Partitioning 3-colored complete graphs into three monochromatic cycles *

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Abstract

We show in this paper that in every 3-coloring of the edges of K^n all but $o(n)$ of its vertices can be partitioned into three monochromatic cycles. From this, using our earlier results, actually it follows that we can partition *all* the vertices into at most 10 monochromatic cycles, improving the best known bounds. If the colors of the three monochromatic cycles must be different then one can cover $(\frac{3}{4} - o(1))n$ vertices and this is close to best possible.

1 Introduction

It was conjectured in [5] that in every r-coloring of a complete graph, the vertex set can be covered by r vertex disjoint monochromatic cycles (where vertices, edges and

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the empty set are accepted as cycles).

Conjecture 1. *(Erdős, Gyárfás, Pyber, [5]) In every* r*-coloring of the edges of* K_n *its vertex set can be partitioned into* r *monochromatic cycles.*

For general r, the $O(r^2 \log r)$ bound of Erdős, Gyárfás, and Pyber [5] has been improved to $O(r \log r)$ by Gyárfás, Ruszinkó, Sárközy and Szemerédi [8]. The case $r = 2$ was conjectured earlier by Lehel and was settled by Łuczak, Rödl and Szemerédi [13] for large n using the Regularity Lemma. Later Allen [1] gave a proof without the Regularity Lemma and recently Bessy and Thomassé [3] found an elementary argument that works for every n.

The main result of this paper confirms Conjecture 1 in asymptotic sense for $r = 3$.

Theorem 1. *In every* 3*-coloring of the edges of* K_n *all but* $o(n)$ *of its vertices can be partitioned into three monochromatic cycles.*

The history of Conjecture 1 suggests that the cycle partition problem is difficult even in the $r = 2$ case. On the other hand, if we relax the problem and allow two monochromatic cycles to intersect in at most one vertex (almost partition), then it becomes easy. Indeed, Gyárfás [6] gave a simple proof that two cycles of distinct colors that intersect in at most one vertex cover the vertex set. Similar result does not seem to be easy for $r \geq 3$ colors.

Combining Theorem 1 with some of our earlier results from [8] we can actually prove that we can partition *all* the vertices into at most 10 monochromatic cycles, improving the best known bounds for $r = 3$.

Theorem 2. *In every* 3*-coloring of the edges of* K_n *the vertices can be partitioned into at most 10 monochromatic cycles.*

Note that in the same way for a general r if one could prove the corresponding asymptotic result as in Theorem 1 (even with a weaker linear bound on the number of cycles needed; unfortunately we are not there yet), then we would have a linear bound overall. This makes the asymptotic result interesting.

In the proof of Theorem 1 our main tools will be the Regularity Lemma [14] and the following lemma. A *connected matching* in a graph G is a matching M such that all edges of M are in the same component of G.

Lemma 1. *If* n *is even then in every* 3*-coloring of the edges of* K_n *the vertex set can be partitioned into three monochromatic connected matchings.*

In fact, we will need the following density version of Lemma 1.

Lemma 2. *For every $\eta > 0$ there exist n_0 and $\varepsilon > 0$ such that for $n \geq n_0$ the following holds. In every 3-edge coloring of a graph G with n vertices and more than $(1-\varepsilon)\binom{n}{2}$ edges there exist 3 monochromatic connected matchings which partition at least $(1-\eta)n$ vertices of G.*

Certain 3-colorings often occur among extremal colorings for Ramsey numbers of triples of paths, triples of even cycles and their analysis is important in the corresponding results, see e.g. [2, 9]. These colorings also play a crucial role in this paper and we call them 4-*partite colorings*, defined as follows.

The vertex set of K_n is partitioned into four non-empty parts $A_1 \cup A_2 \cup A_3 \cup A_4$, $|A_1| \leq |A_2| \leq |A_3| \leq |A_4|$ such that all edges in the complete bipartite graphs $B(A_1, A_2)$ and $B(A_3, A_4)$ are colored 1, in $B(A_1, A_3)$ and $B(A_2, A_4)$ are colored 2, and $B(A_1, A_4)$ and in $B(A_2, A_3)$ are colored 3. Inside each part the edges are colored arbitrarily.

One can easily observe that in a 4-partite coloring that has equal partite classes and within all the four partite classes all edges are colored with color 1, at most 75 percent of the vertices can be covered by three vertex disjoint cycles having *different colors.* Thus Theorem 1 fails if we insist that the monochromatic cycles must have different colors. On the other hand, the proof of Theorem 3 shows that the example is essentially best possible.

