Ramsey-Type Results for Gallai Colorings

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Abstract: A Gallai-coloring of a complete graph is an edge coloring such that no triangle is colored with three distinct colors. Gallai-colorings occur in various contexts such as the theory of partially ordered sets (in Gallai's original paper) or information theory. Gallai-colorings extend 2-colorings of the edges of complete graphs. They actually turn out to be close to

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2-colorings—without being trivial extensions. Here, we give a method to extend some results on 2-colorings to Gallai-colorings, among them known and new, easy and difficult results. The method works for *Gallai-extendible* families that include, for example, double stars and graphs of diameter at most d for $2 \leq d$, or complete bipartite graphs. It follows that every Gallai-colored K_n contains a monochromatic double star with at least $3n+1/4$ vertices, a monochromatic complete bipartite graph on at least $n/2$ vertices, monochromatic subgraphs of diameter two with at least $3n/4$ vertices, etc. The generalizations are not automatic though, for instance, a Gallai-colored complete graph does not necessarily contain a monochromatic star on $n/2$ vertices. It turns out that the extension is possible for graph classes closed under a simple operation called *equalization*. We also investigate Ramsey numbers of graphs in Gallai-colorings with a given number of colors. For any graph H let $RG(r,H)$ be the minimum m such that in every Gallai-coloring of K_m with r colors, there is a monochromatic copy of H. We show that for fixed H, $RG\ (r,H)$ is exponential in r if H is not bipartite; linear in r if H is bipartite but not a star; constant (does not depend on r) if H is a star (and we determine its value). J Graph Theory 64: 233–243, 2010

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1. INTRODUCTION

We consider edge colorings of complete graphs in which no triangle is colored with three distinct colors. In [19] such colorings were called Gallai partitions, in [15] the term Gallai colorings was used. The reason for this terminology stems from its close connection to results of Gallai on comparability graphs [13]. We will use the term *Gallai-coloring* and we assume that Gallai-colorings are colorings on complete graphs. It is useful to keep in mind a particular Gallai-coloring—sometimes called canonical coloring—where all color classes are stars ($V=[n]$ and for all $1 \leq i<j \leq n$ edge ij has color i).

More than just the term, the concept occurs again and again in relation of deep structural properties of fundamental objects. A main result in Gallai's original paper—translated to English and endowed by comments in [22]—can be reformulated in terms of Gallai-colorings. Basic results about comparability graphs can be equivalently discussed in terms of Gallai-colorings, as the theorem below shows. Further occurrences are related to generalizations of the perfect graph theorem [5], or applications in information theory [18].

The following theorem expresses the key property of Gallai-colorings. It is stated implicitly in [13] and appeared in various forms [4, 5, 15]. The following formulation is from [15].

Theorem 1. *Any Gallai-coloring can be obtained by substituting complete graphs with Gallai-colorings into vertices of a 2-colored complete graph on at least two vertices.*

The substituted complete graphs are called *blocks* whereas the 2-colored complete graph into which we substitute is the *base graph*. Substitution in Theorem 1 means replacements of vertices of the base graph by Gallai-colored blocks so that all edges between replaced vertices keep their colors.

Theorem 1 is an important tool for proving results for Gallai-colorings. For example, it was used to extend Lovász's perfect graph theorem to Gallai-colorings, see [5, 19]. In [4] a more refined decomposition of Gallai-colorings was established. In this paper we focus on the following subjects:

- Extending 2-coloring results as black boxes
- Gallai colorings with fixed number of colors

A. Gallai-Extension Using Black Boxes

In [15] Ramsey-type theorems for 2-colorings were extended to Gallai-colorings, using Theorem 1. Here, we have a similar goal, but we accomplish it using a completely different method. Instead of extending the proofs of 2-coloring results, we define a property—we call it *Gallai-extendible*—of families of graphs that automatically carries over 2-coloring results to Gallai-colorings.

