Large Monochromatic Components in Edge Colorings of Graphs: A Survey

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1 Introduction

The aim of this survey is to summarize an area of combinatorics that lies on the border of several areas: Ramsey theory, resolvable block designs, factorizations, fractional matchings and coverings, and partition covers. Unless stated otherwise, coloring means *edge colorings* of graphs; an r-coloring is an assignment of elements of $\{1, 2, \ldots, r\}$ to the edges.

1.1 A Remark of Erdős and Rado and Its Extension

A very simple statement – the leitmotif of the survey – is a remark of Erdős and Rado. It can be phrased in different ways.

Proposition 1.1. *The following statements are equivalent:*

- *Either a graph or its complement is connected.*
- *Every* 2*-colored complete graph has a monochromatic spanning tree.*
- *If two partitions are given on a ground set such that each pair of elements is covered by some block of the partitions then one of the partitions is trivial, i.e., has only one block.*
- *Pairwise intersecting edges of a bipartite multigraph have a common vertex.*

The first two statements are clearly equivalent. The equivalence of the third and fourth follows through duality: the blocks of the two partitions (through duality) become the two partite sets of the bipartite graph and the vertices become

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(possibly multi-)edges. The equivalence of the second and third statements comes from considering the correspondence of blocks of the two partitions with the components of the two colored subgraphs in the 2-coloring of the edges of a complete graph.

Any of the (equivalent) statements formulated in Proposition 1.1 can be proved immediately; in Sect. 2 we overview many extensions of it. Several natural questions arise: can one say more about the monochromatic spanning tree guaranteed by Proposition 1.1; may connectivity be replaced by stronger properties, such as small diameter, higher connectivity (or both). These are discussed in Sect. 2.1. Another important extension is surveyed in Sect. 2.2 when 2-colorings are replaced by Gallai-colorings; these are colorings where the number of colors is not restricted but the requirement is that there is no multicolored (rainbow) triangle in the colorings. It turned out that many results that hold for 2-colorings have extensions, or even "black-box" extensions, to Gallai-colorings as well.

A separate section, Sect. 3, is devoted to r-colorings. The problem was suggested in [24] and the case $r = 3$ was solved there; a minor inaccuracy was corrected in [1]. The problem was rediscovered in [5]. The general result for r colors was proved in [25]. It extends Proposition 1.1 as follows.

Theorem 1.2 ([25]). *The following equivalent statements are true:*

- *In every r-coloring of K_n there is a monochromatic component with at least $n/r - 1$ vertices.*
- *If r partitions are given on a ground set of n elements such that each pair of elements is covered by some block of the partitions then one of the partitions has a block of size at least $n/r - 1$.*
- *If an intersecting r-partite (multi)hypergraph has n edges then it has a vertex of degree at least $n/r - 1$ (intersecting means that any two edges have a vertex in common).*

The equivalence of statements in Theorem 1.2 follows the same way as in Proposition 1.1 and can be proved by two different proof techniques shown in Sects. 3.1 and 3.2. The next subsection gives an important construction showing that Theorem 1.2 is close to best possible.

1.2 Colorings from Affine Planes

Consider an affine plane of order $r - 1$ that is r partitions of a ground set $V, |V| = (r-1)^2$ into blocks of size $r - 1$ so that each pair of elements of V is covered by a unique block. (If $r-1$ is a prime power, affine planes indeed exist.) There is a natural way to color the edges of a complete graph with vertex set V: for $i = 1, 2, \ldots, r$ color the pairs within the blocks of the ith partition class with color i. For example, for $r = 3$ we obtain the 3-coloring of K_4 (a factorization), for $r = 4$ we obtain the 4-coloring of K_9 where each color class is the union of three vertex disjoint triangles. In general, this coloring is an example showing that Theorem 1.2 holds with equality: every monochromatic connected component has $|V|/(r-1) = r-1$ vertices. Further cases of equality are discussed in the next subsection.

1.3 Extending Colorings by Substitutions

A useful way of extending a coloring of a complete graph is to substitute colored complete graphs to its vertices so that the edges between the substituted parts retain their original colors.

In the r-coloring described above, the cardinality of the vertex set is fixed: $|V| = (r-1)^2$. One can easily extend it by substituting complete graphs – usually with arbitrary colorings – into all vertices. For example, to see that Theorem 1.2 is sharp for $n = k(r-1)^2$ (and when affine plane of order $r-1$ exists) just substitute arbitrarily r-colored complete graphs on k vertices into the coloring defined in the previous subsection. If $n \neq k(r-1)^2$ then more subtle substitutions still can be used, these problems are explored in Sect. 3.3.

The colorings defined here and in the previous subsection work only when affine planes exist. On the other hand, if they do not exist then a result of Füredi [21] immediately implies that Theorem 1.2 can be improved (see Sect. 3.2 for more details).

Theorem 1.3. *Suppose that affine planes of order $r-1$ do not exist. Then in every r-coloring of K_n there is a monochromatic component with at least $n(r-1)/r(r-2)$ vertices.*

The first case when Theorem 1.3 applies is $r = 7$.

Problem 1.4. Let $f(n)$ be the cardinality of the largest monochromatic component that must occur in every 7-coloring of K_n. Then, from the previous results, the asymptotic of $f(n)$ is between $6n/35$ and $7n/35$. Improve these bounds!

The asymptotic of $f(n)$ in Problem 1.4 would follow from Füredi's problem ([22], Problem 4.6): to find α where

$$\alpha = \max\{\tau^*(\mathcal{H}) : \mathcal{H} \text{ is intersecting 7-partite hypergraph}\}.$$

In fact, $f(n) \sim n/\alpha$; see Sect. 3.2.

