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Monochromatic Matchings in the Shadow Graph of Almost Complete Hypergraphs

András Gyárfás[1*], Gábor N. Sárközy[1, 2†], and Endre Szemerédi[3]

[1]Computer and Automation Research Institute, Hungarian Academy of Sciences, Budapest, P.O. Box 63, Budapest H-1518, Hungary
gyarfas@sztaki.hu

[2]Computer Science Department, Worcester Polytechnic Institute, Worcester, MA 01609, USA
gsarkozy@cs.wpi.edu

[3]Department of Computer Science, Rutgers University, New Brunswick, NJ 08903, USA
szemered@cs.rutgers.edu

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Abstract. Edge colorings of r-uniform hypergraphs naturally define a multicoloring on the 2-shadow, i.e., on the pairs that are covered by hyperedges. We show that in any $(r-1)$-coloring of the edges of an r-uniform hypergraph with n vertices and at least $(1-\varepsilon)\binom{n}{r}$ edges, the 2-shadow has a monochromatic matching covering all but at most $o(n)$ vertices. This result confirms an earlier conjecture and implies that for any fixed r and sufficiently large n, there is a monochromatic Berge-cycle of length $(1-o(1))n$ in every $(r-1)$-coloring of the edges of $K_n^{(r)}$, the complete r-uniform hypergraph on n vertices.

Keywords: colored complete uniform hypergraphs, monochromatic matchings

1. Introduction

Let $\mathcal{H}$ be an r-uniform hypergraph (a family of some r-element subsets of a set). The *shadow graph* of $\mathcal{H}$ is defined as the graph $\Gamma(\mathcal{H})$ on the same vertex set, where two vertices are adjacent if they are covered by at least one edge of $\mathcal{H}$. A coloring of the edges of an r-uniform hypergraph $\mathcal{H}$, $r \geq 2$, induces a multicoloring on the edges of the shadow graph $\Gamma(\mathcal{H})$ in a natural way; every edge e of $\Gamma(\mathcal{H})$ receives the color of all hyperedges containing e. A subgraph of $\Gamma(\mathcal{H})$ is *monochromatic* if the color sets of its edges have a nonempty intersection.

A set of pairwise disjoint edges of the shadow graph covering $n-o(n)$ vertices is called an *almost perfect matching* of $\Gamma(\mathcal{H})$. Let $K_n^{(r)}$ denote the complete r-uniform

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hypergraph on n vertices. An r-uniform hypergraph is *almost complete,* if it has at least $(1-o(1))\binom{n}{r}$ edges. We call an r-uniform hypergraph $(1-\varepsilon)$-complete if it has at least $(1-\varepsilon)n^r/r!$ edges.

In this paper we prove the following conjecture from [3].

Theorem 1.1. *Assume that $r \geq 2$ is fixed, $\mathcal{H}$ is an almost complete r-uniform hypergraph with n vertices, and its edges are colored with $r-1$ colors. Then the induced multicoloring on $\Gamma(\mathcal{H})$ contains a monochromatic almost perfect matching.*

It is worth noting that Theorem 1.1 does not hold if we color with r colors instead of $r-1$. An example in [3] gives an r-coloring of $K_n^{(r)}$ such that the largest number of vertices covered by any monochromatic matching is not larger than $\frac{(2r-2)n}{2r-1}$. In fact that is conjectured to be the best result ([3]) and proved for $r=3$ ([5]).

It was proved in [3] that Theorem 1.1 implies a stronger result, namely, that the almost perfect monochromatic matching M guaranteed can be *connected* as well which means that the edges of M are in the same component of the hypergraph defined by the edges of the color of M. Moreover, it was shown in [3] how to combine this strengthening of Theorem 1.1 and a "weak version" of the hypergraph Regularity lemma to get a Ramsey-type result for Berge-cycles. An r-uniform Berge-cycle ([1]) of length ℓ is a sequence of distinct vertices $v_1, v_2, \ldots, v_\ell$ together with a set of distinct edges $e_1, \ldots, e_\ell$ such that e_i contains v_i, v_{i+1} ($v_{\ell+1} \equiv v_1$).