Definition. A family $\mathcal{F}$ of finite connected graphs is *Gallai-extendible* if it contains all stars and if for all $F\in\mathcal{F}$ and for all proper nonempty $U\subset V(F)$ the graph $F'=F'(U)$ defined as follows is also in $\mathcal{F}$:

- $V(F')=V(F)$
- $E(F')=E(F)\setminus\{uv:u,v\in U\}\cup\{ux:u\in U,x\notin U,vx\in E(F)$ for some $v\in U\}$.

We will say that F' is the *equalization* of F in U. The conditions that Gallai-extendible families must contain only connected graphs and must contain all stars are somewhat technical. However, it seems that no application can really utilize more general definitions—and in the canonical Gallai coloring every color class is a star.

Our main result, Theorem 2, states that if every 2-colored K_n contains a monochromatic F of a certain order from a Gallai-extendible family then this remains true for Gallai-colorings: every Gallai-colored K_n also contains from the same family a monochromatic F' such that $|V(F')|\geq|V(F)|$.

Theorem 2. *Suppose that $\mathcal{F}$ is a Gallai-extendible family, and that there exists a function $f:\mathbf{N}\to\mathbf{N}$ such that for every n and for every 2-coloring of K_n there is a monochromatic $F\in\mathcal{F}$ with $|V(F)|\geq f(n)$.*

Then, for every n and every Gallai-coloring of K_n there exists a monochromatic $F'\in\mathcal{F}$ such that $|V(F')|\geq f(n)$—with the same function f.

Moreover, such an F' exists in one of the colors used in the base-graph and also with no edge of F' within a block of the base graph.

The proof of Theorem 2 is in Section 2 together with several examples of Gallai-extendible families (Lemma 1). Applying Theorem 2 to these families, we get the following corollaries (the first two were known before, the others are new). If G is a graph, then H is called a spanning subgraph, if $V(H)=V(G)$. Applying Theorem 2 to the family of connected graphs we get

Corollary 1. *Every Gallai-colored complete graph contains a monochromatic spanning tree.*

For 2-colorings, Corollary 1 is the well-known remark of Erdős and Rado—a first exercise in graph theory. For Gallai-colorings it was proved by Bialostocki et al. in [1]. Applying Theorem 2 to the family of graphs having a spanning tree of height at most two, we get

Corollary 2. *Every Gallai-colored complete graph contains a monochromatic spanning tree of height at most two.*

For 2-colorings Corollary 2 is due to [1], for Gallai-colorings it was proved in [15]. Applying Theorem 2 to the family of graphs with diameter at most three, we get

Corollary 3. *Every Gallai-colored complete graph contains a monochromatic spanning subgraph of diameter at most three.*

For 2-colorings Corollary 3 can be found in [1, 23, 24]. Applying Theorem 2 to the family of graphs with diameter at most two, we get

Corollary 4. *Every Gallai-colored K_n contains a monochromatic subgraph of diameter at most two with at least $\lceil 3n/4 \rceil$ vertices. This is best possible for every n.*

For 2-colorings, this is due to Erdős and Fowler [8] (a weaker version with an easy proof is in [14]). The following construction ([8]) shows that Corollary 4 is sharp: consider a 2-coloring of K_4 with both color classes isomorphic to P_4. Then substitute nearly equal vertex sets into this coloring with a total of n vertices. (The colorings within the substituted parts can be arbitrary.) Applying Theorem 2 to the family of graphs containing a spanning double-star (two vertex disjoint stars joined by an edge), we get

Corollary 5. *Every Gallai-colored K_n contains a monochromatic double star with at least $(3n+1)/4$ vertices. This is asymptotically best possible.*