Theorem 3. *In every 3-coloring of the edges of K_n, at least $(\frac{3}{4} - o(1))n$ vertices can be covered by vertex disjoint monochromatic cycles having distinct colors.*

Theorem 3 relies on the following variant of Lemma 1.

Lemma 3. *In every 3-coloring of the edges of K_n vertex disjoint monochromatic connected matchings of distinct colors cover at least $\frac{3n}{4} - 1$ vertices.*

In fact, here again we will need the density version of Lemma 3.

Lemma 4. *For every $\eta > 0$ there exist n_0 and $\varepsilon > 0$ such that for $n \geq n_0$ the following holds. In every 3-edge coloring of a graph G with n vertices and more than $(1-\varepsilon)\binom{n}{2}$ edges vertex disjoint monochromatic connected matchings of distinct colors cover at least $(1-\eta)\frac{3n}{4}$ vertices of G.*

The organization of the paper is as follows. In the next section we present the proofs of Lemmas 1 and 3. Lemma 1 is the key result of the paper because the derivation of Lemma 2 and Theorem 1 from it (as well as the derivation of Theorem 3 and Lemma 4 from Lemma 3) can now be considered as a rather standard application of the Regularity Lemma, as done in [2], [7], [9] and [12]. Therefore in Sections 3 and 4 we just describe these steps briefly. In Section 5 we sketch the proof of Theorem 2.

2 Proofs of Lemmas 1 and 3

Proof of Lemma 1. Take an arbitrary coloring of the edges of K_n with colors, say, 1, 2, and 3. Let G_1, G_2, G_3 be the subgraphs spanned by the edges of colors 1, 2, 3, respectively. First assume that one of the G_i-s, say, G_1 is a connected spanning subgraph of K_n. Then take a maximum matching M_1 in G_1. All the edges in $V(K_n) \setminus V(M_1)$ are colored 2 or 3, thus these vertices are connected in, say, color 2. Take a maximum matching M_2 in color 2. Again, since M_2 is maximal, all edges in $V(K_n) \setminus ((V(M_1) \cup V(M_2)))$ are colored 3. A maximum matching M_3 here will be connected in color 3 and will contain all vertices of $V(K_n) \setminus (V(M_1) \cup V(M_2))$.

Hence from now on we assume that none of G_i-s is connected and spanning. Let H_1 be a largest monochromatic component, say, in color 1, and select a maximum matching $M_1 \subset H_1$ in color 1. It is well-known that $|V(H_1)| \geq \frac{n}{2}$. Let $Y = V(H_1) \setminus V(M_1)$ and $X = [n] \setminus V(H_1)$. Clearly, all edges in the bipartite graph $B(V(H_1), X)$ have color 2 or 3.

Case 1: $|X| \leq |Y|$. Since M_1 is maximum in H_1, edges having both endpoints in Y are colored 2 or 3. Therefore, Y is connected in, say, color 2. Let M_2 a maximum matching in color 2 in the bipartite graph $B(X, Y)$, $Y_1 = Y \setminus V(M_2)$, $X_1 = X \setminus V(M_2)$. If $X_1 \neq \emptyset$ then $B(X_1, Y_1)$ is complete bipartite in color 3. So take a matching M_3 in color 3 of size $|X_1|$ in $B(X_1, Y_1)$. Since $|X_1| \leq |Y_1|$, we covered all vertices in X. If $|X_1| = |Y_1|$ then we are ready. If $|X_1| < |Y_1|$, regardless of $X_1 = \emptyset$ or $X_1 \neq \emptyset$ take a maximum matching in color 2 in $Y_1 \setminus V(M_3)$ and add its edges to M_2. If we did not cover all the vertices in Y_1 then the vertices yet uncovered span a complete graph in color 3. Cover them with a perfect matching and add these edges to M_3. Let $M = M_1 \cup M_2 \cup M_3$. Clearly, we got a partition into matchings and M_1, M_2, M_3 are connected in 1, 2, 3, respectively. Indeed, M_1 is connected because it is entirely in H_1, M_2 is connected because at least one of the endpoints of its edges is in Y which is connected in color 2. M_3 is connected because if $X_1 \neq \emptyset$ then $B(X_1, Y_1)$ is complete bipartite in color 3 and the rest of its edges have both endpoints in Y_1. If $X_1 = \emptyset$ then the edges of M_3 span a complete graph in color 3.

Case 2: $|X| > |Y|$. In this case we reduce the problem to the 4-*partite* case.

