Ramsey and Turán-type problems for non-crossing subgraphs of bipartite geometric graphs

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April 13, 2010

Abstract

Geometric versions of Ramsey-type and Turán-type problems are studied in a special but natural representation of bipartite graphs and similar questions are asked for general representations. A bipartite geometric graph $G(m,n) = [A, B]$ is *simple* if the vertex classes A, B of $G(m,n)$ are represented in R^2 as

$$A = \{(1,0), (2,0), \dots, (m,0)\}, B = \{(1,1), (2,1), \dots, (n,1)\}$$

and the edge ab is the line segment joining $a \in A$ and $b \in B$ in R^2. This and similar representations (two-layer representations) are studied earlier, and from the point of view of edge crossings, this representation is equivalent to others already in the literature, for example to *cyclic bipartite graphs* or to *ordered bipartite graphs* and certainly almost all textbook figures represent bipartite graphs this way.

Subgraphs - paths, trees, double stars, matchings - are called non-crossing if they do not contain edges with common interior point. The choice of these subgraphs are explained by the fact that connected components of non-crossing subgraphs of simple bipartite geometric graphs must be special trees (caterpillars). We concentrate on *balanced* bipartite graphs, where $m = n$.

The maximum number of edges is determined in a simple bipartite geometric graph $G(n,n)$ that does not contain

- non-crossing matchings with $k + 1$ edges
- matchings with $k + 1$ pairwise crossing edges
- non-crossing trees with $k + 1$ vertices

and in the last case it is shown that any graph with more edges than the extremal value contains a non-crossing double star with $k+1$ vertices. The Ramsey number of non-crossing double stars is also determined: in every 2-coloring of a geometric $K_{n,n}$ there is a non-crossing monochromatic double star with at least $\frac{4n}{5}$ vertices and this is best possible in asymptotic sense.

Finding the Turán number of non-crossing paths and the Ramsey number of non-crossing subtrees and paths remain open together with many similar problems where the position of the vertex set of the bipartite graph is less restricted, either in convex or in general position.

1 Introduction

This paper expands a short abstract [10]. Following [22], a *geometric graph* is a graph whose vertices are in the plane in general position and whose edges are straight-line segments joining the vertices. A geometric graph is *convex* , if its vertices form a convex polygon. A subgraph of a geometric graph is *non-crossing* if no two edges have a common interior point.

Analogues of Turán and Ramsey theories have been considered for geometric graphs and for convex geometric graphs, see [22], [3], [1], [16], [14], [15] and its references.

To describe a specific example, an old remark of Erdős and Rado says that in any 2-coloring of the edges of K_n there is a monochromatic spanning tree. It was proved by Bialostocki, Dierker and Voxman in [2] that there is a monochromatic non-crossing spanning tree in every 2-coloring of the convex geometric graph K_n. They conjectured that this remains true for geometric complete graphs in general and their conjecture was proved by Károlyi, Pach and Tóth in [14]. There are several Ramsey-type and Turán-type results for geometric graphs, see [22] chapter 14, [14], [15]. These results show that Ramsey numbers change significantly for paths, cycles by imposing the non-crossing condition. However, an unpublished result of Perles (a proof is in [15]) states that the maximum number of edges in a graph of n vertices that does not contain a path of length k (determined by Erdős and Gallai [7]) remains the same for non-crossing paths in convex geometric graphs.

In this note we consider geometric versions of Ramsey-type and Turán-type problems for *geometric balanced bipartite graphs,* $G(n,n)$, defined as a geometric graph, whose $2n$ vertices are in two disjoint n-element sets A, B, and its edges are some segments ab with $a \in A, b \in B$. The concept is studied earlier, for example in the form of considering A, B as red and blue sets and investigating the properties of red-blue segments, a survey is [18].

