|, q = |V(C_2) \setminus V(C_1)|$, from the choice of $C_1$ $p \geq q$. Set $A = V(C_1) \cap V(C_2)$. Observe that there are no edges of $G$ in the bipartite graph $[V(C_1) \setminus V(C_2), V(C_2) \setminus V(C_1)]$. Thus, if $V(C_2) \setminus V(C_1) \neq \emptyset$ then $p < \frac{3n-1}{4} < n$.

We apply Theorem 2 to the subgraph spanned by $A$ in $G$ with parameters $t = \lfloor \frac{3n-1}{4} \rfloor, n_1 = n - q, n_2 = n - p$. To do this, we need to check that $n_2 = n - p \geq 1$. This is obvious if $q > 0$ since then $p < n$ as noted in the previous paragraph. On the other hand, if $q = 0$, i.e. $V(C_1) = V$, we need another argument, in fact similar to the one used in the proof of Theorem 2. Observe that the largest red matching in $C_1$ is automatically connected, thus we may assume it has $m < n$ edges. Applying the Tutte-Berge formula for the red graph, we can find a set $X \subset V$ whose removal leaves at least $c = 3n - 1 - 2m + |X|$ odd components. Let $H$ be the blue subgraph of $G$ whose vertex set is $V \setminus X$ and whose edge set is the set of blue edges of $G$ that go between the red components of $V \setminus X$. Notice that $|X| \leq n - 1$ otherwise $G$ has at least $c + |X| = 3n - 1 - 2m + 2|X| > 3n - 1$ vertices, contradiction. Thus $|V(H)| \geq 2n$. We show that $H$ is a connected graph. Indeed, otherwise $V(H)$ can be partitioned into two nonempty sets $P, Q$ so that there are no edges in the bipartite subgraph $[P, Q]$ of $H$, w.l.o.g. $|P| \geq n$. If $P$ intersects each of the $c$ red components in $V \setminus X$ then any $v \in V(H) \setminus P$ is nonadjacent in $G$ to at least $c - 1 \geq n$ vertices, one vertex in all components not containing $v$. On the other hand, if $P$ does not intersect a red component then a vertex $v$ from that component is nonadjacent in $G$ to all vertices of $P$. In both cases we find a vertex $v$ nonadjacent to at least $n$ vertices and that contradicts the assumption $\delta(G) > \frac{3(3n-1)}{4}$. We conclude that $H$ is connected (in blue), i.e. we may assume $|V(C_2)| \geq 2n$ implying $p = |V| - |V(C_2)| \leq n - 1$, therefore $n_2 = n - p \geq 1$ as required.

We claim that with our above choices of the parameters $t, n_1, n_2$ we have $|A| = 3n - 1 - p - q \geq R(S_t, n_1 K_2, n_2 K_2)$. Indeed, for $t \leq n_1$ we have to check that $3n - 1 - p - q \geq 2(n - q) + n - p - 1$ which reduces to $q \geq 0$. For $t > n_1$ we have to check $3n - 1 - p - q \geq n - p + n - q - 1 + t$ which reduces to $n \geq t$, obviously true.
for our choice of $t$. Thus by Theorem 2 we have either a vertex with $t$ edges missing from it or a red matching of size $n - p$ or a blue matching of size $n - q$. The first possibility contradicts the minimum degree assumption on $G$. Thus we have one of the other two possibilities when we claim that the matchings are extendible to the required size.

Indeed, assume that $M$ is a red matching of size $n - p$ in $G[A]$. Every edge from $V(C_1) \setminus A$ to $A$ is red. The red degree of any $v \in V(C_1) \setminus A$ towards $A$ is at least $P = \frac{3(3n - 1)}{4} - p$ and we claim that $P \geq 2(n - p) + p$. Indeed, the inequality reduces to $\frac{3(3n - 1)}{4} \geq 2n$ which is obvious. Thus all the $p$ vertices in $V(C_1) \setminus A$ are adjacent to at least $p$ vertices of $A \setminus V(M)$ and that clearly allows to extend $M$ by $p$ red edges to a red matching of size $n$.