Notice that if either $V(H_1)$ or X is in the same color component in color 2 or 3 then we can use an argument similar to the one we used in case $|X| \leq |Y|$ to get the desired partition. Indeed, assume that, say, X is in the same color component in color 2. Since $|V(H_1)| \geq n/2 \geq |X|$, take arbitrary $(|X| - |Y|)/2$ edges from M_1 (note that $|X| - |Y|$ is even, since n is even) and add their $|X| - |Y|$ endpoints to Y. Let Y be extended with new points to get Z, $|Z| = |X|$. Let M_2 be a maximum matching in $B(Z, X)$ in color 2. Since we assumed that X is in the same color component in color 2, M_2 is connected. The yet uncovered vertices in $B(Z, X)$ form a balanced complete bipartite graph in color 3, cover them with a matching in color 3. Those edges in M_1

which do not have endpoints in Z, M_2 and M_3 give the desired partition. The same argument works if H_1 is in the same color component in color 2 or 3.

So we may assume $\emptyset \neq A_1 \neq V(H_1)$ is a subset of vertices of $V(H_1)$ which are in the same color component in color 2. Set $A_2 = V(H_1) \setminus A_1$. If that color component does not extend to X then all edges between A_1 and X are colored 3 which would imply that X is in the same color component in color 3. So let $\emptyset \neq A_3 \neq X$ be the subset of the vertices of X which are in the same color component with A_1 in G_2, $A_4 = X \setminus A_3$. Clearly all edges in $B(A_1, A_4)$ and $B(A_2, A_3)$ are colored 3, else the color component in G_2 containing vertices of $A_1 \cup A_3$ would contain a vertex from $A_2 \cup A_4$ contradicting to the definition of A_i-s. If a single edge in $B(A_1, A_3)$ or $B(A_2, A_4)$ is colored 3 then $B(V(H_1), X)$ is connected in color 3. Therefore, we may assume that all edges in $B(A_1, A_4)$ and $B(A_2, A_3)$ are colored 2. Finally, if a single edge in $B(A_1, A_2)$ or $B(A_3, A_4)$ is colored 2 or 3 then $B(V(H_1), X)$ is connected in color 2 or 3, respectively. Therefore, we may assume that all edges in $B(A_1, A_2)$ and $B(A_3, A_4)$ are colored 1. Thus we have a 4-partite coloring and the proof will be finished by Lemma 5 below. □

We notice that the proof above gives immediately the following (so far we did not have to repeat a color).

Corollary 1. *Let n be even and assume that we have a 3-edge coloring of the edges of K_n that is* not *4-partite. Then $V(K_n)$ can be partitioned into (at most three) monochromatic connected matchings of distinct colors.*

Lemma 5. *Let n be even and assume that we have a 4-partite 3-edge coloring of the edges of K_n. Then $V(K_n)$ can be partitioned into three monochromatic connected matchings.*

Proof of Lemma 5. For transparency we assume first that all $|A_i|$'s are even. A matching is called *crossing* if its edges are go between different A_i's and *inner* if its edges are all within A_i's. A crossing matching C is *proper* with respect to an inner matching M if the vertex set of C intersects any edge of M in two or zero vertices.

Let $a_i(j)$ denote size of a maximum matching in A_i in color j. Here and through the whole proof we consider the size of a matching to be the number of vertices it covers, i.e. twice the number of edges. A matching covering all vertices of X is called perfect in X. The indices will always show the parts in or among which the matching edges are considered, the number in parenthesis is the color. For example, an inner matching $M_3(2)$ is in A_3 and its edges are colored with color 2, a crossing matching $M_{2,4}(3)$ is between A_2, A_4 in color 3.

There are two basic types for the connected components of the required partition into three connected matchings, one is when the components have three different colors, called the *star-like partition*, for example where the three matchings are in the components $A_1 \cup A_4, A_2 \cup A_4, A_3 \cup A_4$ (of color $3, 2, 1$, respectively). The other type is the *path-like partition* that repeats a color, as in the components $A_1 \cup A_3, A_3 \cup A_2, A_2 \cup A_4$ (of colors $2, 3, 2$, respectively.) The three components are referred as the *target components* in both (star-like and path-like) cases.

Claim 1. *If*

$$|A_4| \geq |A_1| + |A_2| + |A_3| - (a_1(3) + a_2(2) + a_3(1)) \tag{1}$$

then there is a star-like partition of K_n.