As convex geometric graphs form a natural subclass of geometric graphs, the following representation, apparently studied first in [6], seems to be a most natural subclass of balanced geometric bipartite graphs $G(n,n)$ (in fact a standard way of drawing bipartite graphs). The partite sets of G in R^2 are $A = \{a_1 = (1,0), a_2 = (2,0), \dots, a_n = (n,0)\}$ and $B = \{b_1 = (1,1), b_2 = (2,1), \dots, b_n = (n,1)\}$ and the edge a_ib_j is the line segment joining $a_i \in A$ and $b_j \in B$. This representation is equivalent (from the point of view of crossings) to *cyclic bipartite graphs* (Brass, Károlyi, Valtr [3]), i. e. convex geometric graphs with $2n$ vertices whose partite classes A, B form two 'intervals'. It is also equivalent to *ordered bipartite graphs* where each element of A precedes each elements of B (Füredi and Hajnal [8], Pach and Tardos [23]). For easier reference we call this representation *a simple* $G(n,n)$ in this paper.

Notice the following characterization of non-crossing subgraphs of simple $G(n,n)$-s, sometimes referred as *biplanar graphs* , apparently first discovered by Harary and Schwenk, [13]. A *caterpillar* is a special tree in which the vertices of degree larger than one form a path.

Proposition 1. *([13],[5]) Every connected component of a non-crossing subgraph of simple $G(n,n)$ is a caterpillar.*

Simple geometric graphs are very similar to the 'cyclic bipartite' graphs considered by Brass, Károlyi and Valtr in [3] and the 'ordered bipartite' graphs considered by Füredi and Hajnal [8], and by Pach and Tardos [23]. However, here we investigate only non-crossing subgraphs and do not go into finer details of ordered subgraphs.

It follows from counting arguments of Mubayi [21] and Liu, Morris, Prince [20] that in every 2-coloring of the edges of the complete bipartite graph $K_{n,n}$ there is a monochromatic double star with at least n vertices and this is a sharp result. (A *double star* is a tree obtained by joining the centers of two disjoint stars by an edge, the base edge of the double star). Here we prove that in the geometric version the situation is different.

Theorem 1. *In every 2-coloring of the simple $K_{n,n}$ there is a non-crossing monochromatic double star with at least $\frac{4n}{5}$ vertices. This bound is asymptotically best possible.*

Theorem 2. *Suppose that $2n \geq k$ and a simple bipartite graph $G = G(n,n)$ does not contain non-crossing double stars with $k+1$ vertices. Then $|E(G)| \leq n(k-1) - \lfloor \frac{(k-1)^2}{4} \rfloor$. This bound is sharp for each pair of integers satisfying $2n \geq k$.*

It is worth noting that in Theorem 2 the maximum number of edges is approximately $\frac{n^2}{2}$ when k is approximately $(2-\sqrt{2})n$. Thus Theorem 1 does not follow from Theorem 2. The construction, showing that Theorem 2 is sharp, does not contain *any* non-crossing subgraph with $k+1$ vertices.

Turán and Ramsey problems are also studied for matchings in geometric graphs. Kupitz ([16], see also Theorem 14.4 in [22]) determined the maximal number of edges in a convex geometric graph with n vertices that does not contain non-crossing matchings with $k+1$ edges. Similar result is proved for matchings with pairwise crossing edges ([4], see also Theorem 14.14 in [22]). These results have a unified form for simple bipartite graphs.

Theorem 3. *Assume that $G = G(n,n)$ is a simple bipartite graph that does not contain a non-crossing (crossing) matching with $k+1$ edges. Then $|E(G)| \leq 2kn - k^2$ and this bound is sharp for both cases.*

For the Ramsey problem, it is known ([14]) that in any 2-coloring of the edges of a geometric complete graph with n vertices there is a monochromatic non-crossing matching with $\lfloor \frac{n+1}{3} \rfloor$ edges and this is sharp. The bipartite version here follows immediately from the well-known result that every geometric $K_{n,n}$ contains a perfect matching [19] so the proof of the next proposition is left to the reader.