Corollary 1.2. *In every $(r-1)$-coloring of the edges of $K_n^{(r)}$ there is a monochromatic Berge-cycle of length at least $(1-o(1))n$.*

Note that in [3] it was conjectured that for sufficiently large n this statement is true with a monochromatic Berge-cycle of length n, i.e., there is a monochromatic *Hamiltonian* Berge-cycle. However, at the moment we are unable to prove this stronger statement.

The way to obtain the above corollary from Theorem 1.1 illustrates a principle due to Łuczak (suggested in [8]): In many cases the task of finding a monochromatic path or cycle can be reduced to the easier task of finding a monochromatic matching via the Regularity lemma. This principle is applied in many recent Ramsey-type results such as [2–7].

2. Proof of Theorem 1.1

Suppose that $\mathcal{H}$ is a hypergraph with vertex set V, $|V|=n$ and $0<\delta<1$ is fixed.

Informally, a sequence L of k distinct vertices of V is a δ-*bounded selection with respect to a family $\mathcal{F}$ of forbidden sets*, if its elements are chosen in k consecutive steps so that in each step there is a set $F \in \mathcal{F}$, $|F| \leq \delta n$ such that vertices from F cannot be included as the next element of the sequence.

Formally, to capture that the forbidden set F for the next vertex of the sequence depends on the choices of previous vertices, we assume that the sets of $\mathcal{F}$ are indexed with certain sequences of vertices, called valid sequences. This is achieved by the following assumptions. The empty sequence $\emptyset$ is valid and there is a set $F(\emptyset) \in \mathcal{F}$. A one-element sequence $x_1 \in V$ is valid if and only if $x_1 \notin F(\emptyset)$. For $1 \leq i < k$

and for each valid sequence $(x_1, x_2, \ldots, x_i)$ of distinct vertices of V there is a set $F(x_1, x_2, \ldots, x_i) \in \mathcal{F}$ and furthermore a sequence $(x_1, x_2, \ldots, x_i, x_{i+1})$ is valid if and only if $x_{i+1} \notin F(x_1, x_2, \ldots, x_i)$.

Now, with the assumptions of the previous paragraph, a sequence $L = (x_1, x_2, \ldots, x_k)$ of k distinct vertices is called a δ-*bounded selection with respect to* $\mathcal{F}$ if L is a valid sequence and each set in $\mathcal{F}$ has at most δn vertices. As a trivial example, every sequence is δ-bounded with respect to $\mathcal{F}$ that contains only empty sets. For the convenience of the reader a slightly less trivial example is described in the next paragraph.

Let $\mathcal{H}$ be a balanced complete tripartite graph on n vertices and let $A(x)$ denote the partite class containing $x \in V$. Then define $\mathcal{F}$ by $F(\emptyset) = \emptyset$ and $F(x_1, x_2, \ldots, x_i) = \cup_{j=1}^{i} A(x_j)$. With respect to these forbidden sets *every* $\frac{1}{3}$-bounded selection $L = (x_1, x_2)$ is an *edge* in $\mathcal{H}$; *every* $\frac{2}{3}$-bounded selection $L = (x_1, x_2, x_3)$ is a *triangle* in $\mathcal{H}$. In fact, no matter how forbidden sets are defined, the numbers $\frac{1}{3}, \frac{2}{3}$ can not be lowered without violating the statements above, that is, for a $\delta < \frac{1}{3}$ not every δ-bounded selection $L = (x_1, x_2)$ will be an edge in $\mathcal{H}$, and similarly, for a $\delta' < \frac{2}{3}$ not every δ'-bounded selection $L = (x_1, x_2, x_3)$ will be a triangle in $\mathcal{H}$.

The following lemma from [3] ensures that in almost complete hypergraphs there *exists* $\mathcal{F}$, such that with respect to $\mathcal{F}$ *every* δ-bounded selection of at most r vertices is contained in many edges of the hypergraph.