Similarly, a blue matching $M$ of size $n - q$ can be extended (in case of $q = 0$ no extension is needed of course) by checking the inequality $Q = \frac{3(3n - 1)}{4} - q \geq 2(n - q) + q$ that in fact reduces to the same inequality as in the previous case and finishes the proof. □

4 Building paths from connected matchings

Here we sketch how to get Theorem 6 from Theorem 4 and the Regularity Lemma [17]. The material of this section is fairly standard by now, so we omit some of the details. Combining the Degree form and the 2-color version of the Regularity Lemma we get the following version. (For these and other variants of the Regularity Lemma see [12].)

**Lemma 7.** [Regularity Lemma – 2-colored Degree form] For every $\varepsilon > 0$ and every integer $m_0$ there is an $M_0 = M_0(\varepsilon, m_0)$ such that for $n \geq M_0$ the following holds. For all graphs $G = G_1 \cup G_2$ with $V(G_1) = V(G_2) = V$, $|V| = n$, and real number $\rho \in [0, 1]$, there is a partition of the vertex-set $V$ into $l + 1$ sets (so-called clusters) $V_0, V_1, \ldots, V_l$, and there are subgraphs $G' = G'_1 \cup G'_2$, $G'_1 \subset G_1$, $G'_2 \subset G_2$ with the following properties:

- $m_0 \leq l \leq M_0$,
- $|V_0| \leq \varepsilon|V|$,
- all clusters $V_i$, $i \geq 1$, are of the same size $L$,
- $\deg_{G'}(v) > \deg_G(v) - (\rho + \varepsilon)|V|$ for all $v \in V$,
- $G'|_{V_i} = \emptyset$ ($V_i$ are independent in $G'$),

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• all pairs $G'_{V_i \times V_j}, 1 \leq i < j \leq l$, are $\varepsilon$-regular, each with a density 0 or exceeding $\rho$.

• all pairs $G'_{sV_i \times V_j}, 1 \leq i < j \leq l, 1 \leq s \leq 2$, are $\varepsilon$-regular.

Let $G$ be a graph on $n \geq n_0$ vertices with $\delta(G) > (\frac{3}{4} + \eta)n$ and consider a 2-coloring $G = G_1 \cup G_2$ of $G$. We apply Lemma 7 for $G$, with $\varepsilon \ll \rho \ll \eta \ll 1$. We get a partition of $V = \bigcup_{0 \leq i \leq l} V_i$. We define the following reduced graph $G^R$: The vertices of $G^R$ are $p_1, \ldots, p_l$, and we have an edge between vertices $p_i$ and $p_j$ if the pair $(V_i, V_j)$ is $\varepsilon$-regular in $G'$ with density exceeding $\rho$. Since in $G'$, $\delta(G') > (\frac{3}{4} + \eta - (\rho + \varepsilon))|V|$, an easy calculation shows that in $G^R$ we have $\delta(G^R) \geq (\frac{3}{4} + \eta - 2\rho)l > \frac{3}{4}l$ (see e.g. [14] for a similar computation). Define an edge-coloring $G^R = G^R_1 \cup G^R_2$ in the following way. The edge $p_ip_j$ is colored with the color that contains more edges from $G'_{V_i \times V_j}$, thus clearly the density of this color is still at least $\rho/2$ in $G'_{V_i \times V_j}$.

We remove at most two vertices from $G^R$ to make sure that the number of vertices has the form $3k - 1$. Then, applying Theorem 4 to the 2-colored $G^R$ we get a connected monochromatic matching saturating at least $\frac{2l}{3}$ vertices of $G^R$. To lift this monochromatic connected matching to a monochromatic path in the original graph can be done by applying the following standard lemma (special case of Lemma 4.2 in [9]) with $c = 2/3$ and with our choices of $\varepsilon, \rho$ and reduced graph $G^R$.

Lemma 8. Assume that for some positive constant $c$ there is a monochromatic connected matching $M$ (say in $G^R_1$) saturating at least $c|V(G^R)|$ vertices of $G^R$. Then in the original $G$ we find a monochromatic path in $G_1$ covering at least $c(1 - 3\varepsilon)n$ vertices.

Acknowledgement. The careful work of a referee and advices of Oliver Riordan is appreciated.

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