Proposition 2. *In every r-coloring of the edges of a geometric graph $K_{n,n}$ there is a monochromatic non-crossing matching with at least $\lceil \frac{n}{r} \rceil$ edges. This bound is best possible.*

2 Open problems

2.1 Problems for simple geometric bipartite graphs.

The 2-color Ramsey problem for paths in balanced bipartite graphs have been solved independently in [11] and in [9]. Concerning the geometric version we ask

Problem 1. *What is the length of the largest monochromatic non-crossing path that exists in every 2-coloring of a simple geometric $K_{n,n}$? The bounds $\frac{n}{2}$ and $\frac{2n}{3}$ follow from the cited result of Perles and from using the graph $H(n,p)$ defined below with suitable p .*

The Turán number of paths in bipartite graphs have been determined in [12]. Concerning the geometric version the straightforward question is

Problem 2. *What is the maximum number of edges in a simple geometric graph $G(n,n)$ that does not contain a non-crossing path of length k? The upper bound is $n(k-1)$ from the cited result of Perles and the lower bound is $n(k-1) - \left\lfloor \frac{(k-1)^2}{4} \right\rfloor$, according to the construction given in the proof of Theorem 2.*

Presently we can not separate the Ramsey number of non-crossing trees (caterpillars) from the Ramsey number of non-crossing double stars.

Problem 3. *What is the order of the largest monochromatic non-crossing subtree (caterpillar) that exists in every 2-coloring of the edges of a simple geometric $K_{n,n}$? The bounds are $\frac{4n}{5}$ and n or $n+1$ (depending of the parity of n). Note that the largest monochromatic non-crossing subforest F has at least $n+1$ vertices because a simple geometric $K_{n,n}$ has non-crossing spanning trees (this remark is valid for any geometric $K_{n,n}$).*

2.2 Problems for general geometric bipartite graphs.

It looks as the 'really geometric' problems arise when the vertex set of $G(n,n)$ is allowed to be in general position or in convex position (with no restriction on the sets A, B). Of course, Proposition 1 limits the non-crossing subgraphs to caterpillars. There are some results in the literature, for example it is known that every geometric $K_{n,n}$ contains a perfect matching [19] and a spanning tree of maximum degree three [17]. It seems that from the following (twelve) problems only one can be answered easily (Proposition 2).

Problem 4. *What is the maximum number of edges in a geometric (convex geometric) $G(n,n)$ that does not contain a non-crossing matching (path, caterpillar) with k edges?*

Problem 5. *What is the maximum number of edges in a non-crossing monochromatic matching (path, caterpillar) contained in every 2-coloring of the edges of a geometric (convex geometric) $K_{n,n}$?*

3 A construction

We shall often use a simple balanced geometric graph $H = H(n,p)$ defined as follows. Fix $1 \le p \le n$, and the vertices of H are $a_i = (i,0), b_i = (i,1)$ for $i = 1, 2, \ldots, n$. The edge set of H is $\{a_i b_j : p+2 \le i+j \le 2n-p\}$.

Lemma 1. *Suppose that F is a non-crossing subgraph of H without isolated vertices. Then $|V(F)| \le 2(n-p)$.*

Proof. Let q be the smallest integer such that $a_q \in V(F)$ and let r be the smallest integer for which $a_q b_r \in E(F)$. Similarly, s is the largest integer such that $a_s \in V(F)$ and t is the largest integer for which $a_s b_t \in E(F)$. Since F has no isolates, q, s, t, r are well defined. Set

$$X = \{a_i : 1 \le i < q\} \cup \{b_j : 1 \le j < r\}, Y = \{a_i : s < i \le n\} \cup \{b_j : t < j \le n\}.$$

We claim that at least p vertices of X and at least p vertices of Y are not in $V(F)$. This is obvious for $q > p$ and for $s < n - p + 1$. Suppose $q \leq p$, now $q - 1$ vertices of $A \cap X$ are not in $V(F)$ from the definition of q. From the definition of H, $a_q b_j \notin E(H)$ for $j = 1, 2, \ldots, p - q + 1$, thus any edge of F incident to b_j would cross the edge $a_q b_r$. Therefore none of the b_j-s are in $V(F)$ for $j = 1, 2, \ldots, p - q + 1$. Thus we have at least $q - 1 + p - q + 1 = p$ vertices not in $V(F)$. A symmetric argument shows that at least p vertices of Y are not in $V(F)$. This proves the claim and the lemma follows from it. $\square$

