On (δ, χ)-bounded families of graphs

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Abstract

A family $\mathcal{F}$ of graphs is said to be (δ, χ)-bounded if there exists a function $f(x)$ satisfying $f(x) \to \infty$ as $x \to \infty$, such that for any graph G from the family, one has $f(\delta(G)) \leq \chi(G)$, where $\delta(G)$ and $\chi(G)$ denotes the minimum degree and chromatic number of G, respectively. Also for any set $\{H_1, H_2, \ldots, H_k\}$ of graphs by $Forb(H_1, H_2, \ldots, H_k)$ we mean the class of graphs that contain no H_i as an induced subgraph for any $i = 1, \ldots, k$. In this paper we first answer affirmatively the question raised by the second author by showing that for any tree T and positive integer ℓ, $Forb(T, K_{\ell,\ell})$ is a (δ, χ)-bounded family. Then we obtain a necessary and sufficient condition for $Forb(H_1, H_2, \ldots, H_k)$ to be a (δ, χ)-bounded family, where $\{H_1, H_2, \ldots, H_k\}$ is any given set of graphs. Next we study (δ, χ)-boundedness of $Forb(\mathcal{C})$ where $\mathcal{C}$ is an infinite collection of graphs. We show that for any positive integer ℓ, $Forb(K_{\ell,\ell}, C_6, C_8, \ldots)$ is (δ, χ)-bounded. Finally we show a similar result when $\mathcal{C}$ is a collection consisting of unicyclic graphs.

1 Introduction

A family $\mathcal{F}$ of graphs is said to be (δ, χ)-bounded if there exists a function $f(x)$ satisfying $f(x) \to \infty$ as $x \to \infty$, such that for any graph G from the family one has $f(\delta(G)) \leq \chi(G)$, where $\delta(G)$ and $\chi(G)$ denotes the minimum degree and chromatic number of G, respectively. Equivalently, the family $\mathcal{F}$ is (δ, χ)-bounded if $\delta(G_n) \to \infty$ implies $\chi(G_n) \to$

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∞ for any sequence $G_1, G_2, \ldots$ with $G_n \in \mathcal{F}$. Motivated by Problem 4.3 in [6], the second author introduced and studied (δ, χ)-bounded families of graphs (under the name of δ-bounded families) in [10]. The so-called color-bound family of graphs mentioned in the related problem of [6] is a family for which there exists a function $f(x)$ satisfying $f(x) \to \infty$ as $x \to \infty$, such that for any graph G from the family one has $f(col(G)) \leq \chi(G)$, where $col(G)$ is defined as $col(G) = \max\{\delta(H) : H \subseteq G\} + 1$. As shown in [10] if we restrict ourselves to hereditary (i.e. closed under taking induced subgraph) families then two concepts (δ, χ)-bounded and color-bound are equivalent. The first specific results concerning (δ, χ)-bounded families appeared in [10] where the following theorem was proved (in a somewhat different but equivalent form).

Theorem 1 *([10]) For any set $\mathcal{C}$ of graphs, $Forb(\mathcal{C})$ is (δ, χ)-bounded if and only if there exists a constant $c = c(\mathcal{C})$ such that for any bipartite graph $H \in Forb(\mathcal{C})$ one has $\delta(H) \leq c$.*

Theorem 1 shows that to decide whether $Forb(\mathcal{C})$ is (δ, χ)-bounded we may restrict ourselves to bipartite graphs. We shall make use of this result in proving the following theorems.

Similar to the concept of (δ, χ)-bounded families is the concept of χ-bounded families. A family $\mathcal{F}$ of graphs is called χ-bounded if for any sequence $G_i \in \mathcal{F}$ such that $\chi(G_i) \to \infty$, it follows that $\omega(G_i) \to \infty$. The first author conjectured [2] (independently by Sumner [9]) the following

Conjecture 1 *For any fixed tree T, $Forb(T)$ is χ-bounded.*

2 Finite (δ, χ)-bounded families

The first result in this section shows that for any tree T and positive integer ℓ, $Forb(T, K_{\ell,\ell})$ is (δ, χ)-bounded which answers affirmatively a problem of [10].

Theorem 2 *For every fixed tree T and fixed integer ℓ, and any sequence $G_i \in Forb(T, K_{\ell,\ell})$, $\delta(G_i) \to \infty$ implies $\chi(G_i) \to \infty$.*

We shall prove Theorem 2 in the following quantified form.