2 2-Colorings

2.1 Type of Spanning Trees, Connectivity, Diameter

Looking at the first form of Proposition 1.1, it is natural to ask what kind of monochromatic spanning trees can be found in every 2-coloring of a complete graph. Bialostocki, Dierker, and Voxman [3] suggested three types: trees of height at most two; trees obtained by subdividing the edges of a star with k edges (a k-octopu s); and trees obtained by identifying an endpoint of a path with the center of a star (a broom).

Theorem 2.1 ([3]). *In every* 2*-coloring of* K_n *there exists a monochromatic spanning* k*-octopus with* $k \geq \lceil (n-1)/2 \rceil$ *and also a monochromatic spanning tree of height at most two.*

The third suggested type, the broom, remained a conjecture until Burr found a proof. Unfortunately Burr's manuscript [11] was not published (although generalizations [16,31] appeared), so it is doubly justified to reproduce Burr's "book-proof" here.

Theorem 2.2 ([11]). *In every* 2*-coloring of* K_n *there exists a monochromatic spanning broom.*

Proof. Assume w.l.o.g. that in a red–blue coloring of a complete graph, the red graph is k-connected and the blue graph is at most k connected. Then the blue graph becomes disconnected after the deletion of a set X of at most k vertices. Since the red graph is k-connected, by a theorem of Dirac (see [43], Exercise 6.66) X can be covered by a red cycle (an edge if $k = 1$). Thus the vertex set of K_n can be covered by a red cycle C and a red complete bipartite graph $G = [A, B]$. Observe that a complete bipartite graph always has a spanning broom such that its starting point is arbitrary. Therefore covering C with a red path then continuing in the complete bipartite graph $[A \setminus C, B \setminus C]$ we can find a red broom. □

Concerning the diameter of a monochromatic connected spanning subgraph, the following result is folkloristic (forgive me if I missed further references).

Theorem 2.3 ([3, 45, 49]). *In every* 2*-coloring of a complete graph there is a monochromatic spanning subgraph of diameter at most three.*

Proof. If vertices u, v are at a distance at least three in red then uv is blue and every other vertex w is adjacent to at least one of u, v in blue. Thus there is a spanning double star in blue. □

How large is the largest monochromatic piece of diameter two? The following coloring shows that one cannot expect more than $3n/4$. Start with the 2-coloring of K_4 where both color classes form a P_4. Substitute nearly equal vertex sets into this coloring to get a total of n vertices. Erdős and Fowler [14] proved that this example is best possible.

Theorem 2.4 ([14]). *In every* 2*-coloring of* K_n *there is a monochromatic subgraph of diameter at most two with at least* $3n/4$ *vertices.*

The proof of Theorem 2.4 is difficult. A weaker result (also best possible) with a very simple proof is the following.

Theorem 2.5 ([26]). *In every 2-coloring of K_n there is a color, say red, and a set W of at least $3n/4$ vertices such that any pair of points in W can be connected by a red path of length at most two.*

Another natural question is the maximum order of a monochromatic k-connected subgraph in 2-colorings of K_n. This question was introduced in [9] and further elaborated in [41, 42].

Example. Let B be the 2-colored complete graph on vertex set [5] with red edges $12, 23, 34, 25, 35$ and with the other edges blue. (Both color classes form a "bull".) Assuming that $n > 4(k-1), k \geq 2$, let $B(n,k)$ be a 2-colored complete graph with n vertices obtained by replacing vertices $1, 2, 3, 4$ in B by arbitrary 2-colored complete graphs of $k-1$ vertices and replacing vertex 5 in B by a 2-colored complete graph of $n - 4(k-1)$ vertices. All edges between the replaced parts retain their original colors from B. Note that $B(n,k)$ denotes a member of a rather large family of graphs. The definition of $B(n,k)$ is used in the case $n = 4(k-1)$ as well, but in this case vertices 1, 4 (2, 3) of B are replaced by red (blue) complete subgraphs (and vertex 5 is deleted). Thus in this case we have just one graph for each k, which we denote by $B(k)$. Observe that the color classes of $B(k)$ form isomorphic graphs and there is no monochromatic k-connected subgraph in $B(k)$.

It is easy to check that in $B(n,k)$ the maximal order of a k-connected monochromatic subgraph is $n - 2(k-1)$. It is conceivable that each $B(n,k)$ is an optimal example for every k; i.e., the following assertion is true.

Conjecture 2.6 ([9]). For $n > 4(k-1)$, every 2-colored K_n has a k-connected monochromatic subgraph with at least $n - 2(k-1)$ vertices.

For $k = 2$ it is easy to prove the conjecture.

Theorem 2.7 ([9]). *For $n \geq 5$ there is a monochromatic 2-connected subgraph with at least $n-2$ vertices in every 2-coloring of K_n.*

Proof. Every 2-coloring of K_5 contains a monochromatic cycle. Proceeding by induction, let (w.l.o.g.) H be a 2-connected red subgraph with $n-3$ vertices in a 2-coloring of K_n. If some vertex of $W = V(K_n) \setminus V(H)$ sends at least two red edges to H then we have a 2-connected red subgraph with $n-2$ vertices. Otherwise the blue edges between $V(H)$ and W determine a 2-connected blue subgraph of at least $n-2$ vertices (either a blue $K_{2,n-4}$ or a blue $K_{3,n-3}$ from which three pairwise disjoint edges are removed). □

Conjecture 2.6 was answered positively in [42] for $k = 3$ and for $n \geq 13k$. In [9] it was remarked that it is enough to prove the conjecture for $4(k-1) < n < 7k-5$. Another related conjecture – the graph $B(k)$ shows that it is sharp if true – is the following.

Conjecture 2.8 ([9]). Every 2-colored K_n contains a monochromatic subgraph that is at least $(n/4)$-connected.

