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Note

Trees in greedy colorings of hypergraphs

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ABSTRACT

It is a well-known proposition that every graph of chromatic number larger than t contains every tree with t edges. The 'standard' reasoning is that such a graph must contain a subgraph of minimum degree at least t. Bohman, Frieze, and Mubayi noticed that, although this argument does not work for hypergraphs, it is still possible that the proposition holds for hypergraphs as well. Indeed, Loh recently proved that every uniform hypergraph of chromatic number larger than t contains every hypertree with t edges.

Here we observe that the basic property of the well-known greedy algorithm immediately implies a much more general result (with a conceptually simpler proof): if the greedy algorithm colors the vertices of an r-uniform hypergraph with more than t colors then the hypergraph contains every r-uniform hypertree with t edges.

Responding to a problem posed in [1], Loh [3] proved the following.

Theorem 1. *Every r-uniform hypergraph with chromatic number larger than t contains a copy of every r-uniform hypertree with t edges.*

In this note we show that Theorem 1 has a much stronger form (with an even simpler proof): the chromatic number can be replaced by the number of colors used by a greedy coloring. The well-known *greedy coloring* is a procedure that takes the vertices of a hypergraph in a given order and colors the current vertex by assigning the smallest positive integer that creates no completely monochromatic edge. The basic property of the greedy algorithm is that a vertex v of color k with $k > 1$ has a set of *witnesses*: edges $e_1, \ldots, e_{k-1} \in E(\mathcal{H})$ containing v such that i is the color of every vertex $w \in e_i \setminus \{v\}$. Call e_i the *witness in color i at v*.

A *hypertree* for $r \geq 2$ is an r-uniform hypergraph that can be built by taking an initial edge and then at each step adding a new edge that intersects the previous hypertree in exactly one vertex, the *root* of the new edge, while the other vertices of the new edge are referred to as the *leaf set*. We refer to these steps as the *defining order* of the tree (clearly the defining order is not unique). Notice that the witnesses at each vertex v in a greedy coloring form an r-uniform hypertree, *a star at v*.

Clearly, 2-uniform hypertrees are the usual trees in graphs. Since the greedy coloring obviously uses at least as many colors as the chromatic number, the next result extends Theorem 1.

Theorem 2. *If a greedy coloring of an r-uniform hypergraph $\mathcal{H}$ uses more than t colors, then $\mathcal{H}$ contains a copy of every r-uniform hypertree T with t edges.*

Proof. Let T be the target hypertree with t edges $e_0, e_1, \ldots, e_{t-1}$ in defining order. First, we define a coloring ψ on $V(T)$ as follows. Color one vertex of e_0 with $t+1$ and all others by t. Then, for $i = 1, \ldots, t-1$, color every vertex of the leaf set of e_i in $\cup_{j=0}^{i} e_j$ with color $t-i$ (notice that $t-i$ is smaller that the color of the root of e_i).

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Consider $\mathcal{H}$ with a greedy coloring φ using more than t colors. We build in $\mathcal{H}$ a φ-colored subtree with edges $h_0, \ldots, h_{t-1}$ that is isomorphic to the ψ-colored T. Let $v \in V(\mathcal{H})$ with $\varphi(v) = t + 1$ and select a witness $h_0 \in E(\mathcal{H})$ of color t at v. Clearly there is a color-preserving bijection f from vertices of e_0 to vertices of h_0 and that can be easily extended as follows. For $1 \leq i \leq t - 1$, suppose that e_i intersects at w the subtree $\cup_{j=0}^{i-1} e_j$. Select a witness $h_i \in E(\mathcal{H})$ of color $t - i$ at $f(w)$. Such a witness exists, and by mapping the leaf set of e_i (in the subtree $\cup_{j=0}^{i} e_j$) to $h_i \setminus \{f(w)\}$ we obviously get a color-preserving extension of f. Since the colors on the witnesses through the extensions are all different, the final f maps T to a subtree of $\mathcal{H}$ isomorphic to it. $\square$

The observant reader may notice that, for $t = 2$, Theorem 2 is essentially Exercise 13.33 in [4]: any hypergraph that does not contain a pair of edges intersecting in exactly one vertex is 2-colorable. The case when T is a path appeared as a lemma in [5]. Furthermore, for $r = 2$, the proof of Theorem 2 gives the following.

Corollary 1. *If a greedy coloring uses more than t colors on a graph G, then G contains every tree T on t edges with all vertices having distinct colors.*

It is worth noting that the coloring of the tree T in Corollary 1 is a special one; namely, T has a root such that on every path from the root the colors are decreasing. However, if the chromatic number of G is exactly $t + 1$, then G contains every tree on $t + 1$ vertices arbitrarily colored with distinct colors (see [2,6]).

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