Theorem 3 *For every tree T and for positive integers ℓ, k there exist a function $f(T, \ell, k)$ with the following property. If G is a graph with $\delta(G) \geq f(T, \ell, k)$ and $\chi(G) \leq k$ then G contains either T or $K_{\ell,\ell}$ as an induced subgraph.*

In Theorem 3 we may assume that the tree T is a complete p-ary tree of hight r, T_p^r, because these trees contain any tree. Using Theorem 1 we note that to prove Theorem 3 it is enough to show the following lemma.

Lemma 1 *For every p, r, l there exists $g(p, r, \ell)$ such that the following is true. Every bipartite graph H with $\delta(H) \geq g(p, r, \ell)$ contains either T_p^r or $K_{\ell,\ell}$ as an induced subgraph.*

Proof. To prove the lemma, we prove slightly more. Call a subtree $T \subseteq H$ a *distance tree* rooted at $v \in V(H)$ if T is rooted at v and for every $w \in V(T)$ the distance of v and w in T is the same as the distance of v and w in H. In other words, in a distance tree T, level i of T, L_i, is a subset of the vertices at distance i from v in H. Notice that - a distance tree of H is an induced subtree of H if and only if $xy \in E(H)$ such that $x \in L_i, y \in L_{i+1}$ implies $xy \in E(T)$ (observe that in this statement it is important that H is a bipartite graph otherwise $xy \in E(H)$ would be possible with $x, y \in L_i$).

We claim that with a suitable $g(p, r, \ell)$ lower bound for $\delta(H)$ every vertex of a bipartite graph H is the root of an induced distance tree T_p^r in H.

The claim is proved by induction on r. For $r = 1$, $g(p, 1, \ell) = p$ is a suitable function for every ℓ, p. Assuming that $g(p, r, \ell)$ is defined for some $r \geq 1$ and for all p, ℓ, define $P = p^{r+1}(\ell - 1)$ and

$$u = g(p, r + 1, \ell) = \max\{g(P, r, \ell), 1 + 2^{Pp^r}(\max\{p - 1, \ell - 1\})\} \tag{1}$$

Suppose that $\delta(H) \geq u$, $v \in V(H)$. By induction, using that $u \geq g(P, r, \ell)$ by (1), we can find an induced distance tree $T = T_P^r$ rooted at v. In fact we shall only extend a subtree T^* of T, defined as follows. Keep p from the P subtrees under the root and repeat this at each vertex of the levels $1, 2, \ldots r - 2$. Finally, at level $r - 1$, keep all of the P children at each vertex. (Here one can refine the proof to get better bounds.) Let L denote the set of vertices of T^* at level r, $L = \cup_{i=1}^{p^r} A_i$ where the vertices of A_i have the same parent in T^*, $|A_i| = P$. Let $X \subseteq V(H) \setminus V(T^*)$ denote the set of vertices adjacent to some vertex of L. (In fact, since T is a distance tree and H is bipartite, $X \subseteq V(H) \setminus V(T^*)$.) Put the vertices of X into equivalence classes, $x \equiv y$ if and only if x, y are adjacent to the same subset of L. There are less than $q = 2^{Pp^r}$ equivalence classes. Delete from X all vertices of those equivalence classes that are adjacent to at least ℓ vertices of L. Since H has no $K_{\ell,\ell}$ subgraph, at most $q(l - 1)$ vertices are deleted. Delete also from X all vertices of those equivalence classes that have at most $p-1$ vertices. During these deletions less than $q(\max\{p - 1, \ell - 1\}) < u - 1$ vertices were deleted, the set of remaining vertices is Y. It follows from (1) that every vertex of L is adjacent to at least one vertex $y \in Y$ - in fact to at least p vertices of Y in the equivalence class of y.

Now we plan selecting vertex $x_i \in A_i$ so that each of them has a set B_i of p neighbors in Y, the B_i-s are pairwise disjoint and no x_i is adjacent to any vertex in B_j if $j \neq i$. Thus $\cup_{i=1}^{p^r} B_i$ extends T^* to the required induced distance tree T_p^{r+1}.

Start with an arbitrary vertex $x_1 \in A_1$. There are at least p neighbors of x_1 in an equivalence class of Y, define B_1 as p of them. At most $\ell - 1$ vertices of L define this class, thus we can select $x_2 \in A_2$ from a different from those. Now take any neighbor of x_2 and repeat the procedure by selecting $x_3 \in A_3$ different from the at most $2(\ell - 1)$ vertices that may define the previous classes. Since $|A_{p^r}| = P > (p^r - 1)(\ell - 1)$, all these steps can be taken. □

Using Theorem 2 we can characterize (δ, χ)-bounded families of the form $Forb(H_1, \ldots, H_k)$ where $\{H_1, \ldots, H_k\}$ is any finite set of graphs. In the following result by a star tree we mean any tree isomorphic to $K_{1,t}$ for some $t \geq 1$.