The following result was needed as a lemma in [36]. It shows that high connectivity and small diameter can be simultaneously required for monochromatic subgraphs with order close to n.

Theorem 2.9 ([36]). *For every k and for every 2-colored K_n there exists $W \subset V(K_n)$ and a color such that $|W| \geq n - 28k$ and any two vertices in W can be connected in that color by k internally vertex disjoint paths, each with length at most three.*

Notice that the paths connecting vertices of W in Theorem 2.9 may leave W, as in Theorem 2.5. Probably Theorem 2.9 can be strengthened, as Theorem 2.4 strengthens Theorem 2.5.

Problem 2.10. Is it possible to strengthen Theorem 2.9 by requiring that the monochromatic paths connecting the pairs of W are completely within W?

2.2 Gallai-Colorings: Substitutions to 2-Colorings

Edge colorings of complete graphs in which no triangles are colored with three distinct colors are called Gallai-colorings in [31]. These colorings are very close to 2-colorings as the following decomposition theorem shows. This result is implicit in Gallai's seminal paper [23] and was refined in [12]. The form below is from [31].

Theorem 2.11. *Every Gallai-coloring can be obtained from a 2-colored complete graph with at least two vertices by substituting Gallai-colored complete graphs into its vertices.*

Theorem 2.11 is a natural tool to extend results from 2-colorings to Gallai-colorings. In [31] several results were extended, most notably Burr's theorem (see Theorem 2.16). Certain properties are not extendible though; there is obviously a monochromatic star with at least $((n-1)/2) + 1$ vertices in every 2-coloring of K_n but this does not extend to Gallai-colorings. Substituting almost equal green complete graphs into the vertices of a 2-colored K_5 in which the red and blue colors form pentagons, we get a Gallai-coloring that shows that the following result is almost sharp (for $n = 5k + 2$ one can be added).

Theorem 2.12 ([31]). *In every Gallai-coloring of K_n there is a monochromatic star with at least $2n/5$ edges.*

In [35] a method was devised that can extend a result from 2-colorings to Gallai-colorings. It works for certain classes of graphs and when it works it provides a "black-box" extension; i.e., one does not need to know the (occasionally very difficult) proof of the 2-coloring result. To define those classes, a family $\mathcal{F}$ of finite connected graphs was called *Gallai-extendible* in [35] if contains all stars and if for all $F \in \mathcal{F}$ and for all proper nonempty $U \subset V(F)$ the graph $F' = F'(U)$ defined as follows is also in $\mathcal{F}$:

- $V(F') = V(F)$.
- $E(F') = E(F) \setminus \{uv : u, v \in U\}) \cup \{ux : u \in U, x \notin U, vx \in E(F)$ for some $v \in U\}$.

Theorem 2.13 ([35]). *Suppose that $\mathcal{F}$ is a Gallai-extendible family, and that there exists a function $f : \mathbf{N} \to \mathbf{N}$ such that for every n and for every 2-coloring of K_n there is a monochromatic $F \in \mathcal{F}$ with $|V(F)| \geq f(n)$.*

Then, for every n and every Gallai-coloring of K_n there exists a monochromatic $F' \in \mathcal{F}$ such that $|V(F')| \geq f(n)$ – with the same function f.

As shown in [35], graphs with spanning trees of height at most $h \geq 2$, graphs of diameter at most d for each $d > 1$, and graphs having a spanning double star are all Gallai-extendible. Therefore Theorems 2.3, 2.4, and Corollary 4.6 have black-box extensions to Gallai-colorings.

Theorem 2.14 ([35]). *In every Gallai-coloring of K_n one can find monochromatic spanning trees of height at most two, monochromatic double stars and monochromatic diameter two subgraphs with at least $3n/4$ vertices.*

Graphs having a spanning complete bipartite subgraphs are also Gallai-extendible, therefore we have the following.

Theorem 2.15 ([35]). *Every Gallai-colored K_n contains a monochromatic complete bipartite subgraph with at least $\lceil (n+1)/2 \rceil$ vertices.*

There are cases when Theorem 2.13 is not applicable (at least directly): brooms (or graphs having spanning brooms) are not Gallai-extendible, however, Theorem 2.2 remains true for Gallai-colorings as shown in [31] (conjectured by Bialostocki in [3]).

Theorem 2.16 ([31]). *In every Gallai-coloring of K_n there exists a monochromatic spanning broom.*

3 Multicolorings: Basic Results and Proof Methods

3.1 Complete Bipartite Graphs: Counting Double Stars

Usually Ramsey numbers are larger than the lower bound coming from the corresponding Turán numbers of the graph in the majority color. However, the following lemma is an exception.

Lemma 3.1 ([25]). *In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic subtree with at least n/r vertices.*

This lemma was obtained in [25] by proving that a majority color class (a color class with the largest number of edges) always has a subtree with at least n/r vertices. A short proof of this is due to Mubayi [45] and Liu, Morris, and Prince [41]. In fact they prove the following stronger statement: if the edges of the complete bipartite graph with n vertices are colored with r colors, there is a monochromatic double star with at least n/r vertices. A *double star* is a tree obtained by joining the centers of two disjoint stars by an edge.