Proof. Let $M_1(3), M_2(2), M_3(1)$ be inner matchings of size $a_1(3), a_2(2), a_3(1)$, respectively, and let M be an arbitrary perfect matching of A_4. Condition (1) ensures that we can select a crossing matching C that is proper with respect to M and matches

$$(A_1 \setminus V(M_1(3))) \cup (A_2 \setminus V(M_2(2))) \cup (A_3 \setminus V(M_3(1)))$$

to A_4. Since the matchings not covered by C, i.e. $M_1(3), M_2(2), M_3(1)$ and the uncovered part of M, are in the same target components, the claim follows. $\square$

So we may assume

$$|A_4| < |A_1| - a_1(3) + |A_2| - a_2(2) + |A_3| - a_3(1). \tag{2}$$

Next notice that inequalities

$$|A_2| - a_3(1) < |A_4| - a_4(2) \tag{3}$$
$$|A_3| - a_4(2) < |A_2| - a_2(3) \tag{4}$$
$$|A_4| - a_2(3) < |A_3| - a_3(1) \tag{5}$$

cannot hold at the same time. Indeed, else their sum gives $0 < 0$, a contradiction. So at least one of these inequalities is violated and we may assume that one of the following cases must hold:

$$|A_2| - a_3(1) \geq |A_4| - a_4(2) \tag{6}$$
$$|A_3| - a_4(2) \geq |A_2| - a_2(3) \tag{7}$$
$$|A_4| - a_2(3) \geq |A_3| - a_3(1) \tag{8}$$

Case 1: (6) holds. Here we will find a path-like partition in the components $A_1 \cup A_3, A_3 \cup A_2, A_2 \cup A_4$ (of colors $2, 3, 2$, respectively.)

Match vertices of A_1 arbitrarily in color 2 to $|A_1|$ vertices of A_3. Denote this matching by $M_{1,3}(2)$. The rest of the vertices in A_3 can be partitioned into three monochromatic matchings, $M_3(1), M_3(2), M_3(3)$. Match the endpoints of the edges in $M_3(1)$ arbitrarily to $|M_3(1)|$ vertices in A_2, obtaining $M_{3,2}(3)$. This is feasible, since by (6)

$$|A_2| \geq |A_4| - a_4(2) + a_3(1) \geq |M_3(1)|.$$

Now take an inner matching $M_4(2)$ of size $a_4(2)$. The yet uncovered $|A_2| - |M_3(1)|$ vertices in A_2 will be matched to vertices in A_4 so that this matching $M_{2,4}(2)$ covers $A_4 \setminus V(M_4(2))$, and it is proper with respect to $M_4(2)$. This is feasible, because by (6)

$$|A_4| - a_4(2) \leq |A_2| - a_3(1) \leq |A_2| - |M_3(1)| = |A_2| - \frac{|M_{3,2}(3)|}{2}.$$

Since the part of $V(K_n)$ uncovered by the crossing matching $M_{1,3}(2) \cup M_{3,2}(3) \cup M_{2,4}(2)$ is covered by $M_3(2) \cup M_3(3) \cup M_4(2)$ which belong to the target components, we have the required partition.

Case 2: (7) holds. Here we define a path-like partition in the components $A_1 \cup A_4, A_4 \cup A_3, A_3 \cup A_2$ (of colors $3, 1, 3$, respectively.)

Let $M_{1,4}(3)$ be an arbitrary crossing matching that maps A_1 to A_4 and partition the uncovered vertices of A_4 into three monochromatic matchings $M_4(1), M_4(2), M_4(3)$.

Subcase 2.1: $|M_4(2)| \leq |A_3| - |A_2|$. Let $M_{2,3}(3)$ be an arbitrary crossing matching that maps A_2 to A_3. Let $M_{4,3}(1)$ be a crossing matching from the uncovered part of A_3 into $A_4 \setminus V(M_{1,4}(3))$ such that it covers $M_4(2)$ and it is proper with respect to $M_4(1) \cup M_4(2) \cup M_4(3)$. This is feasible since

$$M_4(2) \leq |A_3| - |A_2| \leq |A_4| - |A_1|$$

and the vertex set uncovered by the union of the three crossing matchings is covered by matchings in the same target components (by $M_4(1) \cup M_4(3)$).