Corollary 1 *Given a finite set of graphs $\{H_1, H_2, \ldots, H_k\}$. Then $Forb(H_1, H_2, \ldots, H_k)$ is (δ, χ)-bounded if and only if one of the following holds:*
(i) For some i, H_i is a star tree.
(ii) For some i, H_i is a forest and for some $j \neq i$, H_i is complete bipartite graph.

Proof. Set for simplicity $\mathcal{F} = Forb(H_1, H_2, \ldots, H_k)$. First assume that $\mathcal{F}$ is (δ, χ)-bounded. From the well-known fact that for any d and g there are bipartite graphs of minimum degree d and girth g, we obtain that some H_i should be forest. If H_i is star tree then (i) holds. Assume on contrary that none of H_i's is neither star tree nor complete bipartite graph. Then $K_{n,n}$ belongs to $\mathcal{F}$ for some n. But this violates the assumption that $\mathcal{F}$ is (δ, χ)-bounded.

To prove the converse, first note that by a well known fact (see [10]) if H_i is a star tree then $Forb(H_i)$ is (δ, χ)-bounded. Now since $\mathcal{F} \subseteq Forb(H_i)$ then $\mathcal{F}$ too is (δ, χ)-bounded. Now let (ii) hold. We may assume that H_{i_0} is forest and H_{j_0} is an induced subgraph of $K_{\ell,\ell}$ for some l. It is enough to show that $Forb(H_{i_0}, K_{\ell,\ell})$ is (δ, χ)-bounded. If H_{i_0} is a tree then the assertion follows by Theorem [2]. Let $T_1, \ldots, T_k$ be the connected components of H_{i_0} where $k \geq 2$. We add a new vertex v and connect v to each T_i by an edge. The resulting graph is a tree denoted by T. We have $Forb(H_{i_0}, K_{\ell,\ell}) \subseteq Forb(T, K_{\ell,\ell})$ since H_{i_0} is induced subgraph of T. The proof now completes by applying Theorem [2] for $Forb(T, K_{\ell,\ell})$. □

3 Infinite (δ, χ)-bounded families

In the sequel we consider $Forb(H_1, H_2, \ldots)$ where $\{H_1, H_2, \ldots\}$ is any infinite collection of graphs. When at least one of the H_i-s is tree then the related characterization problem is easy. The following corollary is immediate.

Corollary 2 *Let T be any non star tree. Then $Forb(T, H_1, \ldots)$ is (δ, χ)-bounded if and only if at least one of H_i-s is complete bipartite graph.*

When no graph is acyclic in our infinite collection $H_1, H_2, \ldots$ we face with non-trivial problems. The first result in this regard is a result from [8]. They showed that if G is any even-cycle-free graph then $col(G) \leq 2\chi(G) + 1$. This shows that $Forb(C_4, C_6, C_8, \ldots)$ is (δ, χ)-bounded. Another result concerning even-cycles was obtained in [10] where the following theorem has been proved. Note that $d(G)$ stands for the average degree of G.

Theorem 4 *([10]) Let F be any set of even integers, G a graph with $F \subseteq F(G)$ and $A = E \setminus F$ where E is the set of even integers greater than two. Assume that $A = \{g_1, g_2, \ldots\}$. Set $\lambda = 2d(d+1)$ where $d = gcd(g_1 - 2, g_2 - 2, \ldots)$. If $d \geq 4$ then*

$$\chi(G) \geq \frac{d(G)}{\lambda} + 1.$$

In the sequel using a result from [4] we show that for any positive integer ℓ, $Forb(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$ is (δ, χ)-bounded. For this purpose we need to introduce bipartite chordal graphs. A bipartite graph H is said to be bipartite chordal if any cycle of length at least 6 in H has at least one chord. Let H be a bipartite graph with bipartition (X, Y). A vertex v of H is simple if for any $u, u' \in N(v)$ either $N(u) \subseteq N(u')$ or $N(u') \subseteq N(u)$. Suppose that $\mathcal{L} : v_1, v_2, \ldots, v_n$ is a vertex ordering of H. For each $i \geq 1$ denote $H[v_i, v_{i+1}, \ldots, v_n]$ by H_i. An ordering $\mathcal{L}$ is said to be a simple elimination ordering of H if v_i is a simple vertex in H_i for each i. The following theorem first appeared in [4] (see also [5]).