The 2-color version of Corollary 5 is (a special case of) a result in [16], it slightly extends a special case of a result in [7]: in every 2-coloring of K_n there are two points, v,w and a color, say red, such that the size of the union of the closed neighborhoods of v,w in red is at least $(3n+1)/4$. (The slight extension is that one can also guarantee that the edge vw is red.) Corollary 5 is asymptotically best possible, as shown by a standard random graph argument in [7]. Applying Theorem 2 to the family of graphs containing a spanning complete bipartite graph, we get

Corollary 6. *Every Gallai-colored K_n contains a monochromatic complete bipartite subgraph with at least $\lceil (n+1)/2 \rceil$ vertices, and at least one more if n is congruent to* -1 *modulo 4.*

For 2-colorings Corollary 6 follows easily since there is a monochromatic star of the required size. However, for Gallai-colorings there are not always monochromatic stars with $\lceil (n+1)/2 \rceil$ vertices—the largest monochromatic star has $2n/5$ vertices, see [15]. It is worth noting that Corollary 6 is best possible for every n. Paley graphs provide infinitely many examples, but there are simpler 2-colorings that do not contain monochromatic complete bipartite graphs larger than the size claimed in Theorem 6. Consider the vertex set as a regular n-gon and define the red graph by edges xy forming a diagonal of length at most k (if $n=4k$, $4k+1$, $4k+2$) or at most $k+1$ (if $n=4k+3$).

We conclude this part with some remarks on Gallai-extendible families. A *broom* is the union of a path and a star where the end-vertex of the path coincides with the central vertex of the star, and this is the only common vertex of the two. Burr [2] proved that every 2-colored complete graph has a monochromatic spanning broom. Gyárfás and Simonyi [15] extended Burr's theorem to Gallai-colorings. We cannot reprove this result with Theorem 2 as a black box extension of Burr's theorem because brooms are not Gallai-extendible. However—and similar ideas might be useful in other potential applications of Theorem 2—it is possible to combine a key element of Burr's proof with Gallai-extendable families (in our case with $\mathcal{F}_5$) to extend Burr's theorem to Gallai colorings.

B. Gallai Colorings With Given Number of Colors

As mentioned above, in canonical Gallai-colorings each color class is a star, thus Gallai-colorings do not necessarily contain any monochromatic H different from a star (apart from isolated vertices). However, we may define for any graph H a kind of restricted Ramsey number, $RG(r,H)$, the minimum m such that in every Gallai-coloring of K_m with r colors there is a monochromatic copy of H.

It turns out that some classical Ramsey numbers whose order of magnitude seems hopelessly difficult to determine, behave nicely if we restrict ourselves to Gallai-colorings with r-colors. For example, the Ramsey number of a triangle in r-colorings, $R(r,K_3)$ is known to be between bounds far apart (c^r and $\lfloor er! \rfloor + 1$, see for example in [20]) but it is not hard to determine $RG(r,K_3)$ *exactly* as follows.

Theorem 3.

$$RG(r,K_3) = \begin{cases} 5^k + 1 & \text{for } r = 2k \\ 2 \times 5^k + 1 & \text{for } r = 2k+1 \end{cases}$$

In fact—as we were informed by Magnant [21]—Theorem 3 is due to Chung and Graham [6]. Here, we give a simpler proof, using Theorem 1.

It is worth noting that there are several "extremal" colorings for Theorem 3. For example, let G_1 be a black edge and let G_2 be the K_5 partitioned into a red and a blue pentagon. The graphs H_1, H_2 obtained by substituting G_1 (G_2) into vertices of G_2 (G_1) have essentially different 3-colorings and both are extremal for $r=3$ in Theorem 3.

Although one can easily determine some more exact values of $RG(r,H)$ for small graphs H, we conclude with the following two theorems that determine its order of magnitude.