Lemma 2.1. *Assume that* $\varepsilon > 0, \delta = \varepsilon^{2^{-r}}$ *and* $\mathcal{H}$ *is a* $(1-\varepsilon)$*-complete r-uniform hypergraph on n vertices. Then there exists* $\mathcal{F}$*, such that with respect to* $\mathcal{F}$ *there exist* δ*-bounded selections and every* δ*-bounded selection* $L \subset V(\mathcal{H})$ *of at most r vertices has the following property: L is contained in at least* $(1-\delta)\frac{n^{r-|L|}}{(r-|L|)!}$ *edges of* $\mathcal{H}$*. In particular, for* $|L| = r$*, L is an edge of* $\mathcal{H}$*.*

Now we are ready to prove Theorem 1.1 by induction on r. Let $\varepsilon > 0$ be arbitrary, $\mathcal{H}$ is a $(1-\varepsilon)$-complete r-uniform hypergraph with n vertices whose edges are colored with $r-1$ colors. We shall prove that there is a monochromatic matching M in $\Gamma(\mathcal{H})$ covering all but at most αn vertices, where α tends to 0 if ε tends to 0.

Set $p = \sqrt{\delta} n + 1$. For $r = 2$ we have a $(1-\varepsilon)$-complete graph (colored with one color). Select a maximum matching M in the graph. Observe that the set of vertices uncovered by M forms an independent set. Since $\binom{p}{2} > \frac{\delta n^2}{2} > \frac{\varepsilon n^2}{2}$, less than $p = o(n)$ vertices are uncovered by M.

Assume that Theorem 1.1 is true for every $q < r$. Consider an $(r-1)$-coloring of a $(1-\varepsilon)$-complete r-uniform hypergraph $\mathcal{H}$ with $r \geq 3$. Set $\delta = \varepsilon^{2^{-r}}$ and apply Lemma 2.1 to define $\mathcal{F}$. Consider all δ-bounded selections of r vertices with respect to $\mathcal{F}$. Let G be the graph defined on the set of vertices that appear as the first vertices of these δ-bounded selections where the edges are those edges of the shadow graph $\Gamma(\mathcal{H})$ that appear as the first two vertices of these δ-bounded selections. Note that G has minimum degree at least $(1-2\delta)n$. Indeed, if $x_1 \in V(G)$ then at most δn vertices are forbidden for x_2 and another at most δn vertices are not in $V(G)$. On the other hand, for the remaining $(1-2\delta)n$ vertices Lemma 2.1 (with $|L| = 2$) ensures that $\{x_1, x_2\}$ is covered by at least $(1-\delta)\frac{n^{r-2}}{(r-2)!} > 0$ edges of $\mathcal{H}$, thus $x_1 x_2 \in E(G)$.

For any $v \in V(G)$ and $1 \le i \le r-1$, let A_i be the set of vertices w such that color i is not on the edge $vw \in E(G)$, without loss of generality, $|A_1| \le |A_2| \le \cdots \le |A_{r-1}|$. Assume that $y_i \in A_i$ for $1 \le i \le r-1$ are distinct vertices. Then $e = \{v, y_1, \ldots, y_{r-1}\} \notin E(\mathcal{H})$ because no color can be assigned to the r-tuple e. Therefore, with

$$D = \frac{|A_1|(|A_2|-1)(|A_3|-2)\cdots(|A_{r-1}|-r+2)}{(r-1)!},$$

v is contained in at least D r-tuples of $V(\mathcal{H})$ that are not edges of $\mathcal{H}$. However, Lemma 2.1 (with $|L| = 1$) guarantees that the number of these r-tuples is less than

$$\binom{n-1}{r-1} - (1-\delta)\frac{n^{r-1}}{(r-1)!} < \frac{\delta n^{r-1}}{(r-1)!}.$$

Now it follows that

$$\frac{(|A_1|-r+2)^{r-1}}{(r-1)!} < D < \frac{\delta n^{r-1}}{(r-1)!},$$

implying that $|A_1| < n\delta^{\frac{1}{r-1}} + r - 2$. Using this and the minimum degree condition in G, v is adjacent in color 1 to at least $(1-2\delta-\rho)n$ vertices of G, where $\rho = \delta^{\frac{1}{(r-1)}} + \frac{r-2}{n}$.