Lemma 3.2 ([41, 45]). *In every r-coloring of a complete bipartite graph on n vertices there is a monochromatic double star with at least n/r vertices.*

Proof. Suppose that $G = [A, B]$ is an r-colored complete bipartite graph, let $d_i(v)$ denote the degree of v in color i. For any edge ab of color i, $a \in A, b \in B$, set $c(a,b) = d_i(a) + d_i(b)$. Using the Cauchy–Schwartz inequality, we get

$$\begin{aligned}\sum_{ab\in E(G)} c(a,b) &= \sum_{a\in A}\sum_{i=1}^{r} d_i^2(a) + \sum_{b\in B}\sum_{i=1}^{r} d_i^2(b)\\ &\geq |A|r\left(\frac{\sum_{a\in A}\sum_{i=1}^{r} d_i(a)}{|A|r}\right)^2 + |B|r\left(\frac{\sum_{b\in B}\sum_{i=1}^{r} d_i(b)}{|B|r}\right)^2\\ &= |A||B|\left(\frac{|A|+|B|}{r}\right),\end{aligned}$$

therefore for some $a \in A, b \in B$, $c(a,b) \geq |A| + |B|/r$; i.e., there is a monochromatic double star of the required size. □

Lemma 3.2 implies Theorem 1.2 in a stronger form.

Corollary 3.3. *Suppose that the edges of K_n are colored with r colors. Then either all color classes have monochromatic spanning trees or there is a monochromatic double star with at least $n/r - 1$ vertices.*

Proof. Indeed, if a color class does not have a spanning tree, there is a complete bipartite subgraph colored with $r - 1$ colors and Lemma 3.2 concludes the proof. □

It is possible that for $r \geq 3$ the second conclusion of Corollary 3.3 is always true. This problem and some results in this direction can be found in Sect. 4.2.

A possible improvement of Lemma 3.1 is suggested in [6].

Conjecture 3.4. If the edges of a complete bipartite graph $[A, B]$ are colored with r colors then there exists a monochromatic subtree with at least $\lceil |A|/r \rceil + \lceil |B|/r \rceil$ vertices.

For $2 \leq r \leq 4$ Conjecture 3.4 was proved in [6] with an example that shows that, unlike the case of Lemma 3.1, for $r = 5$ the conjectured large monochromatic subgraph is not in the majority color.

3.2 Fractional Transversals: Füredi's Method

To present a very powerful method introduced by Füredi, the notion of fractional covers and matchings is summarized. A *fractional transversal* of a hypergraph is a nonnegative weighting on the vertices such that the sum of the weights over any edge is at least 1. The *value* of a fractional transversal is the sum of the weights over all vertices of the hypergraph. Then $\tau^*(\mathcal{H})$ is the minimum of the values over all fractional transversals of $\mathcal{H}$. A *fractional matching* of a hypergraph is a nonnegative weighting on the edges such that the sum of weights over the edges containing any fixed vertex is at most 1. The *value* of a fractional matching is the sum of the weights over all edges of the hypergraph. Then $\nu^*(\mathcal{H})$ is the maximum of the values over all fractional matchings of $\mathcal{H}$. By LP duality, $\tau^*(\mathcal{H}) = \nu^*(\mathcal{H})$ holds for every hypergraph $\mathcal{H}$.

Assume that the edges of K_n are r-colored. By Theorem 1.2, to find a monochromatic component with at least $n/r - 1$ vertices is equivalent to finding a vertex of degree at least $n/r - 1$ in an intersecting r-partite multihypergraph $\mathcal{H}$ with n edges. Füredi proved [20] that in such hypergraphs $\tau^*(\mathcal{H}) \leq r - 1$. Using the observation that weighting all edges by the reciprocal of the maximum degree of the hypergraph is a fractional matching with value $|E(\mathcal{H})|/(D(\mathcal{H}))$, we get

$$\frac{|E(\mathcal{H})|}{D(\mathcal{H})} \leq \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \leq r - 1, \tag{1}$$

where D is the maximum degree of $\mathcal{H}$. Thus we have $n/r - 1 = |E(\mathcal{H})|/r - 1 \leq D(\mathcal{H})$.

Notice that the above proof uses the LP duality theorem and this is applicable in other variants of the problem; see, for example, Sect. 3.5. Moreover, whenever the nonexistence of affine planes of order $r - 1$ is known, Füredi [21] improved his upper bound $\tau^* \leq r - 1$ by $1/r - 1$ and this leads to Theorem 1.3.

3.3 Fine Tuning

Theorem 1.2 says that in any r-coloring of K_n there is a monochromatic component with at least $n/r - 1$ vertices. We have already seen that this is sharp if $r - 1$ is a prime power and n is divisible by $(r-1)^2$. The first case when one can improve on this (by one) occurs for $r = 3$ and $n = 4k + 2$ ([1]). In [6] the order of the largest monochromatic connected subgraph of K_n has been found for $r = 4, 5$ and for all values of n. It turned out that these values depend on the smallest multicover of affine planes. An *i-cover* of a hypergraph is a nonnegative integer weight assignment to the vertices such that the sum of weights on every edge is at least i. The minimum total weight over all i-covers is the i-cover number of the hypergraph. Let $w(i, q)$ be defined as the minimum of the i-cover numbers over all affine planes of order q. (The i-covers of affine planes are also called affine blocking sets.) For example, a fundamental result of Jamison [10, 37] says that $2q - 1$ points (points on the union of two intersecting lines) is the smallest 1-cover of the Desarguesian affine plane. However, $w(1,9) < 17$ because the Hughes plane of order 9 has a transversal of 16 points [10].

The most general sharp result is obtained by Füredi's method in [21] (similarly as explained in Sect. 3.2). In terms of the parameter $w(i,q)$, it gives a sharp result whenever the number of colors is one less than a power of prime. The result confirms a conjecture of Bierbrauer [7]. It is more convenient to use inverse notation here: let $f(D,r)$ be the maximum n such that there exists an r-coloring of the edges of K_n for which the largest monochromatic connected subgraph has no more than D vertices.