Subcase 2.2: $|M_4(2)| > |A_3| - |A_2|$. Now we match $V(M_4(2))$ arbitrarily into $U \subseteq A_3$ by a crossing matching $M_{4,3}(1)$. This is possible since by (7)

$$A_3 \geq |A_2| - a_2(3) + a_4(2) \geq |A_2| - a_2(3) + |M_4(2)| \geq |M_4(2)|.$$

Then take a matching $M_2(3)$ of size $a_2(3)$ in A_2. There exists a crossing matching $M_{3,2}(3)$ from $A_3 \setminus U$ to A_2 such that it covers $A_2 \setminus V(M_2(3))$ and it is proper with respect to $M_2(3)$ because by (7)

$$|A_2| - |V(M_2(3)| = |A_2| - a_2(3) \leq |A_3| - a_4(2) \leq |A_3| - |M_4(2)| =$$

$$= |A_3| - |U| \leq |A_2|,$$

where the last inequality follows from the subcase condition. The vertex set uncovered by the union of the three crossing matchings is covered by $M_4(1) \cup M_4(3)$ so covered by matchings in the target components.

Case 3: (8) holds. A_4 is partitioned into matchings $M_4(1), M_4(2), M_4(3)$. Here we define three subcases.

Subcase 3.1: $|A_2| + |A_3| - |A_1| \geq |A_4| - (|M_4(1)| + |M_4(2)|)$. Here we use the components $A_1 \cup A_2, A_2 \cup A_4, A_4 \cup A_3$ (of colors $1, 2, 1$, respectively.)

First we take $M_{1,2}(1)$ as an arbitrary crossing matching that matches all vertices of A_1 to A_2. The uncovered part of A_2 is partitioned into matchings $M_2(1), M_2(2), M_2(3)$. Take a matching $M_3(1)$ of size $a_3(1)$ in A_3.

We want to define a crossing matching M^* from $A_3 \cup (A_2 \setminus V(M_{1,2}(1))$ to A_4 such that $M^* = M_{2,4}(2) \cup M_{3,4}(1)$ and has the following two properties. One one hand, we want $M_{2,4}(2)$ to cover $M_2(3)$ and $M_{3,4}(1)$ to cover $A_3 \setminus V(M_3(1))$. This is possible since by (8)

$$|M_2(3)| + |A_3| - a_3(1) \leq |M_2(3)| + |A_4| - a_2(3) \leq |A_4|. \tag{9}$$

On the other hand, we want M^* to cover $M_4(3)$ and this is guaranteed by the condition of the present subcase. Indeed

$$|A_2| - |A_1| + |A_3| \geq |M_4(3)| = |A_4| - (|M_4(1)| + |M_4(2)|). \tag{10}$$

Therefore M^* can be defined with the required properties as a proper matching with respect to $M_2(1) \cup M_2(2) \cup M_4(1) \cup M_4(2)$. Notice that the definition of M^* ensures that the vertices uncovered by $M_{1,2}(1) \cup M^*$ are in the target components. This finishes Subcase 3.1.

Subcase 3.2: $|A_1| + (|A_3| - |A_2|) \geq |M_4(2)|$. Here we use the components $A_1 \cup A_4, A_4 \cup A_3, A_3 \cup A_2$ (of colors $3, 1, 3$, respectively.)

Partition A_4 into matchings $M_4(1), M_4(2), M_4(3)$. First match all vertices of A_2 to A_3 to obtain $M_{2,3}(3)$.

Then $M_{1,4}(3)$ and $M_{3,4}(1)$ are defined so that their union is a crossing matching and proper with respect to $M_4(1) \cup M_4(3)$ and $M_{1,4}(3)$ matches the set A_1 to A_4 and $M_{3,4}(1)$ matches $A_3 \setminus V(M_{2,3}(3))$ to A_4. Since $|A_1| + |A_3| \leq |A_2| + |A_4|$, i.e. $|A_1| + (|A_3| - |A_2|) \leq |A_4|$, there is enough room in A_4 for $M_{1,4}(3)$ and $M_{3,4}(1)$. Moreover, by the subcase condition, we can also ensure that $M_{1,4}(3) \cup M_{3,4}(1)$ covers $M_4(2)$. Therefore the vertices uncovered by $M_{2,3}(3) \cup M_{1,4}(3) \cup M_{3,4}(1)$ are covered by $M_4(1) \cup M_4(3)$, so they are in the target components. This finishes Subcase 3.2.