Theorem 5 *([4]) Let H be a bipartite graph with bipartition (X, Y). Then H is chordal bipartite if and only if it has a simple elimination ordering. Furthermore, suppose that H is chordal bipartite. Then there is a simple ordering $y_1, \ldots, y_m, x_1, \ldots, x_n$ where $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$, such that if x_i and x_k with $i < k$ are both neighbors of some y_j, then $N_{H'}(x_i) \subseteq N_{H'}(x_k)$ where H' is the subgraph of H induced by $\{y_j, \ldots, y_m, x_1, \ldots, x_n\}$.*

Theorem 6 *$Forb(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$ is (δ, χ)-bounded.*

Proof. By Theorem 1 it is enough to show that the minimum degree of any bipartite graph $H \in Forb(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$ is at most $\ell - 1$.

Let H be a bipartite $(K_{\ell,\ell}, C_6, C_8, C_{10}, \ldots)$-free graph with $\delta(H) \geq \ell$. Let $y_1, \ldots, y_m, x_1, \ldots, x_n$ be the simple ordering guaranteed by Theorem 5. The vertex y_1 has at least k neighbors say $z_1, \ldots, z_k$ such that $N(z_1) \subseteq N(z_2) \subseteq \ldots \subseteq N(z_k)$. Now since $d_Y(z_1) \geq k$ so there are k vertices in Y which are all adjacent to z_1. From other side $N(z_1) \subseteq N(z_i)$ for any $i = 1, \ldots, k$. Therefore all these k neighbors of z_1 are also adjacent to z_i for any i. This introduces a subgraph of H isomorphic to $K_{\ell,\ell}$, a contradiction. $\square$

We conclude this section with another (δ, χ)-bounded (infinite) family of graphs. By a unicyclic graph G we mean any connected graph which contains only one cycle. Such a graph is either a cycle or consists of an induced cycle C of length say i and a number of at most i induced subtrees such that each one intersects C in exactly one vertex. We call these subtrees (which intersects C in exactly one vertex) the attaching subtrees of G. Recall from the previous section that T_p^r is the p-ary tree of hight r. For any positive integers p and r by a (p, r)-unicyclic graph we mean any unicyclic graph whose attaching subtrees are subgraph of T_p^r. We also need to introduce some special instances of unicyclic graphs. For any positive integers p, r and even integer i, let us denote the graph consisting of the even cycle C of length i and i vertex disjoint copies of T_p^r which are attached to the cycle C by $U_{i,p,r}$ (to each vertex of C one copy of T_p^r is attached).

Proposition 1 *For any positive integers t, p and r, there exists a constant $c = c(t, p, r)$ such that for any $K_{2,t}$-free bipartite graph H if $\delta(H) \geq c$ then for some even integer i, H contains an induced subgraph isomorphic to $U_{i,p,r}$.*

Proof. Let H be any $K_{2,t}$-free bipartite graph. There are two possibilities for the girth $g(H)$ of H.

Case 1. $g(H) \geq 4r + 3$. Let C be any smallest cycle in H. Since H is bipartite then C has an even length say $i = g(H)$. We prove by induction on k with $0 \leq k \leq i$ that if $\delta(H) \geq g(p, r, t) + 2$ then H contains an induced subgraph isomorphic to the graph obtained by C and k attached copies of T_p^r, where $g(p, r, t)$ is as in Lemma 1. The assertion is trivial for $k = 0$. Assume that it is true for k and we prove it for $k + 1$. By induction hypothesis we may assume that H contains an induced subgraph L consisting of the cycle C plus k copies of T_p^r attached to C. Let v be a vertex of C at which no tree is attached. Let e and e' be two edges on C which are incident with the vertex v. We apply Lemma 1 for $H \setminus \{e, e'\}$. Note that since $\delta(H) \geq g(p, r, t) + 2$ then the degree of v in $H \setminus \{e, e'\}$ is at least $g(p, r, t)$. We find an induced copy of T_p^r grown from v in $H \setminus \{e, e'\}$. Denote this copy of T_p^r by T_0. Consider the union graph $L \cup T_0$. We show that $L \cup T_0$ is induced in H. We only need to show that no vertex of T_0 is adjacent to any vertex of L. The distance of any vertex in T_0 from the farthest vertex in C is at most $r + i/2$. The distance of any vertex in the previous copies of T_p^r in L from C is at most r. Then any two vertices in $T_0 \cup L$ have distance at most $2r + i/2$. Now if there exists an edge between two such vertices we obtain a cycle of length at most $2r + i/2 + 1$ in H. By our condition on the girth of H we obtain $2r + i/2 + 1 < g(H)$, a contradiction. This proves our induction assertion for $k + 1$, in particular the assertion is true for $k = i$. But this means that H contains the cycle C with i copies of T_p^r attached to C in induced form. The latter subgraph is $U_{i,p,r}$. This completes the proof in this case.