Theorem 3.5 ([21]). *Assume that an affine plane of order q exists. Define i for every D by $i = q\lceil D/q\rceil - D$ where $0 \le i < q$. Then, for every $D \ge q^2 - q$,*

$$f(D, q+1) = q^2 \left\lceil \frac{D}{q} \right\rceil - w(i,q).$$

3.4 When Both Methods Work: Local Colorings

The analogue of Theorem 1.2 for local r-colorings was obtained in [32]. A *local r-coloring* of a complete graph is a coloring where the number of colors incident to each vertex is at most r. How large is the largest monochromatic connected subgraph in local r-colorings of K_n?

Let $f(n,r)$ denote the largest m such that in every local r-coloring of the edges of K_n there is a monochromatic connected subgraph with m vertices. Clearly $f(n,r) \le n/r - 1$ whenever Theorem 1.2 is sharp, because r-colorings are special local r-colorings. This function has been also defined implicitly in [3], in connection with mixed Ramsey numbers. In particular, $RM(\mathcal{T}_n, G)$ was defined as the minimum m such that in any edge coloring of K_m there is either a monochromatic tree on n vertices or a totally multicolored copy of G. The special case when G is a star was treated in [4]. Since the requirement of forbidding a multicolored star $K_{1,\,r+1}$ is equivalent to local r-colorings, the next result implies the asymptotic value of $RM(\mathcal{T}_n, K_{1,!r+1})$ (extending the special case $r = 2$ in [4]).

Theorem 3.6 ([32]). $f(n,r) \ge rn/(r^2 - r + 1)$ *with equality if a finite plane of order $r - 1$ exists and $r^2 - r + 1$ divides n.*

The construction for showing that Theorem 3.6 is sharp when indicated is as follows. Consider the points of a finite plane of order $r - 1$ as the vertices of a complete graph, label the lines, and color each pair of vertices by the label of the line going through it. Then substitute each vertex i by a k-element set A_i so that the A_is are pairwise disjoint. The coloring is extended naturally with the proviso that the edges within A_is are colored with some color among the colors that were incident to vertex i. The result is a locally r-colored K_n where $n = k(r^2 - r + 1)$ and the largest monochromatic connected subgraph has $kr = nr/(r^2 - r + 1)$ vertices.

Both methods discussed in Sects. 3.1, and 3.2 can be used to prove Theorem 3.6. The method of counting double stars can be applied through the following theorem.

Theorem 3.7 ([32]). *Assume that the edges of a complete bipartite graph* $G = [A, B]$ *are colored so that the edges incident to any vertex of* A *are colored with at most* p *colors and the edges incident to any vertex of* B *are colored with at most* q *colors. Then there exists a monochromatic double star with at least* $|A|/q + |B|/p$ *vertices.*

A corollary of Theorem 3.7 is an extension of Lemmas 3.1 and 3.2.

Corollary 3.8 ([32]). *In every local* r*-coloring of a complete bipartite graph* G *there exists a monochromatic double star with at least* $|V(G)|/r$ *vertices.*

Proof of Theorem 3.6. If any monochromatic, say red component C satisfies $|C| \geq rn/(r^2 - r + 1)$, we have nothing to prove. Otherwise apply Theorem 3.7 for the complete bipartite graph $[A, B] = [V(C), V(G) \setminus V(C)]$. The edges incident to any $v \in A$ are colored with at most $p = r - 1$ colors and the edges incident to any $v \in B$ are colored with at most $q = r$ colors. Thus, using Theorem 3.7 and $|A| < rn/(r^2 - r + 1)$, there is a monochromatic component of size at least

$$|A|/q + |B|/p = \frac{|A|}{r} + \frac{n - |A|}{r - 1} = \frac{n}{r - 1} - |A|\left(\frac{1}{r - 1} - \frac{1}{r}\right)$$
$$> n\left(\frac{1}{r - 1} - \frac{r}{r^2 - r + 1}\left(\frac{1}{r(r - 1)}\right)\right) = \frac{rn}{r^2 - r + 1}. \qquad \square$$

Proof of Theorem 3.7. Let $d_i(v)$ denote the degree of v in color i. For any edge ab of color i, $a \in A, b \in B$, set $c(a, b) = d_i(a) + d_i(b)$. Let $I(v)$ denote the set of colors on the edges incident to $v \in V(G)$. Then, by using the Cauchy–Swartz inequality and the local coloring conditions, we get

$$\sum_{ab \in E(G)} c(a, b) = \sum_{a \in A} \sum_{i \in I(a)} d_i^2(a) + \sum_{b \in B} \sum_{i \in I(b)} d_i^2(b)$$
$$\geq |A|p\left(\frac{\sum_{a \in A} \sum_{i \in I(a)} d_i(a)}{|A|p}\right)^2 + |B|q\left(\frac{\sum_{b \in B} \sum_{i \in I(b)} d_i(b)}{|B|q}\right)^2$$
$$= |A||B|\left(\frac{|B|}{p} + \frac{|A|}{q}\right),$$

therefore for some $a \in A, b \in B$, $c(a, b) \geq |A|/q + |B|/p$. Since the edges incident to a or b in the color of ab span a monochromatic connected double star with $c(a, b)$ vertices, Theorem 3.7 follows. $\square$

The second proof of Theorem 3.6 follows the argument shown in Sect. 3.2. Assume that the edges of K_n are locally r-colored. Consider the hypergraph H whose vertices are the vertices of K_n and whose edges are the vertex sets of the connected monochromatic components. In the dual of H, H^*, every edge has at most r

vertices and each pair of edges has a nonempty intersection. Füredi proved [20] that in such hypergraphs the fractional transversal number, $\tau^*(H^*) \le r - 1 + (1/r)$. Then, as (1) in Sect. 3.2, we have

$$\frac{|E(H^*)|}{D(H^*)} \le \nu^*(H^*) = \tau^*(H^*) \le r - 1 + \frac{1}{r} \qquad (2)$$

where D is the maximum degree of H^*. Thus $((r|E(H^*)|)/(r^2 - r + 1)) \le D(H^*)$. Noting that $|E(H^*)| = n$ and $D(H^*)$ equals the maximum size of an edge in H (i.e., the maximum size of a connected component in the local r-coloring), the inequality of Theorem 3.6 follows.