In Ramsey problems we may consider H and its (bipartite) complement as a 2-colored simple geometrical graph $K_{n,n}$. In H the order of largest non-crossing tree, path, double star is the same. This is not the case for $\overline{H}$. For example, it is easy to see that for $2p \geq n$, $\overline{H}$ always contains a non-crossing tree with $n + 1$ vertices. This is not true for double stars as the following lemma shows(we need that to show that Theorem 1 is asymptotically sharp).

Lemma 2. *Assume that* $n = 5k + 2, p = 3k + 1$. *Then the largest non-crossing double star of both* H *and* $\overline{H}$ *has* $4k + 2$ *vertices.*

Proof. Any non-crossing subgraph of H without isolated vertices has at most $2(n - p) = 4k + 2$ vertices from Lemma 1. On the other hand, the complete geometric subgraph of H spanned by $\{a_1, \ldots, a_{n-p}, b_{p+1}, \ldots, b_n\}$ clearly has a spanning double star. Thus the statement of the lemma follows for H. To prove it for $\overline{H}$, we need to look at three cases only, according to the indices of the base edge $e = a_i b_j$ of a double star T. If i, j are both at most $n - p$ then T has at most $p + 1 = 3k + 2 < 4k + 2$ vertices. If $1 \leq i \leq n - p, n - p + 1 \leq j$ then $j \leq p + 1 - i$ follows from the definition of H. There are two non-crossing maximal double stars with base e. One is taking the j 'left' neighbors of a_i and the $2p - n + 2 - i + j$ 'right' neighbors of b_j. Now

$$2p-n+2-i+j \leq 2p-n+2-1+p+1-i \leq 3p-n+1 = 3(3k+1)-(5k+2)+1 = 4k+2$$

proving what we want. The other maximal non-crossing double star on e has $p - j + 2$ vertices since there are $p - j - i + 2$ 'right' neighbors of i and i 'left' neighbors of j. Clearly, $p - j + 2 < 4k + 2$ thus this double star is small. Finally, if $n - p + 1 \leq j \leq p + 1 - i, n - p + 1 \leq i \leq p + 1 - j$, then the two maximal non-crossing double stars on e have $2p - n + 2 - i + j$ and $2p - n + 2 - j + i$ vertices. Assume by symmetry that the first is the maximum, then

$$2p - n + 2 - i + j \leq 2p - n + 2 - i + p + 1 - i \leq 3p - n + 2 = 4k + 2. \quad \square$$

4 Proof of Theorems 1,2,3

Proof of Theorem 1. Consider an arbitrary red-blue coloring of the edges of a balanced geometric bipartite graph $G = [A, B]$. Let G_R, G_B denote the red and blue subgraphs of G. Set

$$D = \max\{d_{G_R}(a_1), d_{G_R}(b_n), d_{G_B}(a_1), d_{G_B}(b_n)\}.$$

Assume first that $D \geq (1/2 + 1/10)n$, without loss of generality the maximum is attained at b_n in the red color. Let i denote the smallest index for which a_ib_n is red. If a_i has at least $(1/4 - 1/20)n$ red neighbors in B, we have a red non-crossing double star on a_ib_n spanning at least $(1/4 + 1/2 + 1/20)n = \frac{4n}{5}$ vertices. Otherwise a_i has at least $(3/4 + 1/20)n = \frac{4n}{5}$ blue neighbors in B giving a red star (a special non-crossing double star) that is as large as required.

In the case when $D < (1/2+1/10)n$, assume (w.l.o.g.) that edge a_1b_n is red. Now - from the definition of D - both $d_{G_R}(a_1)$ and $d_{G_R}(b_n)$ are at least $(1/2-1/10)n$ therefore we have a non-crossing red double star on a_1b_n with at least $2(1/2 - 1/10)n = \frac{4n}{5}$ vertices.