Theorem 4. *Assume that H is a fixed graph without isolated vertices. Then $RG(r,H)$ is exponential in r if H is not bipartite and linear in r if H is bipartite and not a star.*

Theorem 5. *If $H=K_{1,p}$ is a star, $p>1$ and $r\geq 3$ then $RG(r,H)=(5p-3)/2$ for odd p, $RG(r,H)=(5p/2)-3$ for even p.*

For completeness of the star case, notice that for $H=K_{1,p}$ we have trivially $RG(1,H)=R(1,H)=p+1$ and $RG(2,H)=R(2,H)$ can be determined easily ($2p-1$ for even p and $2p$ for odd p, [17]). It is also worth noting that while $RG(r,H)$ is constant (does not depend on r), $R(r,H)$ is linear in r (and in p), see [3].

A Gallai-coloring can be also viewed as an anti-Ramsey coloring for C_3, anti-Ramsey colorings for a graph H have been introduced in [9]. This direction has a large literature that we do not touch here. Moreover, Gallai-colorings are also connected to so-called mixed Ramsey numbers, where the aim is to find either a multicolored graph G (in our case a triangle) or a monochromatic graph H. We are aware of some papers in preparation that determine exact values of $RG(r,H)$. Faudree et al. [10] determined the value of $RG(r,H)$ for many bipartite graphs H. Fujita [11] proved that $RG(r,C_5)=2^{r+1}+1$; Fujita and Magnant [12] extended Gallai-colorings to colorings without a rainbow S_3^+, a triangle with a pendant edge.

2. GALLAI-EXTENDIBLE FAMILIES, PROOF OF THEOREM 2

We denote by $\text{dist}_H(u,v)$ the number of edges in a shortest path of H between $u,v\in V(H)$.

Lemma 1. *The following families are Gallai-extendible*:

- $\mathcal{F}_1$, *the family of connected graphs*;
- $\mathcal{F}_2(d)$, *the family of graphs having a spanning tree of height at most d, for any $d\geq 2$—equivalently a root $x\in V(F)$ such that* $\text{dist}(x,v)\leq d$ *for all $v\in V(F)$*;
- $\mathcal{F}_3(d)$, *the family of graphs with diameter at most d for any $d\geq 2$*;
- $\mathcal{F}_4$, *the family of graphs having a spanning double-star—equivalently, two adjacent vertices forming a dominating set*;
- $\mathcal{F}_5$, *the family of graphs containing a spanning complete bipartite graph, that is, the family of graphs F so that $V(F)$ can be partitioned into two nonempty sets A and B so that $ab\in E(F)$ for all $a\in A$, $b\in B$*;

Proof. We prove for all these families $\mathcal{F}$ and every $F\in\mathcal{F}$, and for all proper nonempty $U\subseteq V(F)$ (or $U\in\mathcal{U}_F$) that the graph F' we get after equalizing in U is still in $\mathcal{F}$. Since the five families we consider are closed under the addition of edges and it is immediate from the definition that equalization is a monotonous operation, that is, $F_1\subseteq F_2$ implies $F_1'\subseteq F_2'$, it is sufficient to prove $F'\in\mathcal{F}$ for minimal elements $F\in\mathcal{F}$. Whenever it is comfortable to exploit this fact we will do it: for instance, when checking the statement for $\mathcal{F}_2$ or $\mathcal{F}_4$, F can be chosen to be a tree of height at most two, or a double-star.

For $\mathcal{F}_1$ the statement is immediate noting that the connectivity of F implies that whenever an edge $e=xy\in E(F), F\in\mathcal{F}_1$ disappears, there exists a path of length 2 in F' between its endpoints.

For $\mathcal{F}=\mathcal{F}_2(d)$ or $\mathcal{F}=\mathcal{F}_3(d)$ the following claim will provide the statement:

Claim. *For* $u,v\in V(F)$, $uv\in E(F)$ *we have* $\mathrm{dist}_{F'}(u,v)\leq 2$, *and if* $uv\notin E(F)$ *then* $\mathrm{dist}_{F'}(u,v)\leq \mathrm{dist}_F(u,v)$.