3.5 Hypergraphs

Theorem 1.2 was extended to hypergraphs in [17] as follows. We note here that for hypergraphs there are several notions of connectivity. Unless stated otherwise we consider a hypergraph connected if its *cover graph* – the pairs of vertices that are covered by at least one edge of the hypergraph – spans a connected graph.

Theorem 3.9 ([17]). *In every r-coloring of the edges of the complete t-uniform hypergraph on n vertices, there is a connected monochromatic subhypergraph on at least n/q vertices, where q is the smallest integer satisfying $r \le \sum_{i=0}^{t-1} q^i$. The result is best possible if q is a prime power and n is divisible by q^t.*

The lower bound of Theorem 3.9 comes from Füredi's method. Let $f(n, r, t)$ be defined as the minimum size of a monochromatic component that must be present in any r-coloring of the t-sets of an n-element set. Since here hypergraphs are colored instead of graphs, the equivalent formulations of Theorem 1.2 have to be modified accordingly. Instead of intersecting r-partite (multi)hypergraphs we have *t-wise intersecting* (multi)hypergraphs (i.e., hypergraphs in which any t edges have a common vertex). Then – similarly to the arguments leading to (1) and (2) – one can estimate $f(n, r, t)$ as follows.

Lemma 3.10 ([17]). *$f(n, r, t) \ge n/\tau^*(r, t)$ where*

$$\tau^*(r, t) = \max\{\tau^*(\mathcal{H}) : \mathcal{H} \textit{ is } r\textit{-partite, } t\textit{-wise intersecting hypergraph}\}.$$

The example showing that Theorem 3.9 is sharp when stated is a natural extension of the construction in Sect. 1.2 from affine planes to affine spaces of dimension t. Consider $A(t, q)$, the affine space of dimension t and order q, define color class i by the t-element subsets of points that are within some hyperplane of the ith parallel class of hyperplanes. This coloring can be extended by substituting sets for points of $A(t, q)$ as in Sect. 1.3; in particular, if $n = q^t m$, one can substitute m vertices to all points of $A(t, q)$.

It is worth noting that for $t = 2$ we have $r = q + 1$ and Theorem 3.9 becomes Theorem 1.2. For $t \ge 3$ there are big gaps in the values of r for which Theorem 3.9

provides a sharp answer. For example, if $t = 3$, we get from Theorem 3.9 that for $r \leq 3$ we have a spanning monochromatic connected subhypergraph (i.e., one spanning all vertices) and for $r = 7$ we have one spanning at least $n/2$ vertices. For four five, and six colors Theorem 3.9 provides the same lower bound ($n/2$). The value of $\tau^*(4,3)$ was determined in [25] and the values $\tau^*(5,3)$, $\tau^*(6,3)$ in [29]. Through Lemma 3.10 it follows that

Theorem 3.11.

$$f(n,4,3) \geq \frac{3n}{4}[25], \quad f(n,5,3) \geq \frac{5n}{7}, \quad f(n,6,3) \geq \frac{2n}{3}[29].$$

In fact, Theorem 3.11 is sharp for infinitely many n (when the fractions in the lower bounds are integers).

4 Multicolorings: Type of Components

It would be interesting to know more about the structure of the largest monochromatic components. In the basic extremal colorings (Sect. 1.2) the components are complete graphs and after substitutions (Sect. 1.3) the components are balanced complete partite graphs. Thus it is expected that extremal colorings have strong connectivity properties.

4.1 Components with Large Matching

In Ramsey-type applications, for example, in [19, 30], and others, it turned out that the problem of finding a large matching in a monochromatic component can be applied to Ramsey problems concerning paths and cycles. Let $g(n,r)$ be the maximum m such that in every r-coloring of K_n there is a monochromatic component with a matching that covers at least m vertices. There are two natural upper bounds for $g(n,r)$. From the constructions showing that Theorem 1.2 is sharp (at least asymptotically) it follows that $g(n,r) \leq n/r - 1$ for infinitely many n and r. Since the Ramsey number of matchings was determined long ago by Cockayne and Lorimer [13], it follows that $g(n,r) \leq 2n/r + 1$. The two bounds coincide for $r = 3$ and in [30] it was proved that indeed, $g(n,3)$ is asymptotic to $n/2$ and this was a very important step to determine exactly the three color Ramsey number of paths. The following is probably a difficult problem (even for $r = 4$).

Problem 4.1. Is $g(n,r)$ asymptotic to $n/r - 1$?

The affirmative answer would imply (through the regularity lemma, applying a principle introduced by Łuczak in [44]) that the r-color Ramsey number of P_n is asymptotic to $(r-1)n$ and would probably be useful in many other applications as well.

4.2 Double Stars

Another type of component that emerged from the first proof of Theorem 1.2 is the double star. Corollary 3.3 states that in r-colorings of K_n where at least one color class is disconnected, there is a monochromatic double star with at least $n/(r-1)$ vertices. Perhaps this statement remains true for all colorings.

Problem 4.2. For $r \geq 3$, is there a monochromatic double star of size asymptotic to $n/(r-1)$ in every r-coloring of K_n?

However, even the following problem is open.

Problem 4.3. Is there a constant d (perhaps $d = 3$) such that in every r-coloring of K_n there is a monochromatic subgraph of diameter at most d with at least $n/(r-1)$ vertices?