Case 2. $g(H) \le 4r+2$. In this case we prove a stronger claim as follows. If H is any $K_{2,t}$-free bipartite graph and $\delta(H) \ge (4r+2)(t-1)(\max\{r+1, p^{r+1}\}) + 1$ with $g(H) = i$ then H contains any graph G which is obtained by attaching k trees $T_1, \ldots, T_k$ to the cycle of length i such that any T_j is a subtree of T_p^r and k is any integer with $0 \le k \le i$. It is clear that if we prove this claim then the main assertion is also proved.

Now let G be any graph obtained by the above method. We prove the claim by induction on the order of G. If G consists of only a cycle then its length is i and any smallest cycle of H is isomorphic to G. Assume now that G contains at least one vertex of degree one and let v be any such vertex of G. Set $G' = G \setminus v$. We may assume that H contains an induced copy of G'. Denote this copy of G' in H by the very G'. Let $u \in G'$ be the neighbor of v in G. It is enough to show that there exists a vertex in $H \setminus G'$ adjacent to u but not adjacent to any vertex of G'. Define two subsets as follows: $A = \{a \in V(G') : au \in E(G')\}$, $B = \{b \in V(H) \setminus V(G') : bu \in E(H)\}$.

It is clear that $A \cup B$ is independent. Let $C = V(G') \setminus A \setminus \{u\}$. The number of edges between B and C is at most $(t-1)|C|$. We claim that there is a vertex say $z \in B$ which is not adjacent to any vertex of C, since otherwise there will be at least $|B|$ edges between B and C. This leads us to $|B| \le (t-1)|C|$. From other side for the order of C we have $|C| \le (4r+2)(\max\{r+1, p^{r+1}\})$. Let $n_{p,r} = (4r+2)(\max\{r+1, p^{r+1}\})$. We have therefore $|B| \le (t-1)(n_{p,r} - |A| - 1)$ and $|A| + |B| \le (t-1)n_{p,r}$. But $|A| + |B| = d(u) > (t-1)n_{p,r}$, a contradiction. Therefore there is a vertex z that is adjacent to u in H but not adjacent to $G' \setminus \{u\}$. By adding the edge uz to G' we obtain an induced subgraph of H isomorphic to G, as desired.

Finally by taking $c = \max\{g(p,r,t) + 2\ ,\ (4r+2)(t-1)(\max\{r+1, p^{r+1}\}) + 1\}$ the proof completes. □

Using Proposition 1 and Theorem 1 we obtain the following result.

Theorem 7 *Fix positive integers $t \ge 2$, p and r. For any $i = 1, 2, 3, \ldots$, let G_i be any (p,r)-unicyclic graph whose cycle has length $2i+2$. Then $Forb(K_{2,t}, G_1, G_2, \ldots)$ is (δ, χ)-bounded.*

4 Concluding remarks

If a family $\mathcal{F}$ is both (δ, χ)-bounded and χ-bounded then it satisfies the following stronger result. For any sequence $G_1, G_2, \ldots$ with $G_i \in \mathcal{F}$ if $\delta(G_i) \to \infty$ then $\omega(G_i) \to \infty$. Let us call any family satisfying the latter property, (δ, ω)-bounded family.

The following result of Rödl (originally unpublished) which was later appeared in Kierstead and Rödl ([7] Theorem 2.3) proves the weaker form of Conjecture 1.

Theorem 8 *For every fixed tree T and fixed integer ℓ, and any sequence $G_i \in Forb(T, K_{\ell,\ell})$, $\chi(G_i) \to \infty$ implies $\omega(G_i) \to \infty$.*

Combination of Theorem 3 with Theorem 8 shows that $Forb(T, K_{\ell,\ell})$ is (δ, ω)-bounded.

As we noted before the class of even-hole-free graphs is (δ, χ)-bounded. It was proved in [1] that if G is even-hole-free graph then $\chi(G) \leq 2\omega(G) + 1$. This implies that $Forb(C_4, C_6, \ldots)$ too is (δ, ω)-bounded.

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