Indeed, if $uv\in E(F)$, then either at least one of u and v is not in U, and then $uv\in E(F')$, or $u,v\in U$, and then—from the connectivity of F and the fact that U is a proper subset of $V(F)$—they have a common F'-neighbor. The first part is proved.

To prove the second part, let P be a shortest path in F between u and v, $|E(P)|\geq 2$. Then P can be subdivided to subpaths induced by U and other subpaths (there must be others, since otherwise replace P by a two-path from u to v). Define the path P' in F' between u and v by replacing the subpaths in U by an arbitrary vertex in the subpath—in the special case when u or v is on the subpath, replace it by u or v. Since all vertices of U have the same neighbors outside U, P' will indeed be a path in F', and $|E(P')|\leq|E(P)|$, as claimed.

Now if $F\in\mathcal{F}_2(d)$ ($d\geq 2$), apply the claim to the root x and all other vertices $v\in V(F)$ to get that $F'\in\mathcal{F}_2(d)$. Similarly, if $F\in\mathcal{F}_3(d)$, apply the claim to all pairs $u,v\in V(F)$.

If $F\in\mathcal{F}_4$, let $xy\in E(F)$ be such that $V(F)$ consists of neighbors of x and neighbors of y. If neither x nor y are in U, no edge is deleted at equalization and there is nothing to prove. Similarly, if exactly one of them is in U, say $x\in U$, $y\notin U$, then $xy\in F$ implies that y is adjacent in F' with every vertex in U, and the vertices that are not in U remain neighbors of x or y in F' as well.

It remains to check $F'\in\mathcal{F}_4$ if both $x,y\in U$. This is also easy, because every vertex of F is adjacent to at least one of x and y, and therefore in F' every vertex of U is adjacent to every vertex in $V(F)\setminus U$. We are then done because a complete bipartite graph contains a spanning double-star.

Let $F\in\mathcal{F}_5$. If one of the two classes, say A is disjoint of U, $F\subseteq F'$, so the statement is obvious. If now U meets both A and B in a vertex a and b, respectively, we are also done, since all $A\cup B$ is F-adjacent with either a or b, so all vertices of $U\cap(A\cup B)$ are F'-adjacent with all vertices of $(A\cup B)\setminus U$, and both of these sets are nonempty, finishing the proof for this class. ∎

Proof of Theorem 2. Suppose that $\mathcal{F}$ is a Gallai-extendible family and c is a Gallai-coloring of K_n. By Theorem 1, c can be obtained by substituting Gallai-colored complete graphs into the vertices $\{v_1, v_2, \ldots, v_k\}$ of a base graph B with a red–blue coloring, $k \geq 2$. Suppose that B is connected in red (in fact, we shall use only that B has no isolated vertex in red). The vertex sets of the substituted complete graphs give a partition $\mathcal{U}$ on $V(K_n)$.

Let c' be the 2-coloring of K_n obtained from c by recoloring all edges within all blocks of the partition $\mathcal{U}$ to the red color. In the coloring c', by the assumption of Theorem 2, K_n has a subgraph $F \in \mathcal{F}$ with $|V(F)| \geq f(n)$, such that F is monochromatic in c'. If F is blue then F is a monochromatic subgraph in c as well and the proof is finished.

Thus we may assume that $F \subseteq E_{c'}(\text{red})$ (the red edges in c'). If $V(F) \subseteq U$ for some $U \in \mathcal{U}$ then—using that B has no isolated vertex in the red color—we can select a star S in K_n such that S is red in c, its center $v \notin U$ and its leaf set is U. Now $S \in \mathcal{F}$ (because $\mathcal{F}$ contains all stars) and $|V(S)| > |V(F)|$, finishing the proof.