For $r = 3$ the affirmative answer to Problem 4.3 follows from a result of Mubayi.

Theorem 4.4 ([45]). *In every* 3*-coloring of* K_n *there is a monochromatic subgraph of diameter at most four with at least* $n/2$ *vertices.*

The best known estimate for double stars is the following.

Theorem 4.5 ([33]). *For* $r \geq 2$ *there is a monochromatic double star with at least* $(n(r+1)+r-1)/r^2$ *vertices in any* r*-coloring of the edges of* K_n*.*

Corollary 4.6 ([33]). *In every* 2*-coloring of* K_n *there is a monochromatic double star with at least* $(3n+1)/4$ *vertices.*

Corollary 4.6 is close to best possible, 2-colorings of K_n where the largest monochromatic double star is asymptotic to $3n/4$ and can be obtained from random graphs or from Paley graphs. In [15] the existence of such a 2-coloring was proved by the random method. However, for $r \geq 3$ the random method does not provide a good upper bound for $f(n, r)$.

Observing that a double star has diameter at most three, the bound in Theorem 4.5 provides a slight improvement (for $r \geq 3$) of the following result of Mubayi.

Theorem 4.7 ([45]). *There is a monochromatic subgraph of diameter at most three with at least* $n/(r-1+1/r)$ *vertices in every* r*-coloring of* K_n*.*

5 Variations

We finish the survey by showing some variations of the basic theme in chronological order.

5.1 Vertex-Coverings by Components

A well-known conjecture, frequently cited as the Lovász–Ryser conjecture, states the following extension of Theorem 1.2. It is stated in three forms to parallel Theorem 1.2. $\tau(\mathcal{H})$ denotes the transversal number, the minimum number of vertices needed to intersect all edges of $\mathcal{H}$.

Conjecture 5.1. The following equivalent statements are true:

- In every r-coloring of K_n, $V(K_n)$ can be covered by the vertex sets of at most $r-1$ monochromatic components.
- If r partitions are given on a ground set of n elements such that each pair of elements is covered by some block of the partitions then the ground set can be covered by at most $r-1$ blocks.
- For every intersecting r-partite (multi)hypergraph $\mathcal{H}$, $\tau(\mathcal{H}) \le r-1$.

Conjecture 5.1 is proved for $r \le 4$ in [25] and for $r = 5$ in [48]. Related problems can be found in a recent survey by Kano and Li [38].

5.2 Coloring by Group Elements

Bialostocki and Dierker conjectured that Proposition 1.1 can be generalized as follows. In every coloring of the edges of K_{n+1} with colors in $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ there is a spanning tree with color sum zero modulo n (to get Proposition 1.1, use 0, 1 as two colors). The conjecture is proved for n prime in [2] and for general n in [18], [47]. In fact, the proof of Schriver and Seymour in [47] works for hypergraphs as well. An r-uniform hypertree $\mathcal{T}$ is a connected r-uniform hypergraph with p edges on $p(r-1)+1$ vertices. Notice that for $r = 2$ we get the usual definition of a tree in graphs.

Theorem 5.2 ([47]). *Suppose that $\mathcal{K}$ is the complete r-uniform hypergraph on $p(r-1)+1$ vertices and the edges of $\mathcal{K}$ are labeled with an Abelian group of order p. Then $\mathcal{K}$ has a spanning hypertree with total weight zero.*

5.3 Coloring Geometric Graphs

Following [46], a *geometric graph* is a graph whose vertices are in the plane in general position and whose edges are straight-line segments joining the vertices. A geometric graph is *convex* if its vertices form a convex polygon. A subgraph of a geometric graph is *noncrossing* if no two edges have a common interior point. Ramsey-type problems for geometric graphs were first studied in [39] and [40]. The following result of Károlyi, Pach, and Tóth (a geometric generalization of Proposition 1.1) was conjectured by Bialostocki and Erdős (see [3] with a proof for convex geometric graphs).

Theorem 5.3 ([39]). *In every 2-coloring of a geometric complete graph there is a non-crossing monochromatic spanning tree.*

Proof. The nice inductive proof of Theorem 5.3 from [39] is as follows. We may assume that vertices $P_1, \dots, P_n$ of the geometric K_n have strictly increasing x-coordinates. Set $L(i) = \{P_j : 1 \le j \le i\}$, $R(i) = \{P_j : i < j \le n\}$. We may also assume that the edges along the convex hull (meaning, really, the perimeter of the convex hull) of K_n have the same color, say red, otherwise induction works by removing a point P_j of the convex hull where two colors meet. Induction also works if for any i, $2 \le i \le n-1$, the monochromatic spanning trees in $L(i)$, $R(i)$ have the same color. Thus these spanning trees switch colors at each i, moreover for $i = 2$ the switch is from red to blue and for $i = n-1$ the switch is from blue to red, otherwise a red edge along the convex hull from P_1 or from P_n would define red noncrossing spanning trees. The conclusion is that for some i, $2 \le i \le n-2$, there is a red–blue switch at i and blue–red switch at $i+1$. Taking a (red) edge along the convex hull that joins the left (red) tree at i with the right (red) tree at $i+1$ results in a red noncrossing spanning tree. □

One can ask whether Lemmas 3.1 and 3.2 have geometric versions as well. The simplest case is when the complete bipartite graph is balanced and drawn with partite sets $A = \{(1,0),(2,0),\dots,(n,0)\}$ and $B = \{(1,1),(2,1),\dots,(n,1)\}$ (and the edge ab is the line segment joining $a \in A$ and $b \in B$ in R^2). Call this representation a *simple geometric* $K_{n,n}$.