We may assume that the conditions of the previous two subcases are violated therefore adding their negations we get $2|A_3| < |A_4| - |M_4(1)|$ so we have

$$|A_2| + |A_3| \leq 2|A_3| < |A_4| - |M_4(1)| \leq |A_4| \tag{11}$$

and this, combined with (2) gives

$$|A_1| > |M_4(1)|. \tag{12}$$

Subcase 3.3: $a_3(3) \geq |A_3| - |A_2|$ (or $a_3(2) \geq |A_3| - |A_1|$). This condition ensures a crossing matching $M_{2,3}(3)$ that matches the set A_2 to A_3 so that the uncovered part of A_3 has a perfect matching $M_3(3)$. On the other hand, condition (12) ensures that the set A_1 can be matched to A_4 properly by $M_{1,4}(3)$ with respect to $M_4(2) \cup M_4(3)$ so that it covers $V(M_4(1))$. Now matchings $M_{2,3}(3) \cup M_3(3)$, $M_{1,4}(3)$ and the uncovered edges of $M_4(2)$ are three matchings and the edges uncovered by these are in $M_4(3)$ i.e. in a target component. The condition $a_3(2) \geq |A_3| - |A_1|$ is completely similar, just using crossing matchings from A_1 to A_3, A_2 to A_4 respectively. This finishes Subcase 3.3.

Subcase 3.4: We may assume that the inequalities of Subcase 3.3 are violated as well and thus we have the

$$a_3(3) < |A_3| - |A_2| = x \tag{13}$$
$$a_3(2) < |A_3| - |A_1| = y \tag{14}$$

upper bounds in two colors for the maximum monochromatic matching in the 3-colored complete graph spanned by A_3. Now we will use the following Theorem of Cockayne and Lorimer [4] to get a lower bound z for $a_3(1)$, in terms of $|A_3|, x, y$.

Theorem 4. *(Cockayne and Lorimer, [4]) Assume that $n_1, n_2, n_3 \geq 1$ are integers such that $n_1 = max(n_1, n_2, n_3)$. Then for $n \geq n_1 + 1 + \sum_{i=1}^{3}(n_i - 1)$ every 3-colored K_n contains a matching of color i with n_i edges for some $i \in \{1, 2, 3\}$.*

Using the notation that size of a matching is twice the number of edges (as we did in the proof), an easy computation from Theorem 4 gives that $z = |A_3| - \frac{x+y}{2} + 1$ if $z \geq x, y$ (i.e. z is the maximum among x, y, z). Therefore in this case

$$a_3(1) \geq z > |A_3| - \frac{x+y}{2}. \tag{15}$$

Substituting x, y to (15) we get

$$a_3(1) > |A_3| - \frac{2|A_3| - |A_1| - |A_2|}{2} = \frac{|A_1| + |A_2|}{2}, \tag{16}$$

Now choose a matching $M_3(1)$ of size $a_3(1)$ in A_3. Using (11), $|A_1| \leq |A_2|$ and (16)

$$\begin{aligned} |A_4| &> |A_2| + |A_3| \geq \frac{|A_1| + |A_2|}{2} + |A_3| = |A_1| + |A_2| + \left(|A_3| - \frac{|A_1| + |A_2|}{2}\right) \\ &\geq |A_1| + |A_2| + |A_3 \setminus V(M_3(1))| = |A_1 + |A_2| + |A_3| - a_3(1), \end{aligned}$$

thus Claim 1 finishes the proof.

If z is not maximum then from $y \geq x$ the maximum is y and from Theorem 4, $z = |A_3| - (x + 2y) + 2$. Thus here

$$a_3(1) \geq z > |A_3| - (x + 2y). \tag{17}$$

Substituting x, y to (17)

$$a_3(1) > |A_3| - (2(|A_3| - |A_1|) + |A_3| - |A_2|) = 2|A_1| + |A_2| - |A_3|. \tag{18}$$

Now choose a matching $M_3(1)$ of size $a_3(1)$ in A_3. Using (11) we get

$$|A_4| > 2|A_3| > 2|A_3| - |A_1| = |A_1| + |A_2| + (|A_3| - (2|A_1| + |A_2| - |A_3|)).$$

If $2|A_1| + |A_2| - |A_3|$ is negative then $|A_4| > |A_1| + |A_2| + |A_3|$, otherwise by (18), $|A_4| > |A_1| + |A_2| + (|A_3| - a_3(1))$. In both cases Claim 1 finishes the proof.

The reader who followed the proof probably agrees that the cases when two or four of the $|A_i|$'s are odd can be treated easily from the following general remark. The inequalities used in the proofs are either sharp and then determine the parity of both sides or there is a slack of at least one and that can be used to adjust the proof. □

Proof of Lemma 3.