Thus, we may assume that $V(F)$ is not a subset of any block of $\mathcal{U}$. Now equalize F in the blocks of $\mathcal{U}$ one after the other. Since F is connected and $V(F)$ is not a subset of some block, eventually all recolored edges will be deleted during the equalizations. We claim that the graph F' resulting from the equalization process is a subgraph of $E_c(\text{red})$. Indeed, equalization adds an edge ux $(u \in U)$ only if $x \notin U$, and there exists $v \in U$, $vx \in E(F)$. Since $E(F) \subseteq E_{c'}(\text{red})$, and vx is not a recolored edge, $vx \in E_c(\text{red})$ follows. Since every block sends only edges of one and the same color to every vertex, $ux \in E_c(\text{red})$ as well, confirming the claim.

Since $\mathcal{F}$ is Gallai-extendible, $F' \in \mathcal{F}$, and clearly $|V(F')| \geq |V(F)| \geq f(n)$. Now the proof is finished (the extra property stated about F' is obvious). ■

3. PROOF OF THEOREMS 3, 4, 5

Proof of Theorem 3. Let $f(r)$ denote the function one less than the claimed value of $RG(r, K_3)$. Observe that

$$f(r) \geq 2f(r-1) \tag{1}$$

for $r \geq 2$ with equality for odd r, and

$$f(r) = 5f(r-2) \tag{2}$$

for $r \geq 3$

To show that $RG(r, K_3) > f(r)$ let G_1 be a 1-colored K_2 and let G_2 be a 2-colored K_5 with both colors forming a pentagon. Recursively construct G_r for odd $r \geq 3$ by substituting two identically colored G_{r-1}'s into the two vertices of G_1 (colored with a different color). Similarly, for even $r \geq 4$, let G_r be defined by substituting five identically colored G_{r-2}'s into the vertices of G_2 (colored with two different colors). The r-coloring defined on G_r is a Gallai-coloring, clearly has $f(r)$ vertices and contains no monochromatic triangles.

We prove by induction that if a Gallai-coloring of K with r-colors and without monochromatic triangles is given then $|V(K)| \leq f(r)$. Using Theorem 1, the coloring of K can be obtained by substitution into a 2-colored nontrivial base graph B. In our case clearly $2 \leq |V(B)| \leq 5$.

Case 1: $|V(B)|=2$. Since there are no monochromatic triangles, the graphs substituted cannot contain any edge colored with the color of the base edge, therefore, by induction, they have at most $f(r-1)$ vertices. Thus

$$|V(K)|\leq 2f(r-1)\leq f(r)$$

using (1).

Case 2: $|V(B)|=3$. The base graph has no monochromatic triangle so it has an edge b_1b_2 whose color is used only once (as a color on a base edge). Then the graphs substituted into b_1,b_2 must be colored with at most $r-2$ colors and the graph substituted into the third vertex must be colored with at most $r-1$ colors. Thus

$$|V(K)|\leq 2f(r-2)+f(r-1)\leq f(r-1)+f(r-1)=2f(r-1)\leq f(r)$$

using (1) twice.

Case 3: $4\leq|V(B)|\leq 5$. The base graph has no monochromatic triangle so each vertex in the base is incident to edges of both colors. Therefore

$$|V(K)|\leq|V(B)|f(r-2)\leq 5f(r-2)=f(r)$$

using (2). ■

Proof of Theorem 4. First, we give an upper bound on $RG(r,H)$ that is exponential in r by showing $RG(r,H)\leq t^{(n-1)r+1}$ where $t=R(2,H)-1$ and $n=|V(H)|$. We shall assume that $|V(H)|\geq 3$, therefore $n\geq 3, t\geq 2$. Suppose indirectly that a Gallai-coloring with r colors is given on K, $|V(K)|\geq t^{(n-1)r+1}$, but there is no monochromatic H. The base graph B of this coloring has no monochromatic H therefore $|V(B)|\leq R(2,H)-1=t$. This implies that some of the graphs, say G_1, substituted into B has at least $t^{(n-1)r}$ vertices. Let v_1 be an arbitrary vertex of K not in $V(G_1)$. Note that every edge from v_1 to $V(G_1)$ has the same color. Iterating this process with G_1 in the role of K, one can define a sequence of vertices $v_1,v_2,\dots,v_{(n-1)r+1}$ such that for every fixed i and $j>i$ the colors of the edges v_i,v_j are the same. By the pigeonhole principle there is a subsequence of n vertices spanning a monochromatic complete subgraph $K_n\subset K$ and clearly H is a monochromatic subgraph of K_n—a contradiction. Thus, for any—in particular non-bipartite—H we proved an upper bound exponential in r.