The argument above can be repeated for any $v \in V(G)$, showing that $V(G) = \cup_{i=1}^{r-1} X_i$ where $v \in X_i$ has the property that at least $(1-2\delta-\rho)n$ edges of G of color i are incident to v.

Let M_i be a maximum matching in color i in the subgraph of G induced by $V(G) \setminus X_i$ and set $Y_i = V(G) \setminus (V(M_i) \cup X_i)$. Observe that, from the choice of M_i, no edge of G within Y_i is colored with color i.

If

$$p(i-1) + \frac{|X_i|}{1-\delta} \ge |Y_i|$$

holds for some i, $1 \le i \le r-1$, then we have the required large matching in color i. Indeed, almost every edge of G incident to X_i has color i thus M_i can be extended to a matching that misses at most $p(i-1) + (3\delta + \rho)n = o(n)$ vertices of G.

Assume that $p(i-1) + \frac{|X_i|}{1-\delta} < |Y_i|$ for every i, $1 \le i \le r-1$. This implies that

$$\begin{aligned}
\sum_{i=1}^{r-1} |Y_i| &> p\sum_{i=1}^{r-1}(i-1) + \frac{1}{1-\delta}\sum_{i=1}^{r-1}|X_i| \\
&\ge p\binom{r-1}{2} + \frac{|V(G)|}{1-\delta} \\
&\ge p\binom{r-1}{2} + \frac{(1-\delta)n}{1-\delta} \\
&= p\binom{r-1}{2} + n.
\end{aligned}$$

We claim that this inequality implies $\left|Y_i \cap Y_j\right| \geq p$ for some $1 \leq i < j \leq r-1$. Indeed, otherwise

$$n \geq \left|\cup_{i=1}^{r-1} Y_i\right| > \sum_{i=1}^{r-1} |Y_i| - \binom{r-1}{2} p,$$

contradicting to the inequality above. This proves the claim.

Select Y_i, Y_j from the claim, without loss of generality, $|Y_{r-2} \cap Y_{r-1}| \geq p$. From the definition of p, $\binom{p}{2} > \frac{\delta n^2}{2}$ follows, implying from the minimum degree condition in G that there are $x, y \in Y_{r-2} \cap Y_{r-1}$ such that $xy \in E(G)$ — this implies that the colors $r-2, r-1$ are not among the colors of xy. Notice that for $r = 3$ we have a contradiction on this branch of the proof, $xy \in E(G)$ thus $\{x, y\}$ is covered by an edge of $\mathcal{H}$ (a triple) but it cannot have a color. Thus we may assume that $r \geq 4$.

Consider the $(r-2)$-uniform colored hypergraph $\mathcal{H}^*$ with edge set $\{e \setminus \{x, y\} : e \in \mathcal{H}$ and $\{x, y\} \subset e\}$, where the color of the edge $e \setminus \{x, y\}$ is by definition the color of the edge e in $\mathcal{H}$. Note that x, y are the first two vertices in a δ-bounded selection of Lemma 2.1 thus $\mathcal{H}^*$ is a $(1-\delta)$-complete hypergraph. Moreover, since $x, y \in Y_{r-2} \cap Y_{r-1}$, $\mathcal{H}^*$ is colored with $r-3$ colors (colors $r-1$ and $r-2$ cannot be used). Thus, since $r-2 \geq 2$ - induction applies, $\mathcal{H}^*$ has an almost perfect monochromatic matching M in its shadow graph. Observing that M is a monochromatic matching in the shadow graph of $\mathcal{H}$ as well, the proof is finished. ∎

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