Since the proof is very straightforward, we do not address parity problems. By Corollary 1 we may assume that we have a 4-partite coloring (using the same notation as in the previous proof). Notice that equations

$$2|A_1| + a_2(1) + a_3(2) + a_4(3) < \frac{3n}{4} \tag{19}$$

$$a_1(1) + 2|A_2| + a_3(3) + a_4(2) < \frac{3n}{4} \tag{20}$$

$$a_1(2) + a_2(3) + 2|A_3| + a_4(1) < \frac{3n}{4} \tag{21}$$

$$a_1(3) + a_2(2) + a_3(1) + 2|A_4| < \frac{3n}{4}, \tag{22}$$

do not hold at the same time. Else summing them we get

$$\sum_{\substack{1 \leq i \leq 4 \\ 1 \leq j \leq 3}} a_i(j) < n,$$

a contradiction, because the union of perfect matchings within the A_i-s cover all n vertices.

We may assume that some, say the first, of the four (symmetric) inequalities fails, i.e.,

$$2|A_1| + a_2(1) + a_2(3) + a_4(3) \geq \frac{3n}{4}.$$

Select matchings $M_2(1), M_3(2), M_4(3)$ of size $a_2(1), a_3(2), a_4(3)$ in A_2, A_3, A_4, respectively.

If $|A_1| \geq |A_2| - a_2(1) + |A_3| - a_3(2) + |A_4| - a_4(3)$, then similarly to the case of Claim 1 we have a star-like partition, i.e., we cover perfectly all the vertices and all colors are different. Otherwise let M be a matching from A_1 to $B = (A_2 \cup A_3 \cup A_4) \setminus (V(M_2(1)) \cup V(M_3(2)) \cup V(M_4(3)))$. Clearly, $M \cup M_2(1) \cup M_3(2) \cup M_4(3)$ is a union of three connected monochromatic matchings in colors $1, 2, 3$ and is of size $2|A_1| + a_2(1) + a_2(3) + a_4(3) \geq \frac{3n}{4}$. □

3 Moving from complete graphs to almost complete ones

In the past few years the authors and others worked out technical steps of extending results from the complete graph K_n to $(1-\epsilon)$-dense graphs, that have at least $(1-\epsilon)\binom{n}{2}$ edges. Here we apply the method in [9] that replaces the $(1-\epsilon)$-dense graph by a more convenient subgraph H described in the next lemma.

Lemma 6. *(Gyárfás, Ruszinkó, Sárközy, Szemerédi, [9]) Assume that G_n is $(1-\varepsilon)$-dense. Then G_n has a subgraph H with at least $(1-\sqrt{\varepsilon})n$ vertices such that: A. $\Delta(\overline{H}) < \sqrt{\varepsilon}n$; B. $\delta(H) \geq (1-2\sqrt{\varepsilon})n$; C. H is $(1-2\sqrt{\varepsilon})$-dense.*

Another useful lemma from [9] is as follows.

Lemma 7. *(Gyárfás, Ruszinkó, Sárközy, Szemerédi, [9]) Assume $\Delta(\overline{G_n}) < \sqrt{\varepsilon}n$ and $H = [A, B]$ is a bipartite subgraph of G_n with $2\sqrt{\varepsilon}n < |A| \leq |B|$. Then H is a connected subgraph of G_n and contains a matching of size at least $|A| - \sqrt{\varepsilon}n$. Moreover, if only $2\sqrt{\varepsilon}n < |B|$ and $A \neq \emptyset$ is assumed then there is a subgraph H' which is connected and covers A and all but at most $\sqrt{\varepsilon}n$ vertices of B.*

We also need the density version of Theorem 4 of Cockayne and Lorimer [4], proved in [9].

Theorem 5. *(Gyárfás, Ruszinkó, Sárközy, Szemerédi, [9]) Assume that n_1, n_2, n_3 are nonnegative integers such that we have $n_1 = max(n_1, n_2, n_3)$, s is a nonnegative* *integer and G is a graph on n vertices such that for each $v \in V(G)$, $d(v) \geq n-1-s$. If*

$$n \geq s + n_1 + 1 + \sum_{i=1}^{3}(n_i - 1)$$

then, in any 3-coloring of the edges of G there is a matching of color i with n_i edges for some $i \in \{1, 2, 3\}$.