It follows from Lemma 2 that the bound is asymptotically sharp. □

Proof of Theorem 2. We claim first that a geometric $G(n, n)$ that contains no non-crossing double star with $k + 1$ vertices, has at most the claimed number of edges. The proof is by induction n, keeping k fixed. The cases $2n = k$ and $2n - 1 = k$ are obvious. For the induction step, assume that there is no non-crossing double star with $k + 1$ vertices in a geometric graph $G = G(n, n)$ for $2n \geq k + 2$. Select an edge $e = a_ib_j \in E(G)$ with $|i - j|$ is as large as possible. From the choice of e, the edges of G incident to e form a non-crossing double star, thus, from the assumption on G, we have $d_G(a_i) + d_G(b_j) \leq k$. Deleting the vertices a_i, b_j with its incident edges we delete at most $k - 1$ edges and get a balanced geometric graph $F = G(n - 1, n - 1)$. By the inductive hypothesis

$$|E(G)| \leq k-1+|E(F)| \leq k-1+(n-1)(k-1) - \left\lfloor \frac{(k-1)^2}{4} \right\rfloor = n(k-1) - \left\lfloor \frac{(k-1)^2}{4} \right\rfloor$$

proving the claim.

To see that the maximum can be attained, we use the graph $H(n, p)$ for even k. Set $p = n - \frac{k}{2}$, from Lemma 1, the largest non-crossing double star in H has at most $2(n-p) = k$ vertices. On the other hand, $H(n, p)$ has $n^2 - 2\binom{p+1}{2} = n(k-1) - \lfloor \frac{(k-1)^2}{4} \rfloor$ vertices.

For odd k we modify $H(n, p)$ to $H'(n, p)$ so that its edge set is $\{a_ib_j : p+2 \leq i+j \leq 2n - p - 1\}$. Set $p = n - \frac{k-1}{2}$, now the largest non-crossing double star of $H'(n, p)$ has at most $2(n-p)+1 = k$ vertices and $H'(n,p)$ has $n^2 - \binom{p+1}{2} - \binom{p}{2} = n(k-1) - \frac{(k-1)^2}{4}$ edges. □

Proof of Theorem 3. To prepare the proof, we define matchings in the geometric $K_{n,n}$ having pairwise non-crossing (pairwise crossing) edges. For $i = 0, 1, 2, \dots, n-k-1$ let M_i denote the matching with edge set $\{a_1, b_{1+i}, a_2 b_{2+i} \dots, a_{n-i} b_n\}$ and let N_i denote the edge set $\{a_n, b_{n-i}, a_{n-1} b_{n-1-i} \dots, a_{i+1} b_1\}$. Since $N_0 = M_0$, we have $2(n-k-1)+1$ edge-disjoint non-crossing matchings. Assume that $G = G(n,n)$ is a geometric graph containing no non-crossing matching with $k+1$ edges. Then at most k edges of G can be selected from each of the matchings above, thus $|E(G)| \leq k(2n-2k-1)+m$ where m is the number of edges of $K_{n,n}$ not covered by the union of the matchings M_i, N_i. Since $m = 2\binom{k+1}{2}$ we get that

$$|E(G)| \leq k(2n-2k-1) + (k+1)k = 2kn - k^2$$

as desired. This proves the extremal result for non-crossing matchings. The proof for the crossing matching is similar, since one can replace the non-crossing matchings M_i, N_i by matchings containing pairwise crossing edges. To see that both results are sharp, consider k-element sets $X \subset A, Y \subset B$ and define the graph whose edges are incident to $X \cup Y$. This graph has $2kn - k^2$ edges and for $X = \{a_1, \dots, a_k\}, Y = \{b_1, \dots, b_k\}$ it does not contain a non-crossing matching with $k+1$ edges; for $X = \{a_1, \dots, a_k\}, Y = \{b_{n-k+1}, \dots, b_n\}$ it does not contain a matching with $k+1$ pairwise crossing edges. □

Acknowledgement. Thanks to Jani Pach and to Alberto Marquez for pointing to references I was not aware of. I am also grateful to M. Ruszinkó for conversations on the subject and for the proof of Lemma 1 that is simpler than my original proof. Careful reading of a referee is also appreciated.