For a bipartite H assume that both color classes of H have at most n vertices. We show that $RG(r,H)\leq pt(n-1)$, where $p=(n-1)r+2$ (and t is as defined earlier), providing an upper bound linear in r. Indeed, suppose indirectly that a Gallai-coloring with r colors is given on K, $|V(K)|\geq pt(n-1)$ but there is no monochromatic H. The base graph of the Gallai-coloring has at most t vertices, otherwise we have a monochromatic H. Applying the same argument as in the previous paragraph, we find that there is a graph G_1, substituted to some vertex of the base graph, such that $|V(G_1)|\geq|V(K)|/t\geq p(n-1)$. If $|V(K)\setminus V(G_1)|\geq 2n-1$ then—by the pigeonhole principle—we can select $X\subset V(K)\setminus V(G_1)$ so that $|X|=n$ and $[X,V(G_1)]$ is a monochromatic complete bipartite graph—this graph contains H and the proof is finished. We conclude that $|V(G_1)| \geq pt(n-1)-2(n-1)=(pt-2)(n-1)$. Select $v_1 \in V(K)\setminus V(G_1)$ and iterate the argument: into some vertex of the base graph of the Gallai-coloring on G_1 a graph G_2 is substituted with at least $(|V(G_1)|/t) \geq (p-1)(n-1)$ vertices. Selecting $v_2 \in V(G_1)\setminus V(G_2)$ we continue until $T=\{v_1, v_2, \ldots, v_{p-1}\}$ is defined. There is still at least $2(n-1)>n$ vertices in G_{p-1} thus selecting $Y \subset V(G_{p-1})$ with $|Y|=n$, we have a complete bipartite graph $[Y,T]$ such that from each $v \in T$ all edges from Y to v are colored with the same color. Since $|T|=p-1=(n-1)r+1$, by the pigeonhole principle there is $Z \subset T$ such that $|Z|=n$ and $[Y,Z]$ is a monochromatic complete bipartite graph which obviously contains a monochromatic H—a contradiction. Thus, for bipartite H we have an upper bound linear in r.

Lower bounds of the same order of magnitude can be easily given. For a non-bipartite H it is obvious that $RG(r,H)>2^r$ because we can easily define a suitable Gallai-coloring with r colors by repeatedly joining with a new color two identically colored complete graphs of the same size.

If H is bipartite and not a star, it contains two *independent*, that is, vertex-disjoint edges. Then we have $RG(r,H)>r+1$ because the canonical Gallai-coloring of K_{r+1} with r colors (where color class i is a star with i edges) does not have a monochromatic H. ■

Proof of Theorem 5. Assume $H=K_{1,p}, p>1, r\geq 3$. We use a construction and a result from [15]. To see that the claimed values of $RG(r,H)$ cannot be lowered, let C be a K_5 colored with red and blue so that both color classes form a pentagon. For odd p substitute a green $K_{(p-1)/2}$ to each vertex of C. For even p substitute $K_{p/2}$ into one vertex of C and $K_{(p/2)-1}$ to the other four vertices of C. The claimed upper bound of $RG(r,H)$ for odd p follows immediately from the following result of [15] (Theorem 3.1): any Gallai-coloring of K contains a monochromatic $K_{1,p}$ with $p \geq 2|V(K)|/5$ edges. For even p one has to gain one over that bound and that can be easily obtained by modifying the (easy) proof there. ■

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