To transform the proof of Lemma 1 to the proof of Lemma 2 we do the following. We start with a 3-edge colored $(1-\epsilon)$-dense graph G_n and we find there a subgraph H described in Lemma 6. From now on we basically follow the steps of the proof of Lemma 1. For example, the proof of Lemma 1 starts with the case when G_1, the graph with edges of color 1, is a connected spanning subgraph. In the density version this should be replaced by requiring that $|V(G_1)| \geq (1-2\sqrt{\epsilon})n$. Then it is used that in a complete graph colored with two colors 2, 3 one of the colors, say color 2 spans a connected subgraph. In the density version this is replaced by asking that almost all vertices form a connected component in color 2.

Similarly, we reduce the proof in the density version to 4-partite colorings where $A_i \neq \emptyset$ is replaced by $|A_i| \geq c\sqrt{\epsilon}n$ with some small constant c. Inequalities (2), (6), (7), (8) and others governing the proof should be used with a slack of $2\sqrt{\varepsilon}n$. We use Lemma 7 to find matchings between parts of desired size. In the final step of the proof (Subcase 3.4) instead of using Theorem 4 of Cockayne and Lorimer [4] inside A_3 we should use its density version, Theorem 5.

One can derive Lemma 4 with similar modifications of the proof of Lemma 3. □

4 Moving from connected matchings to cycles

In this section we sketch how we can derive Theorem 1 from Lemma 2. This material is fairly standard by now, it has been applied and much discussed in [2], [7], [9] and [12] among others. Therefore here we just give a brief overview, the missing details can be found in these papers.

We apply the edge-colored version of the Regularity Lemma to a 3-colored K_n with a small enough ε, we define the reduced graph G^R and we introduce a majority coloring in G^R. Using Lemma 2 we find 3 monochromatic connected matchings which partition most of the vertices of G^R. Then we turn a monochromatic (say red) connected matching in G^R into a red cycle in K_n with the following standard procedure. We go back to the original graph and first we find short red vertex disjoint connecting paths between clusters of consecutive edges in the matching (using the fact that it is connected in red). These connecting paths will be parts of the final cycle and we remove the internal vertices. We remove a small number of exceptional vertices from each matching edge to guarantee super-regularity and to make sure that the bipartite graph corresponding to the matching edge is balanced. Then in each matching edge we find a red Hamiltonian path to close the red cycle (the existence of this Hamiltonian path is well-known; see e.g. Lemma 2 in [8] or actually this is a very special case of the Blow-up Lemma [11]).

5 Proof of Theorem 2

In this section we sketch the proof of Theorem 2. The proof is using Theorem 1 and results from [8], hence we omit some of the details.

Again just as in Section 4 we apply the edge-colored version of the Regularity Lemma to a 3-colored K_n with a small enough ε, we define the reduced graph G^R and we introduce a majority coloring in G^R. We will need the concept of a *half dense* matching from [8]: a matching M in a graph G is called k-half dense if one can label its edges as $x_1y_1, \ldots, x_{|M|}y_{|M|}$ so that each vertex of $X = \{x_1, \ldots, x_{|M|}\}$ (called the strong end points) is adjacent in G to at least k vertices of $Y = \{y_1, \ldots, y_{|M|}\}$.

Following the proof technique in [8] with $r = 3$, we find the 10 monochromatic cycles in the following steps.

- Step 1: We find a sufficiently large monochromatic (say red), half-dense connected matching M in G^R (more precisely an $(l/48)$-half dense matching where l is the number of vertices in G^R).
- Step 2: We remove the vertices of M from G^R and we go back to the original graph (instead of the reduced graph). We apply Theorem 1 with a small enough ε to cover most of the leftover graph by three monochromatic vertex disjoint cycles.
- Step 3: Applying a lemma about cycle covers of r-colored unbalanced complete bipartite graphs (Lemma 8 from [8], here we use it with $r = 3$) we combine the leftover vertices with some vertices of the clusters associated with vertices of M (by using 6 cycles).
- Step 4: Finally after some adjustments through alternating paths with respect to M, we find a red cycle spanning the remaining vertices of M.

For completeness we restate here the lemma needed in Step 3.

Lemma 8. *(Gyárfás, Ruszinkó, Sárközy, Szemerédi, [8]) There exists a constant* n_0 *such that the following is true. Assume that the edges of the complete bipartite graph* $K(A, B)$ *are colored with* r *colors. If* $|A| \geq n_0$, $|B| \leq |A|/(8r)^{8(r+1)}$, *then* B *can be covered by at most* $2r$ *vertex disjoint monochromatic cycles.*

Indeed, we may apply this lemma in Step 3 if ε is sufficiently small. All the other missing details can be found in [8]. The only major difference here is that in Step 2 we apply Theorem 1 instead of the approach of greedily removing monochromatic cycles used in [8]. □

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