References

[1] N. Alon, P. Erdős, Disjoint edges in geometric graphs, *Discrete and Computational Geometry* **4** (1989) 287 - 290.

[2] A. Bialostocki, P. Dierker, W. Voxman, Either a graph or its complement is connected: a continuing saga, manuscript (2001)

[3] P. Brass, Gy. Károlyi, P. Valtr, A Turán-type extremal theorey of convex geometric graphs, *in Discrete and computational geometry - Algorithms Combin. Springer, Berlin* **25** (2003) 275-300.

[4] V. Capoyleas, J. Pach, A Turán-type theorem on chords of a convex polygon, *J. of Combinatorial Thoey B* **56** (1992), 9-15.

[5] P. Eades, B.D.McKay, N.C.Wormald, On an edge crossing problem, *Proc. 9th Austr. Computer Science Conference* (1986) 327-334.

[6] P. Eades, S. H. Whitesides, Drawing graphs in two layers, *Theoretical Computer Science* **131.**(1994), 361-374.

[7] P. Erdős, T. Gallai, *On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar.* **10.** (1959) 337-356.

[8] Z. Füredi, P. Hajnal, Davenport-Schinzel theory of matrices, *Discrete Mathematics* **103** (1992) 233 - 251.

[9] R. J. Faudree, R. H. Schelp, Path-path Ramsey-type numbers for the complete bipartite graphs, *Jornal of Comb. Theorey B.* **19** (1975) 161-173.

[10] A. Gyárfás. Ramsey and Turán-type problems in bipartite geometric graphs, *Electronic Notes in Discrete Mathematics* **31** (2008) 253-254.

[11] A. Gyárfás, J. Lehel, A Ramsey-type problem in directed and bipartite graphs, *Periodica Math. Hung.* **3** (1973) 299-304.

[12] A. Gyárfás, C. C. Rousseau, R. H. Schelp, An extremal problem for paths in bipartite graphs, *J. of Graph Theory* **8.** (1984) 83-95.

[13] F. Harary, A. Schwenk, A new crossing number for bipartite graphs, *Utilitas Mathematica* **1.** (1972) 203-209.

[14] G. Károlyi, J. Pach, G. Tóth, Ramsey-type results for geometric graphs, I., *Discrete and Computational Geometry* **18** (1997) 247-255.

[15] G. Károlyi, J. Pach, G. Tóth, P. Valtr, Ramsey-type results for geometric graphs, II., *Discrete and Computational Geometry* **20** (1998) 375-388.

[16] Y. Kupitz, On pairs of disjoint segments in convex position in the plane, Convexity and Graph theory (Jerusalem 1981) *Ann. of Discrete Mathematics* **20.** (1984) 203-208.

[17] A. Kaneko, On the maximum degree of bipartite embedding of trees in the plane, *Discrete and Computational Geometry, LNCS* **1763** (2000) 166-177.

[18] A. Kaneko, M. Kano, Discrete geometry on red and blue points in the plane - a survey, *Discrete and Computational geometry, Algorithms and Combinatorics 25, Springer, Berlin, 2003*, 551-570.

[19] L. C. Larson, Problem solving through problems, Springer, 1983

[20] H. Liu, R. Morris, N. Prince, Highly connected monochromatic subgraphs of multicoloured graphs, submitted.

[21] D. Mubayi, Generalizing the Ramsey problem through diameter, *Electronic Journal of Combinatorics,* **9.** (2002) R41.

[22] J. Pach, P. K. Agarwal, Combinatorial Geometry, *Wiley-Interscience* (1995)

[23] J. Pach, G. Tardos, Forbidden paths and cycles in ordered graphs and matrices, Israel J. Math. **155** (2006) 359-380.