It is possible that (for two colors) Lemma 3.1 extends to simple geometric $K_{n,n}$ (perhaps even for arbitrary drawings of $K_{n,n}$).

Problem 5.4 ([27, 28]). In every 2-coloring of a simple geometric $K_{n,n}$ there is a noncrossing monochromatic subtree (a caterpillar) with at least n vertices.

However, the stronger result, Lemma 3.2 does not extend but has the following geometric version.

Theorem 5.5 ([27, 28]). *In every 2-coloring of a simple geometric $K_{n,n}$ there is a noncrossing monochromatic double star with at least $4n/5$ vertices. This bound is asymptotically best possible.*

5.4 Coloring Noncomplete Graphs

Can one extend some of the results above from complete graphs to arbitrary graphs? Somewhat surprisingly, the answer is yes. Theorem 1.2 can be extended to arbitrary graphs as follows. Let $\alpha(G)$ denote the cardinality of a largest independent set of G.

Theorem 5.6. *The following equivalent statements are true:*

- *In every r-coloring of a graph G with n vertices there is a monochromatic component with at least $n/((r-1)\,\alpha\,(G))$ vertices.*
- *If r partitions are given on a ground set of n elements such that among any $\alpha+1$ elements at least one pair is covered by some block of the partitions then one of the partitions has a block of size at least $n/((r-1)\alpha)$.*
- *If an r-partite hypergraph has n edges and among them at most α are pairwise disjoint then it has a vertex of degree at least $n/((r-1)\alpha)$.*

Proof. The equivalence of the statements can be proved by the same translation process as in Theorem 1.2. Their proof is again by Füredi's method, using his result in a form that is more general than in the previous applications. Let $\nu(\mathcal{H})$ denote the maximum number of pairwise disjoint edges in a hypergraph $\mathcal{H}$. □

Theorem 5.7 ([20]). *If an r-uniform hypergraph $\mathcal{H}$ does not contain a projective plane of order $r-1$ than $\tau^*(\mathcal{H}) \le (r-1)\nu(\mathcal{H})$.*

To see that the third statement of Theorem 5.6 holds, let $\mathcal{H}$ be an r-partite hypergraph with n edges and $\nu(\mathcal{H}) \le \alpha$. Since a finite plane of order $r-1$ is obviously not r-partite, Theorem 5.7 applies and – as in previous applications – (1) in Sect. 3.2 and (2) in Sect. 3.4,

$$\frac{|E(\mathcal{H})|}{D(\mathcal{H})} \le \nu^*(\mathcal{H}) = \tau^*(\mathcal{H}) \le (r-1)\alpha,$$

where D is the maximum degree of $\mathcal{H}$. Thus we have

$$\frac{n}{(r-1)\alpha} = \frac{|E(\mathcal{H})|}{(r-1)\alpha} \le D(\mathcal{H}).$$

□

Theorem 5.6 may give hope that results mentioned so far for coloring complete graphs can have (hopefully nice) extensions or at least analogues for coloring graphs with fixed independence number. It looks as if this area is rather unexplored; almost all previous results can be the subjects of investigation. The test cases can very well be graphs with $\alpha(G) = 2$.

One particular attempt is started in [34] to extend Gallai-colorings to arbitrary graphs as edge colorings without multicolored triangles. Suppose that we have a Gallai-coloring of a graph G with $\alpha(G) = 2$. Let $f(n)$ be the minimum order of the largest monochromatic connected subgraph over all such colorings of graphs with n vertices. Clearly, by looking at the union of two disjoint complete graphs, $f(n) \le n/2$. At first sight it is not clear that $f(n)$ is linear; it turns out [34] that it is,

$$\frac{n}{5} \le f(n) \le \frac{3n}{8}$$

but not with coefficient $\frac{1}{2}$. In general, $f(n,\alpha)$ is within reasonable limits.

Theorem 5.8 ([34]). $(\alpha^2 + \alpha - 1)^{-1} n \leq f(n, \alpha) \leq (cn \log \alpha)/\alpha^2$

The following quick proof of the linearity of $f(n)$ (with a coefficient weaker than in Theorem 5.8) points to a far-reaching generalization. Let G be a graph with n vertices with a Gallai-coloring. By Ramsey's theorem every set of $k = R(3, \alpha(G) + 1)$ vertices contains a triangle. By easy counting this implies that G has at least cn^3 triangles, where c depends only on α. To each triangle T assign an edge of T whose color is repeated in T. By the pigeonhole principle, some $xy \in E(G)$ is assigned to $cn^3/\binom{n}{2} \geq 2cn$ triangles $T_i = xyz_i$. Since in each T_i there are two edges in the color of xy, say in red, the graph spanned by the red edges in the union of the $\{x, y, z_i\}$ is connected, and has at least $2cn + 2$ vertices.

With the idea of the proof above, Theorem 5.8 can be extended to hypergraphs and also to colorings that do not contain any multicolored copy of a fixed hypergraph F (in Gallai-colorings $F = K_3$). As for graphs, for a hypergraph $\mathcal{H}$, $\alpha(\mathcal{H})$ denotes the maximum cardinality of $S \subset V(\mathcal{H})$ such that no edges of $\mathcal{H}$ are completely in S.

Theorem 5.9 ([34]). *Suppose that the edges of an r-uniform hypergraph $\mathcal{H}$ are colored so that $\mathcal{H}$ does not contain multicolored copies of an r-uniform hypergraph F. Then there is a monochromatic connected subhypergraph $\mathcal{H}_1 \subseteq \mathcal{H}$ such that $|V(\mathcal{H}_1)| \geq c|V(\mathcal{H})|$, where c depends only on $\mathcal{F}$, r, and $\alpha(\mathcal{H})$ (thus does not depend on $\mathcal{H}$).*

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